APPROXIMATION ORDER OF C^3 QUARTIC B-SPLINE APPROXIMATION OF CIRCULAR ARC

SUNG CHUL BAE[1] AND YOUNG JOON AHN[2†]

[1]DEPARTMENT OF MATHEMATICS EDUCATION, KOREA UNIVERSITY, SEOUL, 136-701, KOREA

[2]DEPARTMENT OF MATHEMATICS EDUCATION, CHOSUN UNIVERSITY, GWANGJU, 501–759, KOREA
E-mail address: ahn@chosun.ac.kr

ABSTRACT. In this paper, we present a C^3 quartic B-spline approximation of circular arcs. The Hausdorff distance between the C^3 quartic B-spline curve and the circular arc is obtained in closed form. Using this error analysis, we show that the approximation order of our approximation method is six. For a given circular arc and error tolerance we find the C^3 quartic B-spline curve having the minimum number of control points within the tolerance. The algorithm yielding the C^3 quartic B-spline approximation of a circular arc is also presented.

1. INTRODUCTION

The circle approximation is one of the most simple and challenging problems in the field of CAGD(Computer Aided Geometric Design). If a circular arc is subdivided with the same length, then all subdivided arcs are congruent. Therefore, if one arc is approximated by a Bézier curve, then all arcs can be approximated by the same method. This is the reason why the circle approximation is simple. However, a reduction in the error and an increase in the continuity of the approximation curve remain as problems to be solved. In the last thirty years, the focus of the circle approximation has been to find the spline approximation that has the highest possible orders of approximation and continuity.

Since de Boor [1] showed the existence of the G^2 cubic spline approximation of a planar curve with approximation order six, many studies have been carried out on the circle approximation using a Bézier or spline curve with a higher order of approximation or continuity. The methods for the circle or ellipse approximation by a G^1 quadratic [2] or G^k cubic spline curve [3, 4, 5] for $k = 1, 2$ have been presented with increasingly smaller error. Floater [6, 7] found a G^2 quadratic spline approximation of a conic with approximation order four and a G^{n-1} spline approximation of odd degree n of a conic with approximation order $2n$, which can be

naturally applied to the circle approximation. The approximation methods of a circular arc by a quartic or quintic spline with approximation order eight or ten and with G^k-continuity for $k = 1, 2, 3$ have been developed [8, 9, 10]. The error bound of a quartic Bézier approximation of a circular arc has been further reduced [11, 12, 13, 14, 15]. Moreover, the circle approximation using LN(linear normal) Bézier curves has been used to obtain the offset approximation [16, 17, 18, 19, 20].

The spline approximation of a circular arc is obtained by merging the Bézier approximation of a segment of the circular arc and its rotations. If the approximate spline is G^k-continuous at the junction point of two consecutive Bézier segments, the number of control points of the spline can be reduced by k. Thus, the continuity order of the approximate spline curve is an important factor. The geometric continuity G^k cannot imply the continuity C^k for $k \geq 2$ in general. Yang and Ye [21] presented a C^2 cubic spline approximation of a circular arc, but there is no C^3 spline approximation whose Bézier segments are all congruent in previous works on circle approximations. This is the motivation of our work. Since any C^3 spline curve composed of two or more Bézier segments should have a degree of at least four, we present a C^3 quartic B-spline approximation of a circular arc in this paper. The exact form of the Hausdorff distance between the circular arc and the C^3 quartic B-spline approximation is obtained. Using this Hausdorff distance, we present an algorithm yielding the C^3 quartic B-spline approximation with the smallest number of control points and an error less than a given tolerance.

The C^1 quadratic spline approximation of a circular arc can be easily obtained, and it has approximation order four [2]. Yang and Ye [21] showed that the C^2 cubic B-spline approximation of a circular arc has approximation order four as well. Thus, we are interested in the approximation order of our C^3 quartic B-spline approximation of a circular arc. In this paper, we show that its approximation order is six, which is a very interesting result.

