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FIXED POINT THEOREMS IN ORDERED DUALISTIC PARTIAL METRIC SPACES

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Abstract. In this article, we introduce the concept of ordered dualistic partial metric spaces and establish an order relation on quasi dualistic partial metric spaces. Later on, using this order relation, we prove fixed point theorems for single and multivalued mappings. We support our results with some illustrative examples.

1. introduction and preliminaries

In recent times, Fixed Point Theory has become one of the most useful branches of Nonlinear Analysis, mainly due to its possible applications in several areas. For instance, different classes of matrix, differential and integral equations can be solved using the appropriate techniques in this field.

In 1994, Matthews [6] added a new concept in the literature of metric spaces which is known as Partial metric space and obtained a fixed point theorem in Partial metric spaces. After some years, O'Neill [9] coined the idea of dualistic partial metric by extending the range $\mathbb{R}_0^+$ to $\mathbb{R}$. Then in 2004, Oltra and Valero [8] come up with Banach fixed point theorem for complete dualistic partial metric spaces.

Matthews, in [6] discussed the relationship between partial metric and quasi metric and justify this relation by giving various examples. Then

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moving on the same pattern, Oltra and Valero in [8] developed the relationship between dualistic partial metric and quasi metric. Recently, Oltra and Valero [8], Altun and Simsek [2] developed some fixed point theorems in complete dualistic partial metric space. In this article, we introduce an order on quasi dualistic partial metric and prove that it is a partial order induced by φ. In section 2 and 3 we use this partial order to prove a fixed point theorem for single valued non decreasing mappings. Moreover, we prove some fixed point results for multivalued mappings satisfying order induced by φ.

We recall some mathematical basics and results to make this paper self sufficient.

Throughout this paper, we denote $(0, \infty)$ by $\mathbb{R}^+$, $[0, \infty)$ by $\mathbb{R}_0^+$, $(-\infty, +\infty)$ by $\mathbb{R}$ and set of natural numbers by $\mathbb{N}$. Let $T : X \to X$ be a self map, a point $x \in X$ is called a fixed point of T if $x = T(x)$.

Define a sequence $\{x_n\}$ in X by a simple iterative method such that

$$x_n = T(x_{n-1}), \text{ where } n \in \mathbb{N}.$$

This particular sequence is known as Picard iterative sequence.

Definition 1.1. [9] Let X be a non-empty set. The function $D : X \times X \to \mathbb{R}$ is said to be dualistic partial metric if it satisfies following properties for all $x, y, z \in X$.

1. $x = y \Leftrightarrow D(x, x) = D(y, y) = D(x, y)$
2. $D(x, x) \le D(x, y)$
3. $D(x, y) = D(y, x)$
4. $D(x, z) \le D(x, y) + D(y, z) - D(y, y)$

The pair (X, D) is called dualistic partial metric space.

Note that if $\mathbb{R}$ is replaced by $\mathbb{R}_0^+$, then D is known as partial metric on X.If (X, D) is a dualistic partial metric space, then $d_D : X \times X \to \mathbb{R}_0^+$ defined by

$$d_D(x, y) = D(x, y) - D(x, x). \tag{1}$$

is called quasi metric on X such that $\tau(D) = \tau(d_D)$ for all $x, y \in X$. Moreover, if d_D is quasi metric on X , then $d_D^s(x, y) = \max\{d_D(x, y), d_D(y, x)\}$ defines a metric on X.

Remark 1.2. It is obvious that every partial metric is dualistic partial metric but converse is not true. To support this comment, define $D_\vee$:

$\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$D_\vee(x, y) = x \vee y = \sup\{x, y\} \text{ for all } x, y \in \mathbb{R}.$$

It is easy to check that $D_\vee$ is a dualistic partial metric. Note that $D_\vee$ is not a partial metric, because $D_\vee(-1, -2) = -1 \notin \mathbb{R}_0^+$. However, the restriction of $D_\vee$ to $\mathbb{R}_0^+$, $D_\vee|_{\mathbb{R}_0^+}$, is a partial metric.

Following [9], each dualistic partial metric D on X generates a T_0 topology $\tau(D)$ on X. The elements of the topology $\tau(D)$ are open balls of the form $\{B_D(x, \varepsilon) : x \in X, \varepsilon > 0\}$ where $B_D(x, \varepsilon) = \{y \in M : D(x, y) < \varepsilon + D(x, x)\}$

Definition 1.3. [9] Let (X, D) be a dualistic partial metric space, then

(1) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, D) converges to a point $x \in X$ if and only if $D(x, x) = \lim_{n\to\infty} D(x, x_n)$.
(2) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, D) is called a Cauchy sequence if $\lim_{n,m\to\infty} D(x_n, x_m)$ exists and is finite.
(3) A dualistic partial metric space (X, D) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges, with respect to $\tau(D)$, to a point $x \in X$ such that $D(x, x) = \lim_{n,m\to\infty} D(x_n, x_m)$.

