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Generalizations of Polynomials in Chebyshev Form

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Abstract

Arbitrary polynomial of degree n can be written in Chebyshev form. In this paper, we generalize this Chebyshev form and study its root distributions.

Keywords: Polynomial, Chebyshev Form, Root Distribution

1. Introduction

Chebyshev polynomials are of great importance in many areas of mathematics, particularly approximation theory. Many papers and books[1,2] have been written about these polynomials. Let $T_n(x)$ be the Chebyshev polynomial of the first kind. These polynomials satisfy the recurrence relations

$$T_0(x) = 1, \quad T_1(x) = x,$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad (n \geq 1)$$

Arbitrary polynomial of degree n can be written in terms of the Chebyshev polynomials of the first kind. Such a polynomial $p_n(x)$ is of the form

$$p_n(x) = \sum_{k=0}^{n} a_k T_k(x)$$

Define for a constant c,

$$p_n(c,x) = \sum_{k=0}^{n-1} a_k T_k(x) + cT_n(x)$$

Then $p_n(c, x)$ is a generalization of $p_n(x)$ and $p_n(a_n, x) = p_n(x)$. If x_0 is a root of $p_n(c, x)$, i.e., $p_n(c, x_0)=0$, then

$$0 = \sum_{k=0}^{n-1} a_k T_k(x_0) + cT_n(x_0)$$
$$= p_n(x_0) + (c - a_n)T_n(x_0)$$

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and so c is uniquely determined by

$$c = -\frac{p_n(x_0)}{T_n(x_0)} + a_n$$

provided that $T_n(x_0) \neq 0$. In this case

$$p_n(c,x) = \sum_{k=0}^{n-1} a_k T_k(x) + \left(-\frac{p_n(x_0)}{T_n(x_0)} + a_n\right) T_n(x)$$
$$= p_n(x) - \frac{p_n(x_0)}{T_n(x_0)} T_n(x),$$

and the roots (including x_0) of $p_n(c, x)=0$ satisfy

$$p_n(x) = \frac{p_n(x_0)}{T_n(x_0)} T_n(x)$$

2. Results and Examples

We summarize the above in Section 1 as follows.

Proposition 1 Let $p_n(x)$ be a polynomial of degree n. Write

$$p_n(x) = \sum_{k=0}^{n} a_k T_k(x)$$

Define

$$p_n(c,x) = \sum_{k=0}^{n-1} a_k T_k(x) + cT_n(x)$$

If x_0 is a root of $p_n(c, x)$, then all roots of $p_n(c, x_0)=0$ satisfy

$$p_n(x) = \frac{p_n(x_0)}{T_n(x_0)} T_n(x)$$

As an example of Proposition 1, we consider the polynomial

$$p_n(x) = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Using (1.5.32) in p. 55 of [2], we may compute that

$$(1+x)^n = {\sum_{k=0}^{n}}' A_k T_k(x)$$

where $\sum'$ means a sum with the first term halved and

$$A_k = \frac{1}{2^{k-1}} \left(\binom{n}{k} + \sum_{j=1}^{\left[\frac{n-k}{2}\right]} \frac{\binom{k+2j}{j}\binom{n}{k+2j}}{2^{2j}} \right)$$
$$= \frac{1}{2^{n-1}} \frac{\Gamma(1+2n)}{\Gamma(1-k+n)\Gamma(1+k+n)}$$
$$= \frac{1}{2^{n-1}} ({}_2F_1(n+k, n-k, 2n+1; 1)),$$

where ${}_2F_1(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n$, $|x| < 1$, is the hypergeometric function and $(\theta)_n = \theta(\theta+1)\ldots(\theta+n-1)$ for $n > 0$ and $(\theta)_0 = 1$. So

$$p_n(x) = (1+x)^n = \frac{1}{2^{n-1}} {\sum_{k=0}^{n}}' {}_2F_1(n+k, n-k, 2n+1; 1)\, T_k(x)$$

Suppose that x_0 is a root of

$$p_n(c, x) = \frac{1}{2^{n-1}} {\sum_{k=0}^{n-1}}' {}_2F_1(n+k, n-k, 2n+1; 1)\, T_k(x) + c T_n(x) = 0$$

Then by Proposition 1, all roots of $p_n(c, x) = 0$ satisfy

$$(1+x)^n = \frac{(1+x_0)^n}{T_n(x_0)} T_n(x)$$

We consider a special case when $x_0 = -1/2$, and study root distribution of $p_n(c, x) = 0$ when $n = 6k$ or $6k+3$. Since for $k \geq 0$,

$$T_n\left(\frac{1}{2}\right) = \begin{cases} 1, & n = 6k, \\ -1, & n = 6k+3, \\ \frac{1}{2}, & n = 6k+1, 6k+5, \\ -\frac{1}{2}, & n = 6k+2, 6k+4, \end{cases}$$

we have

$$(1+x)^n = \begin{cases} \frac{1}{2^n} T_n(x), & n = 6k, 6k+3, \\ \frac{-1}{2^{n-1}} T_n(x), & \text{otherwise.} \end{cases}$$

Proposition 2 Let $p_n(x) = (1+x)^n$. With the same notations used above, if $x_0 = -1/2$ and $n = 6k$ or $6k+3$, then all real roots of $p_n(c, x) = 0$ lie in $(-\infty, -1/2]$ and there are exactly $n/3$ or $n/3+1$ real roots in $(-1, -1/2]$.

Proof Let

$$f_1(x) := (1+x)^n, \quad f_2(x) := \frac{1}{2^n} T_n(x)$$

and $x = \cosh\theta$, $\theta \geq 0$ so that $x \geq 1$. Then $T_n(x) = \cosh n\theta$, and

$$f_1(x) - f_2(x) = \frac{1}{2^n}\left((2 + e^{\theta} + e^{-\theta})^n - \frac{e^{n\theta} + e^{-n\theta}}{2} \right) > 0$$

So there are no real roots in $[1, \infty)$. If $-1 \leq x \leq 1$, then $|T_n(x) \leq 1|$ and so

$$|f_2(x)| \leq \frac{1}{2^n}$$

The equation $|f_1(x)| = |f_2(x)|$ implies that $-1 \leq x \leq -1/2$. We observe that $f_1(x)$ is increasing for $x \geq -1$ and, on the interval $[-1, 1]$ all of the extrema of $f_2(x)$ have values that are either $1/2^n$ or $-1/2^n$. Since

$$f_1\left(-\frac{1}{2}\right) = f_2\left(-\frac{1}{2}\right) = \frac{1}{2^{n,}}$$

and $f_2(x)$ have $n/3$ roots

$$f_2(-1) = \begin{cases} \frac{1}{2^n}, & n = 6k, \\ -\frac{1}{2^n}, & n = 6k+3, \end{cases}$$

in $(-1, -1/2)$, $f_1(x) = f_2(x)$ has exactly $n/3$ or $n/3+1$ real roots in $(-1, -1/2]$.

Remark In case of $x_0 = -1/2$ and $n = 6k$ or $6k+3$, it seems that there is only one real root in $(-\infty, -1]$. Other cases can be considered similarly.

References

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[2] T. J. Rivlin, "Chebyshev polynomials: From approximation theory to algebra and number theory", New York: John Wiley and Sons, 1990.