Our manuscript is organized as follows. In section 2, we find the C^3 quartic uniform B-spline approximation of a circular arc, which is obtained by merging the quartic Bézier approximation and its rotations. In section 3, we present the Hausdorff distance between the C^3 quartic B-spline approximation and the circular arc in closed form, and prove that it has the approximation order six. In section 4, the algorithm yielding the C^3 quartic B-spline approximation with the smallest number of control points within the tolerance is obtained. We summarize our work in section 5.

2. C^3 QUARTIC UNIFORM B-SPLINE CURVE APPROXIMATION OF CIRCULAR ARCS

In this section, we find the approximation of a circular arc by a C^3 quartic uniform B-spline curve whose Bézier segments are all congruent.

Let **c** be the unit circular arc of angle $0 < \alpha < \pi$ expressed by

$$\mathbf{c}(\theta) = (\cos \theta, \sin \theta) \qquad \text{for } \theta \in [0, \alpha]$$

and $\mathbf{p}$ be the quartic Bézier approximation of the circular arc with the control points $\mathbf{p}_0, \mathbf{p}_1, \cdots,$ $\mathbf{p}_4$ expressed by

$$\mathbf{p}(t) = \sum_{i=0}^{4} \mathbf{p}_i B_i^4(t),$$

where $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$ for $i = 0, 1, \cdots, n$ is the Bernstein polynomial of degree n [22, 23]. Since the circular arc is symmetric, the quartic Bézier curve is restricted to be symmetric. Let $\mathbf{b}$ be the spline curve constructed by $\mathbf{p}$ and its rotations $R\mathbf{p}, R^2\mathbf{p}, \cdots,$ where R is the rotation operator by the angle α, and $R^i\mathbf{p}$ is the rotated curve of $\mathbf{p}$ by the angle $i\alpha$. Then, the C^3-continuity of $\mathbf{b}$ at least implies that $\mathbf{p}$ is the G^1 endpoint interpolation of the circular arc. Thus, the control points of $\mathbf{p}$ are

$$
\begin{aligned}
\mathbf{p}_0 &= (1, 0) \\
\mathbf{p}_1 &= (1, h) \\
\mathbf{p}_2 &= r(\cos\frac{\alpha}{2}, \sin\frac{\alpha}{2}) \\
\mathbf{p}_3 &= (\cos\alpha, \sin\alpha) + h(\sin\alpha, -\cos\alpha) \\
\mathbf{p}_4 &= (\cos\alpha, \sin\alpha).
\end{aligned}
\tag{2.1}
$$

For a given circular arc $\mathbf{c}$ of angle ϕ and a given positive integer m, we will construct the quartic B-spline approximation $\mathbf{b}$ by the composition of the quartic Bézier curves $\mathbf{p}, R\mathbf{p}, R^2\mathbf{p}$, $\cdots, R^{m-1}\mathbf{p}$, where $\alpha = \phi/m$. Consider the continuity of $\mathbf{b}$ at each junction point. Two quartic Bézier curves $\mathbf{p}$ and $R\mathbf{p}$ meet at $\mathbf{p}_4 = R\mathbf{p}_0$ and are symmetric with respect to the line L passing through the origin and $\mathbf{p}_4$, as shown in Fig. 1. The curve $\mathbf{b}$ is C^2-continuous at $\mathbf{p}_4$ if and only if $\mathbf{p}''(1) = (R\mathbf{p})''(0)$ or

$$\Delta\mathbf{p}_3 - \Delta\mathbf{p}_2 = R(\Delta\mathbf{p}_2 - \Delta\mathbf{p}_1),$$

where $\Delta\mathbf{p}_i = \mathbf{p}_{i+1} - \mathbf{p}_i$. Its geometric meaning is that the ratio of the distances from $\mathbf{p}_3$ and $\mathbf{p}_2$ to the line L is $1 : 2$, as shown in Fig. 1, which is equivalent to

$$r\sin\frac{\alpha}{2} = 2h. \tag{2.2}$$

The curve $\mathbf{b}$ is C^3-continuous at $\mathbf{p}_4$ if and only if $\mathbf{p}'''(1) = (R\mathbf{p})'''(0)$ or

$$\Delta\mathbf{p}_3 - 2\Delta\mathbf{p}_2 + \Delta\mathbf{p}_1 = R(\Delta\mathbf{p}_2 - 2\Delta\mathbf{p}_1 + \Delta\mathbf{p}_0).$$