Following lemma will be helpful in the sequel.

Lemma 1.4. [9, 11]

(1) *A dualistic partial metric* (X, D) *is complete if and only if the metric space* (X, d_D^s) *is complete.*
(2) *A sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* X *converges to a point* $x \in X$*, with respect to* $\tau(d_D^s)$ *if and only if* $\lim_{n\to\infty} D(x, x_n) = D(x, x) = \lim_{n\to\infty} D(x_n, x_m)$.
(3) *If* $\lim_{n\to\infty} x_n = \upsilon$ *such that* $D(\upsilon, \upsilon) = 0$ *then* $\lim_{n\to\infty} D(x_n, y) = D(\upsilon, y)$ *for every* $y \in X$.

Definition 1.5. Let (X, D) be a dualistic partial metric space. A sequence $\{x_n\}$ in X is said to be 0-Cauchy sequence if $\lim_{n\to\infty} D(x_n, x_m) = 0$ and (X, D) is said to be 0-complete if every 0-Cauchy sequence converges in X.

Definition 1.6. [4] Let A, B be two nonempty subsets of an ordered set X, the relation between A and B is defined as follows:
If for every $b \in B$, there exists $a \in A$ such that $a \preceq b$, then $A \prec_2 B$.

Example 1.7. if $A = [0, 2]$, $B = \left[\frac{1}{4}, 1\right]$, then $A \prec_2 B$.

Remark 1.8. [4] *The relation $\prec_2$ is reflexive and transitive, but are not antisymmetric. For instance, let $X = \mathbb{R}$, $A = [0, 3]$, $B = [0, 1] \cup [2, 3]$, then $A \prec_2 B$ and $B \prec_2 A$, but $A \neq B$. Hence, $\prec_2$ is not partial order on 2^X.*

Definition 1.9. Let M be a nonempty set. Then $(X, \preceq, D)$ is said to be an ordered dualistic partial metric space if:

(i) $(X, \preceq)$ is a partially ordered set.
(ii) (X, D) is a dualistic partial metric space.

A sequence in a set X is called monotone sequence if either it is increasing or decreasing sequence.

Definition 1.10. [4] A multi-valued mapping $T : X \to 2^X$ is called order closed if for monotone sequences $\{u_n\}$, $\{v_n\}$ in X, $u_n \to u_0$, $v_n \to v_0$ and $v_n \in T(u_n)$ imply $v_0 \in T(u_0)$.

Dualistic version of this definition is given by

Definition 1.11. [2] Let $(X, \preceq, D)$ be an ordered dualistic partial metric space. A multivalued mapping $T : X \to 2^X$ is called D-order closed if for monotone sequences, $\{u_n\}$, $\{v_n\} \subseteq X$, $\lim_{n\to\infty} D(u_n, u_0) = D(u_0, u_0)$, $\lim_{n\to\infty} D(v_n, v_0) = D(v_0, v_0)$ and $v_n \in T(u_n)$ imply $v_0 \in T(u_0)$.

2. Fixed point for single-valued mappings

In this section, we shall prove a fixed point theorem for single-valued mappings in an ordered dualistic partial metric space. We begin with the following lemma.

Lemma 2.1. *Let (X, D) be a dualistic partial metric space and $\varphi : X \to R$ be a mapping. Define the relation $\preceq$ on X as follows;*

$$p \preceq q \Leftrightarrow D(p, q) - D(p, p) \leq \varphi(p) - \varphi(q). \tag{2}$$

Then $\preceq$ is an order on X, called order induced byφ.

Proof. As $0 \leq 0$ this implies

$$D(p, p) - D(p, p) \leq \varphi(p) - \varphi(p) \ \Rightarrow\ p \preceq p$$

so $\preceq$ is reflexive.
Now if $p \preceq q$ and $q \preceq p$, we will prove that $p = q$ for this

$$\text{Since } p \preceq q \Leftrightarrow D(p,q) - D(p,p) \leq \varphi(p) - \varphi(q). \tag{3}$$

and

$$q \preceq p \Leftrightarrow D(q,p) - D(q,q) \leq \varphi(q) - \varphi(p). \tag{4}$$