Its geometric meaning is that the perpendicular foot of $\mathbf{p}_2$ to the line L is the internally dividing point of the two perpendicular feet of $\mathbf{p}_1$ and $\mathbf{p}_3$ to the line L in the proportion of $1 : 2$, as shown in Fig. 1, which is equivalent to

$$\cos\alpha + h\sin\alpha = 3r\cos\frac{\alpha}{2} - 2. \tag{2.3}$$

Solving the linear system of two equations in Eqs. (2.2)–(2.3), we have

$$r = \frac{2 + \cos\alpha}{\cos\frac{\alpha}{2}(2 + \cos^2\frac{\alpha}{2})} \qquad \text{and} \qquad h = \frac{\tan\frac{\alpha}{2}(2 + \cos\alpha)}{2(2 + \cos^2\frac{\alpha}{2})}, \tag{2.4}$$
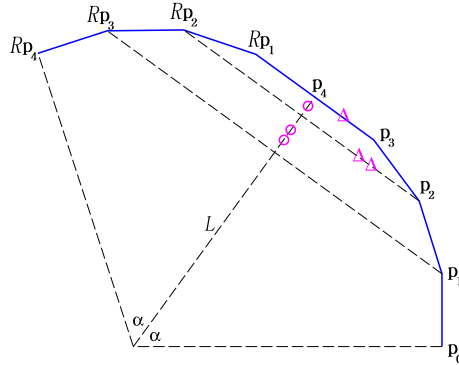
FIGURE 1. The geometric meaning of $\mathbf{p}''(1) = (R\mathbf{p})''(0)$ and $\mathbf{p}'''(1) = (R\mathbf{p})'''(0)$.

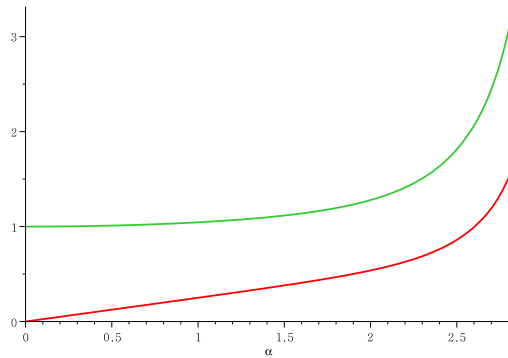as shown in Fig. 2 and the curve composed of $\mathbf{p}$ and $R\mathbf{p}$ can be a C^3-continuous curve.

FIGURE 2. h (red) and r (green) for $\alpha \in (0, 0.9\pi]$.

PROPOSITION 2.1. *If* $\mathbf{p}(t)$, $t \in [0, 1]$ *is the quartic Bézier curve with the control points* $\mathbf{p}_0, \mathbf{p}_1, \cdots, \mathbf{p}_4$ *satisfying Eq. (2.4), then for any positive integer* m, *the curve* $\mathbf{b}(t)$, $0 \le t \le m$, *defined by*

$$\mathbf{b}(t) = \begin{cases} \mathbf{p}(t) & \text{for } t \in [0, 1] \\ R^{\lfloor t \rfloor}\mathbf{p}(t - \lfloor t \rfloor) & \text{for } t \in (1, m] \end{cases}$$

is C^3-*continuous, where* $\lfloor x \rfloor$ *is the greatest integer less than* x.

For $i = 0, 1, \cdots, m - 1$, the control points of the quartic Bézier curve $R^i\mathbf{p}$ are $R^i\mathbf{p}_j$ for $j = 0, 1, \cdots, 4$. Thus, the C^3 curve $\mathbf{b}(t)$, $t \in [0, m]$ can be represented in the form of a quartic
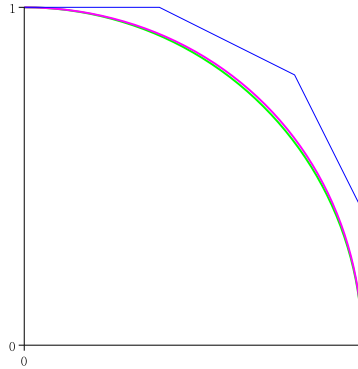
FIGURE 3. The quartic Bézier approximation (magenta) $\mathbf{p}$ of a quarter circle (green) and its control polygon (blue) satisfying Eqs. (2.1) and (2.4).

B-spline curve

$$\mathbf{b}(t) = \sum_{i=0}^{4m} \mathbf{b}_i N_i^4(t)$$

with the control points

$$\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, R\mathbf{p}_1, R\mathbf{p}_2, \cdots, R^{m-1}\mathbf{p}_3, R^{m-1}\mathbf{p}_4$$

and the knot vector $\mathbf{t} = (t_i)_{i=0}^{4m+5}$ satisfying $t_0 = 0$,

$$t_{4i+j} = i \quad \text{for } i = 0, 1, \cdots, m \text{ and } j = 1, 2, 3, 4,$$

and $t_{4m+5} = m$, where the B-spline basis functions of degree j are defined by

$$N_i^0(t) = \begin{cases} 1 & \text{if } t_i \leq t < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

for $i = 0, 1, \cdots$ and

$$N_i^j(t) = \frac{t - t_i}{t_{i+j} - t_i} N_i^{j-1}(t) + \frac{t_{i+j+1} - t}{t_{i+j+1} - t_{i+1}} N_{i+1}^{j-1}(t)$$

for $j = 1, 2, \cdots$ recursively [24, 25]. The domain interval $[0, m]$ and the knot vector $\mathbf{t}$ can be transformed into any interval and a new knot vector by a linear functional.

Since $\mathbf{b}$ is C^3-continuous and has multiple knots in the domain interior, it can be knot-removable. Using the reverse process of knot insertion algorithm [26, 27, 28, 29], we can obtain the C^3 quartic uniform B-spline approximation

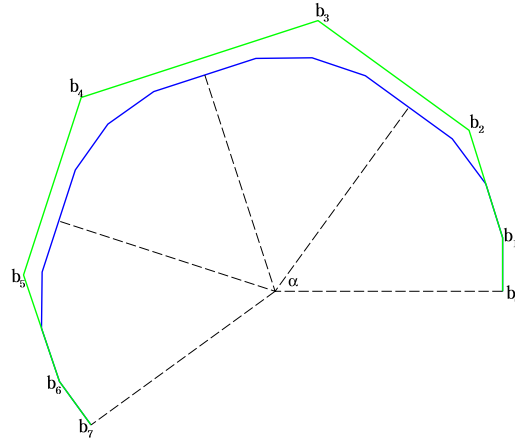$$\mathbf{b}(t) = \sum_{i=0}^{m+3} \mathbf{b}_i N_i^4(t)$$

FIGURE 4. For $\phi = 1.2\pi$ and $m = 4$, the de Boor points $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \cdots,$
$\mathbf{b}_{m+3}$ (green) and the Bézier points $\mathbf{p}_0, \mathbf{p}_1, \cdots, \mathbf{p}_{4m}$ (blue).