Adding (3) and (4), we get

$$D(p,q) - D(p,p) + D(q,p) - D(q,q) \leq 0$$

Using definition of d_D, we have

$$d_D(p,q) + d_D(q,p) \leq 0. \tag{5}$$

Since $d_D(p,q)$ and $d_D(q,p)$ are non-negative, (5) leads to

$$d_D(p,q) = d_D(q,p) = 0. \tag{6}$$

Since d_D is a quasi metric, so (6) entails $p = q$. Thus $\preceq$ is an anti-symmetric relation.
Lastly, if $p \preceq q$ and $q \preceq r$, we show that $p \preceq r$. From condition (2), we have

$$p \preceq q \Leftrightarrow D(p,q) - D(p,p) \leq \varphi(p) - \varphi(q). \tag{7}$$

and

$$q \preceq r \Leftrightarrow D(q,r) - D(q,q) \leq \varphi(q) - \varphi(r). \tag{8}$$

Adding (7) and (8), we obtain

$$D(p,q) - D(p,p) + D(q,r) - D(q,q) \leq \varphi(p) - \varphi(r)$$

Implies

$$d_D(p,q) + d_D(q,r) \leq \varphi(p) - \varphi(r).$$

By triangular inequality

$$d_D(p,r) \leq d_D(p,q) + d_D(q,r).$$

Thus

$$d_D(p,r) \leq d_D(p,q) + d_D(q,r) \leq \varphi(p) - \varphi(r). \tag{9}$$

Inequality (9) entails,

$$D(p,r) - D(p,p) \leq \varphi(p) - \varphi(r) \ \Rightarrow p \preceq r.$$

So, $\preceq$ is transitive. Hence $\preceq$ is a partial order on X. □

It can be observed from the Lemma 2.1 that φ is a decreasing function for the special case when $\preceq$ is equal to usual order $\leq$.
Next we discuss the existence of the order $\preceq$ defined in Lemma 2.1 through an example.

Example 2.2. Let $X = \mathbb{R}$ and define $D_\vee$ by $D_\vee(u, v) = u \vee v$ for all $u, v \in X$. Consider the function φ defined by $\varphi(u) = u^2 - u$ for all $u \in X$. Now

$$\begin{aligned} u &\preceq v \Leftrightarrow D(u,v) - D(u,u) \leq \varphi(u) - \varphi(v) \\ u &\preceq v \Leftrightarrow u \vee v - u \vee u \leq u^2 - u + v - v^2 \\ u &\preceq y \Leftrightarrow u \vee v - u \leq u^2 - u + v - v^2 \\ u &\preceq v \Leftrightarrow u \vee v \leq u^2 - v^2 + v. \end{aligned}$$

We get two relations from this,
either

$$u \preceq v \Leftrightarrow \varphi(v) \leq \varphi(u).$$

or

$$u \preceq v \Leftrightarrow v^2 \leq u^2.$$

Our first result.

Theorem 2.3. *Let (X, D) be a 0-complete dualistic partial metric space, $\varphi : X \to \mathbb{R}$ be a bounded above function and $\preceq$ be an order induced by φ, and $h : X \to X$ is a $\tau(D)$-continuous non decreasing function with $h(u_0) \preceq u_0$ for some $u_0 \in X$. Then h has a fixed point in (X, D).*

Proof. Suppose that $h(u_0) \preceq u_0$ for some $u_0 \in X$ and define a Picard sequence in X by $u_n = h(u_{n-1})$ for all $n \in \mathbb{N}$. Since $u_1 = h(u_0) \preceq u_0$, so $u_1 \preceq u_0$.
And h is non-decreasing, therefore, $u_1 \preceq u_0$ implies $h(u_1) \preceq h(u_0)$ that is $u_2 \preceq u_1$.
this in turn implies that $h(u_2) \preceq h(u_1)$, thus $u_3 \preceq u_2$. Continuing in a similar manner, we get

$$u_0 \succeq u_1 \succeq u_2 \succeq u_3 \succeq ... \succeq u_n \succeq ....$$

Now by definition of φ, we deduce that

$$\varphi(u_0) \leq \varphi(u_1) \leq \varphi(u_2) \leq \varphi(u_3) \leq ... \leq \varphi(u_n) \leq ...$$

Since φ is bounded above, thus, $\{\varphi(u_n)\}_{n=1}^{\infty}$ is monotone bounded sequence and hence convergent sequence. Consequently, $\{\varphi(u_n)\}_{n=1}^{\infty}$ is a Cauchy sequence, for $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|\varphi(u_n) - \varphi(u_m)| < \varepsilon, \text{for } n > m > n_0.$$