with the new control points

$$
\begin{aligned}
\mathbf{b}_0 &= \mathbf{p}_0 \\
\mathbf{b}_1 &= \mathbf{p}_1 \\
\mathbf{b}_2 &= 2\mathbf{p}_2 - \mathbf{p}_1 \\
\mathbf{b}_i &= \frac{3}{2 + \cos\alpha} R^{i-2} \mathbf{p}_2 \qquad \text{for } i = 3, \cdots, m \\
\mathbf{b}_{m+1} &= R^{m-1}(2\mathbf{p}_2 - \mathbf{p}_3) \\
\mathbf{b}_{m+2} &= R^{m-1}\mathbf{p}_3 \\
\mathbf{b}_{m+3} &= R^{m-1}\mathbf{p}_4
\end{aligned}
\tag{2.5}
$$

and the new knot vector $\mathbf{t} = (t_i)_{i=0}^{m+8}$ satisfying $t_0 = \cdots = t_4 = 0$,

$$
t_i = i - 4 \qquad \text{for } i = 5, \cdots, m+3,
\tag{2.6}
$$

and $t_{m+4} = \cdots = t_{m+8} = m$.

3. APPROXIMATION ORDER OF C^3 QUARTIC B-SPLINE APPROXIMATION OF CIRCULAR ARC

In this section, we find the Hausdorff distance $d_H(\mathbf{c}, \mathbf{b})$ between the circular arc $\mathbf{c}$ of angle α and the quartic Bézier approximation $\mathbf{b}$ having the control points in Eqs. (2.1) and (2.4), which is the maximum of

$$
\psi(t) = |\|\mathbf{p}(t)\| - 1| \qquad \text{for } t \in [0, 1]
$$

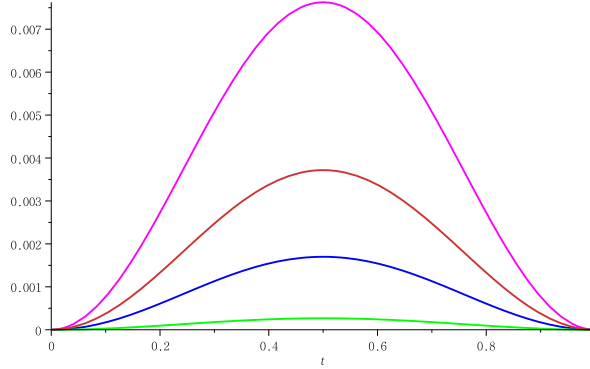Using this error function, we obtain the following error analysis.

FIGURE 5. Error functions $\psi(t)$, $t \in [0, 1]$ for α from $\frac{3}{10}\pi$ (bottom) to $\frac{\pi}{2}$ (top).

PROPOSITION 3.1. *The Hausdorff distance $d_H(\mathbf{c}, \mathbf{b})$ between the circular arc $\mathbf{c}$ of angle α and the quartic Bézier approximation $\mathbf{b}$ having the control points in Eqs. (2.1) and (2.4) is*

$$d_H(\mathbf{c}, \mathbf{b}) = \frac{(5 - \cos\frac{\alpha}{2})(1 - \cos\frac{\alpha}{2})^3}{8(\cos\frac{\alpha}{2})(2 + \cos^2\frac{\alpha}{2})}, \tag{3.1}$$

and its approximation order is six.

Proof. Let $\psi_1(t) = \|\mathbf{p}(t)\|^2 - 1$. Then, by Eqs. (2.1) and (2.4), we have

$$\psi_1(t) = \frac{4\sin^6\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}(2 + \cos^2\frac{\alpha}{2})^2}(t^2 - t)^2\{\sin^2\frac{\alpha}{2}(t^2 - t)^2 - 2(t^2 - t) + 1\}$$

which is a polynomial of degree eight. Since $\psi_1(t) \geq 0$ for $t \in [0, 1]$, we obtain $\psi(t) = \sqrt{\psi_1(t) + 1} - 1$. It follows from $\psi'(t) = \frac{1}{2\sqrt{\psi_1(t)+1}} \cdot \psi_1'(t)$ that $\psi(t)$ and $\psi_1(t)$ have the same critical points. Since