On the other hand, since $u_n \preceq u_m$ from condition (2), we get

$$u_n \preceq u_m \Leftrightarrow D(u_n, u_m) - D(u_n, u_n) \leq \varphi(u_n) - \varphi(u_m).$$

Which implies

$$D(u_n, u_m) - D(u_n, u_n) \leq |\varphi(u_n) - \varphi(u_m)| < \varepsilon.$$

(1), implies

$$d_D(u_n, u_m) < \varepsilon.$$

Since $d_D^s(x, y) = \max\{d_D(x, y), d_D(y, x)\}$, we get

$$d_D^s(u_n, u_m) < \varepsilon.$$

This implies that $\{u_n\}$ is a Cauchy sequence in (X, d_D^s). Since (X, D) is a complete dualistic partial metric space so by Lemma 1.4, the metric space (M, d_D^s) is also complete. So there exists $v \in X$ such that

$$\lim_{n\to\infty} d_D^s(u_n, v) = 0.$$

Again by using Lemma 1.4, we obtain

$$D(v, v) = \lim_{n\to\infty} D(u_n, v) = \lim_{n,m\to\infty} D(u_n, u_m).$$

As

$$\lim_{n,m\to\infty} d_D(u_n, u_m) = 0.$$

Which leads us to

$$\lim_{n,m\to\infty} D(u_n, u_m) = \lim_{n\to\infty} D(u_n, u_n).$$

Now, since (X, D) is a 0-complete dualistic partial metric space, so $\lim_{n\to\infty} D(u_n, u_m) = 0$, this implies that

$$\lim_{n\to\infty} D(u_n, v) = 0.$$

This shows that $\{u_n\}$ is a 0-Cauchy sequence in (X, D) which converges to v. Since h is a $\tau(D)$-continuous, therefore, $v = h(v)$, which completes the proof. □

If we assume that $\varphi(X)$ is compact in $\mathbb{R}$ instead of boundedness of $\varphi(X)$ in Theorem 2.3, we can have the following theorem.

Theorem 2.4. *Let (X, D) be a 0-complete dualistic partial metric space , $\varphi : X \to \mathbb{R}$ be a function such that $\varphi(X)$ is compact and $\preceq$ be an order induced by φ, and $h : X \to X$ is a $\tau(D)$-continuous non-decreasing function with $h(u_0) \preceq u_0$ for some $u_0 \in X$. Then h has a fixed point in (X, D).*

Example 2.5. Let $X = \mathbb{R} - \{0\}$ and consider $\varphi(w) = 1 - \dfrac{1}{w^2}$ for all $w \in X$, then $\varphi(w) = 1 - \dfrac{1}{w^2} < 1$, so it is bounded above. Define $D_\vee$ by $D_\vee(w, v) = w \vee v$ for all $w, v \in X$ and let $\preceq$ be an order as defined in Lemma 2.1. Clearly, $(X, \preceq, D_\vee)$ is a complete ordered dualistic partial metric space. Now,

$$w \preceq v \Leftrightarrow D(w, v) - D(w, w) \leq \varphi(w) - \varphi(v).$$

This implies either

$$w \preceq v \Leftrightarrow 0 \leq \frac{1}{v^2} - \frac{1}{w^2}$$

or

$$w \preceq v \Leftrightarrow v - w \leq \frac{1}{v^2} - \frac{1}{w^2}.$$

Let the mapping $\hbar : X \to X$ is defined by

$$\hbar(w) = \begin{cases} w^2 - 1 & \text{if } w \in (-\infty, -1); \\ w & \text{if } w \in [-1, \infty). \end{cases}$$

Then $\hbar$ is non-decreasing, for if $\hbar(w) = w$, then the result is obvious and if $\hbar(w) = w^2 - 1$, then

$$\hbar(w) \preceq \hbar(v) \Leftrightarrow \hbar(w) \vee \hbar(v) \leq \hbar(w) + \frac{1}{(\hbar(v))^2} - \frac{1}{(\hbar(w))^2}.$$

This implies either $\hbar(w) \preceq \hbar(v) \Leftrightarrow v^2 \leq w^2 + \dfrac{1}{(v^2-1)^2} - \dfrac{1}{(w^2-1)^2}$ for when $\hbar(w) \vee \hbar(v) = \hbar(v)$ or $\hbar(w) \preceq \hbar(v) \Leftrightarrow 0 \leq \dfrac{1}{(v^2-1)^2} - \dfrac{1}{(w^2-1)^2}$ for when $\hbar(w) \vee \hbar(v) = \hbar w$ In both cases we have

$$\hbar(w) \preceq \hbar(v) \Leftrightarrow w \preceq v.$$

Further take $u_0 = \frac{1}{2}$, $\hbar(u_0) = 3$ which implies $\hbar(u_0) \preceq u_0$. So hypotheses of theorem 2.3 are satisfied. Thus $\hbar$ has a fixed point.

3. Fixed points for multivalued mappings

In this section, we present a fixed point theorem for multivalued mappings in an ordered dualistic partial metric space. Let X be a dualistic partial metric space and 2^X represents family of all non-empty subsets of X.

THEOREM 3.1. *Let (X, D) be a complete dualistic partial metric space and $\varphi : X \to \mathbb{R}$ be a bounded above function. Let $\preceq$ be an order induced by φ, $T : X \to 2^X$ be a D-order closed mapping with $T(x_0) \prec_2 \{x_0\}$ for some $x_0 \in X$, and*

$$x \preceq y \text{ implies } T(x) \prec_2 T(y), \tag{10}$$

for all $x, y \in X$. Then T has a fixed point in X.

Proof. Since $T(x)$ is non-empty set and $T(x_0) \prec_2 \{x_0\}$ for some $x_0 \in X$. We can choose $x_1 \in T(x_0)$ such that $x_1 \preceq x_0$, by condition (10), we get $T(x_1) \prec_2 T(x_0)$. For every $x_1 \in T(x_0)$ there is $x_2 \in T(x_1)$ such that $x_2 \preceq x_1$ which implies $T(x_2) \prec_2 T(x_1)$. Again for every $x_2 \in T(x_1)$, there exists $x_3 \in T(x_2)$ such that $x_3 \preceq x_2$ and this implies that $T(x_3) \prec_2 T(x_2)$. Continuing in a similar manner, we get a monotone sequence

$$x_0 \succeq x_1 \succeq x_2 \succeq x_3 \succeq ... \succeq x_n \succeq ....$$

Now by definition of φ, we deduce that

$$\varphi(x_0) \leq \varphi(x_1) \leq \varphi(x_2) \leq \varphi(x_3) \leq ... \leq \varphi(x_n) \leq ....$$

Since φ is bounded above, So $\{\varphi(x_n)\}_{n=1}^{\infty}$ is monotone bounded above sequence and hence convergent sequence. Thus $\{\varphi(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence, so for $\varepsilon > 0$ there exists n_0 such that for $n > m > n_0$, $|\varphi(x_n) - \varphi(x_m)| < \varepsilon$. On the other hand, since $x_n \preceq x_m$, from condition (2), we obtain

$$x_n \preceq x_m \Leftrightarrow D(x_n, x_m) - D(x_n, x_n) \leq \varphi(x_n) - \varphi(x_m).$$

Which implies that

$$D(x_n, x_m) - D(x_n, x_n) \leq |\varphi(x_n) - \varphi(x_m)| < \varepsilon.$$

(1) entails

$$d_D(x_n, x_m) < \varepsilon.$$

Since $d_D^s(x, y) = \max\{d_D(x, y), d_D(y, x)\}$, therefore,

$$d_D^s(x_n, x_m) < \varepsilon.$$

This implies that $\{x_n\}$ is a Cauchy sequence in complete metric space (X, d_D^s). Since (X, D) is a complete dualistic partial metric space, so by Lemma 1.4, the metric space (M, d_D^s) is also complete. Thus there exists $v \in X$ such that

$$\lim_{n\to\infty} d_D^s(x_n, v) = 0.$$

Again using Lemma 1.4, we get

$$D(v,v) = \lim_{n\to\infty} D(x_n, v) = \lim_{n,m\to\infty} D(x_n, x_m).$$

Since T is a D-order closed map and $x_{n+1} \in T(x_n)$. Thus, $v \in T(v)$ and hence v is a fixed point of T. □

EXAMPLE 3.2. Let $X = \mathbb{R}^2$ and define multivalued mapping T by

$$T(x,y) = \begin{cases} \{(0,0),(2,3)\} & \text{if } xy \geq 0; \\ \{(\frac{xy}{x^3+y^3}, \frac{xy}{x^3+y^3}), (1+\frac{xy}{x^3+y^3}, 1+\frac{xy}{x^3+y^3})\} & \text{if } xy < 0. \end{cases}$$

Then T is an ordered closed mapping and for all $(x,y), (u,v) \in \mathbb{R}^2$.

$$(x,y) \preceq (u,v) \Leftrightarrow T(x,y) \prec_2 T(u,v).$$

Further $T(x_0) \prec_2 \{x_0\}$. Hence T satisfies all the conditions of Theorem 3.1 and it has a fixed point.

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