$$\psi_1'(t) = \frac{8\sin^6\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}(2 + \cos^2\frac{\alpha}{2})^2}(2t - 1)(t^2 - t)\{2\sin^2\frac{\alpha}{2}(t^2 - t)^2 - 3(t^2 - t) + 1\}$$

has seven zeros, $0, \frac{1}{2}, 1$, and

$$\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{3 \pm \sqrt{9 - 8\sin^2\frac{\alpha}{2}}}{\sin^2\frac{\alpha}{2}}},$$

and the zeros other than $\frac{1}{2}$ are not contained in the open interval $(0, 1)$, $\psi(t)$ has a unique maximum at $\frac{1}{2}$, as shown in Fig. 5, and

$$d_H(\mathbf{c}, \mathbf{b}) = \psi(\frac{1}{2}) = \frac{(5 - \cos\frac{\alpha}{2})(1 - \cos\frac{\alpha}{2})^3}{8(\cos\frac{\alpha}{2})(2 + \cos^2\frac{\alpha}{2})}.$$
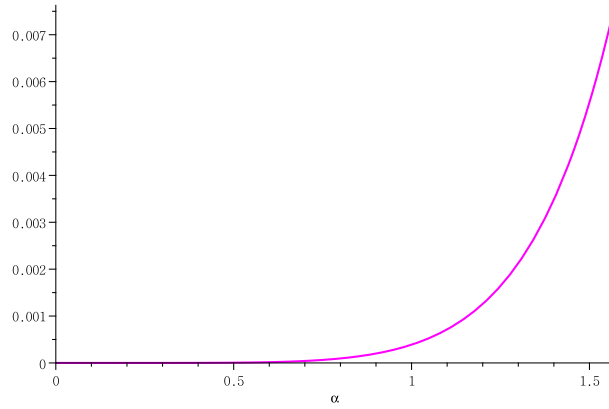
FIGURE 6. Hausdorff distance $d_H(\mathbf{c}, \mathbf{b}) = \varepsilon(\alpha)$ for $\alpha \in (0, \frac{\pi}{2}]$.

By series extension, we have

$$d_H(\mathbf{c}, \mathbf{b}) = \frac{1}{3 \cdot 2^{10}} \alpha^6 + \mathcal{O}(\alpha^8),$$

which means that the approximation order is six. $\square$

The Hausdorff distance $d_H(\mathbf{c}, \mathbf{b})$ in Eq. (3.1) is dependent only on the angle α of the circular arc, as shown in Fig. 6; therefore, we denote it by $\varepsilon(\alpha)$.

4. APPROXIMATION ALGORITHM AND NUMERICAL EXAMPLE

For a given circular arc $\mathbf{c}$ of angle ϕ and a given tolerance TOL, if $d_H(\mathbf{c}, \mathbf{b}) = \varepsilon(\phi)$ is greater than TOL, then the circular arc can be approximated by the quartic B-spline curve, which consists of at least two Bézier segments, i.e., $m \geq 2$. Using the error analysis in Eq. (3.1), we can find the smallest integer m satisfying

$$\varepsilon\left(\frac{\phi}{m}\right) < TOL.$$

Letting $\alpha = \phi/m$, the quartic uniform B-spline curve $\mathbf{b}$ satisfying Eq. (2.5) is a C^3 approximation of the circular arc $\mathbf{c}$ having an error less than TOL. Now, we present an algorithm yielding the C^3 quartic uniform B-spline approximation $\mathbf{b}$ of the circular arc of the angle ϕ
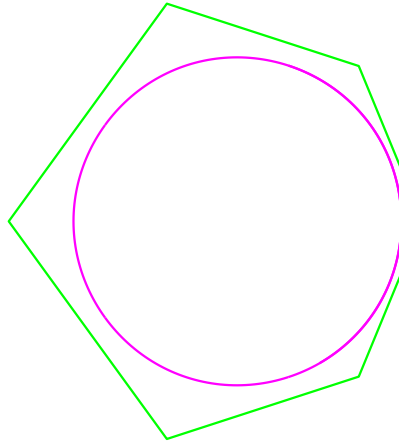
FIGURE 7. For a full circle ($\phi = 2\pi$) and $TOL = 0.005$, our algorithm yields $m = 5$ and the C^3 quartic B-spline approximation (magenta) $\mathbf{b}$ with the control polygon (green) $\mathbf{b}_0, \mathbf{b}_1, \cdots, \mathbf{b}_8$ within the error tolerance.

having the minimum number of control points within the error tolerance TOL as follows.

ALGORITHM
- input : the angle ϕ and tolerance TOL
- find the smallest integer m satisfying $\varepsilon(\frac{\phi}{m}) < TOL$
- put $\alpha = \frac{\phi}{m}$
- find the control points $\mathbf{b}_0, \mathbf{b}_1, \cdots, \mathbf{b}_{m+3}$ by Eq. (2.5)
  and the knot vector $\mathbf{t} = (t_i)_{i=0}^{m+8}$ by Eq. (2.6)
- output : $m, \mathbf{b}_0, \mathbf{b}_1, \cdots, \mathbf{b}_{m+3}$, and $\mathbf{t} = (t_i)_{i=0}^{m+8}$

For example, if a full circle and $TOL = 0.005$ are given, then this algorithm yields $m = 5$ and the control points $\mathbf{b}_0, \mathbf{b}_1, \cdots, \mathbf{b}_8$ of the C^3 quartic uniform B-spline approximation $\mathbf{b}$, as shown in Fig. 7. The exact Hausdorff distance is $d_H(\mathbf{c}, \mathbf{b}) = 0.0017$. Using the algorithm, Table 1 summarizes the required minimum number of control points of the C^3 quartic uniform B-spline approximation of the full circle within the error tolerance for $TOL = 10^{-1}, 10^{-2}, \cdots, 10^{-5}$.

5. CONCLUSION

In this paper, we presented the C^3 quartic uniform B-spline approximation of a circular arc. Since the order of continuity is three, it can reduce the number of control points of the quartic B-spline curve obtained by merging of the quartic Bézier approximate curve and its rotations, which is an advantage of our approximation method. Another advantage is that the Hausdorff distance between any circular arc and its C^3 quartic B-spline approximation is obtained in closed form. Using this closed form, we proved that our quartic B-spline approximation has

TABLE 1. The required minimum number of control points of the C^3 quartic B-spline approximation of the full circle ($\phi = 2\pi$) within the given tolerance TOL.

TOL	number of control points
10^{-1}	7
10^{-2}	8
10^{-3}	10
10^{-4}	12
10^{-5}	16

the approximation order six. Moreover, we obtained the C^3 quartic B-spline approximation having the minimum number of control points within the error tolerance and the algorithm yielding the C^3 quartic B-spline approximation.

REFERENCES

[1] C. de Boor and M. Sabin, *High accuracy geometric Hermite interpolation*, Comp. Aided Geom. Desi., **4** (1987), 269–278.

[2] K. Mørken, *Best approximation of circle segments by quadratic Bézier curves*, Curves and Surfaces, Academic Press, (1990), 387–396.

[3] T. Dokken, M. Dæhlen, T. Lyche, and K. Mørken, *Good approximation of circles by curvature-continuous Bézier curves*, Comp. Aided Geom. Desi., **7** (1990), 33–41.

[4] M. Goldapp, *Approximation of circular arcs by cubic polynomials*, Comp. Aided Geom. Desi., **8** (1991), 227–238.

[5] T. Dokken, *Controlling the shape of the error in cubic ellipse approximation*, Curve and Surface Design, Saint-Malo, Nashboro Press (2002), 113–122.

[6] M. Floater, *High-order approximation of conic sections by quadratic splines*, Comp. Aided Geom. Desi., **12**(6) (1995) 617–637.

[7] M. Floater. *An $O(h^{2n})$ Hermite approximation for conic sections*, Comp. Aided Geom. Desi., **14** (1997) 135–151.

[8] T. Dokken. *Aspects of intersection algorithms and approximation*, PhD thesis, University of Oslo, (1997).

[9] Y. J. Ahn and H. O. Kim. *Approximation of circular arcs by Bézier curves*, J. Comp. Appl. Math., *81* (1997) 145–163.

[10] L. Fang. *Circular arc approximation by quintic polynomial curves*, Comp. Aided Geom. Desi., **15** (1998), 843–861.

[11] S. H. Kim and Y. J. Ahn. *Approximation of circular arcs by quartic Bézier curves*, Comp. Aided Desi., **39**(6) (2007), 490–493.

[12] S. Hur and T. Kim. *The best G^1 cubic and G^2 quartic Bézier approximations of circular arcs*, J. Comp. Appl. Math., **236** (2011), 1183–1192.

[13] Z. Liu, J. Tan, X. Chen, and L. Zhang. *An approximation method to circular arcs*, Appl. Math. Comp., **15** (2012), 1306–1311.

[14] S. W. Kim and Y. J. Ahn. *Circle approximation by quartic G^2 spline using alternation of error function*, J. KSIAM, **17** (2013), 171–179.

[15] B. Kovač and E. Žagar. *Some new G^1 quartic parametric approximants of circular arcs*, Appl. Math. Comp., **239** (2014), 254–264.

[16] I.-K. Lee, M.-S. Kim, and G. Elber. *Planar curve offset based on circle approximation*, Comp. Aided Desi., **28** (1996), 617–630.

[17] G. Elber, I.-K. Lee, and M.-S. Kim. *Comparing offset curve approximation methods*, IEEE Comp. Grap. Appl., **17**(3) (1997), 62–71.

[18] Y. J. Ahn, Y. S. Kim, and Y. Shin. *Approximation of circular arcs and offset curves by Bézier curves of high degree*, J. Comp. Appl. Math., **167** (2004), 405–416.

[19] Y. J. Ahn, C. M. Hoffmann, and Y. S. Kim. *Curvature-continuous offset approximation based on circle approximation using quadratic Bézier biarcs*, Comp. Aided Desi., **43** (2011), 1011–1017.

[20] Y. J. Ahn and C. M. Hoffmann. *Circle approximation using LN Bézier curves of even degree and its application*, J. Math. Anal. Appl., **40** (2014), 257–266.

[21] W. Yang and X. Ye. *Approximation of circular arcs by C^2 cubic polynomial B-splines*, 10th IEEE Intern. Conf. Comput.-Aided Des. Comput. Graph., (2007), 417–420.

[22] B.-G. Lee, Y. Park, and J. Yoo. *Application of Legendre–Bernstein basis transformations to degree elevation and degree reduction*, Comp. Aided Geom. Desi., **19** (2002), 709–718.

[23] H. Sunwoo. *Matrix representation for multi-degree reduction of Bézier curves*, Comp. Aided Geom. Desi., **22** (2005), 261–273.

[24] L. Piegl and W. Tiller. *The NURBS book*. Springer Science & Business Media, (2012).

[25] G. Farin. *Curves and Surfaces for CAGD*. Morgan-Kaufmann, San Francisco, (2002).

[26] W. Böhm. *Inserting new knots into B-spline curves*, Comp. Aided Desi., **4** (1980), 199–201.

[27] E. Cohen, T. Lyche, and R. Riesenfeld. *Discrete B-splines and subdivision techniques in Computer-Aided Geometric Design and Computer Graphics*, Comp. Grap. Image Proc., **14** (1980), 87–111.

[28] W. Böhm, G. Farin, and J. Kahmann. *A survey of curve and surface methods in CAGD*, Comp. Aided Geom. Desi., **1** (1984), 1–60.

[29] K. Mørken, M. Reomers, and C. Schulz. *Computing intersections of planar spline curves using knot insertion*, Comp. Aided Geom. Desi., **26** (2009), 351–366.