



Original Article

Separative Power of an Optimised Concurrent Gas Centrifuge

Sergey Bogovalov* and Vladimir Borman

National Research Nuclear University (MEPHI), Kashirskoje shosse, 31, Moscow, 115409, Russia

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ABSTRACT

The problem of separation of isotopes in a concurrent gas centrifuge is solved analytically for an arbitrary binary mixture of isotopes. The separative power of the optimised concurrent gas centrifuges for the uranium isotopes equals to $\delta U = 12.7 (V/700 \text{ m/s})^2 (300 \text{ K}/T)(L/1 \text{ m}) \text{ kg} \cdot \text{SWU}/\text{yr}$, where L and V are the length and linear velocity of the rotor of the gas centrifuge and T is the temperature. This equation agrees well with the empirically determined separative power of optimised counter-current gas centrifuges.

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1. Introduction

Separation of heavy isotopes in gas centrifuges (hereafter GCs) has been used for industrial production of enriched uranium from the middle of the past century. It is likely that this method of isotope separation will remain the most efficient, from the economical point of view, for the next few decades. Despite the long history of using this method, a lot of important problems of the physics of the isotope separation remain unsolved. The problem of the separative power of GCs is the most important among them. Knowledge or estimation of the separative power of GCs is necessary for design of efficient GCs and important for experts dealing with the problem of nonproliferation of the separation technology.

An attempt to estimate the separative power of GCs has been made, starting with Dirac [1]. He has shown that the separative power δU_{\max} of any GC can not exceed the value

$$\delta U_{\max} = \frac{\pi \rho D L}{2} \left(\frac{\Delta M V^2}{2RT} \right)^2, \quad (1)$$

where ρD is the density of uranium hexafluoride (UF_6) times the coefficient of self-diffusion of uranium isotopes ^{238}U and ^{235}U . ΔM is the mass difference between two uranium isotopes, R is the gas-law constant, T is the gas temperature, L is the length of the GC rotor, and V is the linear velocity of the rotor rotation.

In the early 1960s an Onsager group from US developed a theory called the *pancake approximation* that reduced the

* Corresponding author.

E-mail address: svbogovalov@mephi.ru (S. Bogovalov).
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problem of the GC gas dynamics to the solution of linear elliptical equations of the sixth-order partial derivatives for two variables [2,3]. This approach gave the following equation for the separative power of the GC:

$$\delta U = (0.038V - 11.5)L, \text{ kg} \cdot \text{SWU/yr.} \quad (2)$$

It is important to note that, in contrast to Eq. (1) where the separative power increases as V^4 , in Eq. (2) the separative power grows linearly with V .

Experimental data collected with the help of a large number of Russian GCs have shone new light on this question. According to Senchenkov [4], the separative power is defined by the following empirical equation:

$$\delta U = 12L \left(\frac{V}{700 \text{ m/s}} \right)^2 \left(\frac{2a}{12 \text{ cm}} \right)^{0.4}, \text{ kg} \cdot \text{SWU/yr,} \quad (3)$$

where L is measured in meters. Recently this result has been well confirmed by more extended experimental data [5].

The proportionality of δU to V^2 in the empirical Eq. (3) dramatically contradicts simple theoretical arguments. Let coefficient q be defined as the ratio of the concentration of U235 in the product flux over the concentration in the waste flux. At relatively small q the separative power equals

$$\delta U = \theta(1 - \theta) \frac{F(q - 1)^2}{2}, \quad (4)$$

where $\theta = P/F$ is the ratio of the product mass flux P over feed mass flux F [6]. Radial separation in the centrifugal field gives the following dependence of q on V

$$q = \exp \frac{\Delta M \gamma V^2}{2RT}, \quad (5)$$

which unambiguously gives, $\delta U \sim V^4$. This dependence takes place in Eq. (1) but does not agree with the experiment. For many years, this problem has remained a challenge for specialists. Recently a new equation defining the separative power of GC has been proposed in Kemp [7].

$$\delta U = \left(\frac{V^2 L}{33000} \right) e_E, \text{ kg} \cdot \text{SWU/yr,} \quad (6)$$

where V is measured in meters per second, L is rotor length in meters and e_E is some numerical coefficient. This equation already correctly reproduces the empirical law [Eq. (3)]. Nevertheless, the dependence of the optimised separative power of the GC on the parameters remains an open problem up to now. The solution of this problem is important from the practical point of view. Simple estimates show that the maximal possible separative power defined by Eq. (1) is four to five times higher than the optimal separative power [Eq. (3)] defined experimentally at $V = 700 \text{ m/s}$ and $2a = 12 \text{ cm}$. This dramatic difference is due to the different dependence of the separative power on V . In this connection a few fundamental questions arise. What are the physical reasons for V^2 dependence in Eq. (3)? What factors limit the growth of δU with V ? Is it possible to dispose these factors and to increase the separative power of the gas centrifuges a few times at the same velocity and length of the rotor? In other words, is it possible to design a gas centrifuge a few times more efficient than existing ones? Indeed, Eq. (3) is not a fundamental law of nature, which makes gas centrifuges with higher separative power impossible.

To answer these questions it is necessary to perform a huge amount of computational work on numerical simulation and optimisation of the counter-current gas centrifuges which are used for industrial enrichment of uranium. Even in this case the success is not guaranteed. The gas flow in the counter-current centrifuges is so complicated that it is difficult to understand the connection between the characteristics of the flow and the final optimized separative power. Therefore, it is reasonable to consider a gas centrifuge with much simpler gas flow which allows us to consider the problem analytically. In this case we have a chance to specify the nature of the dependence of the optimised separative power on the parameters and to find a guidance line for understanding this dependence in the case of the counter-current centrifuge. That is why we propose to answer the specified questions in the model of the concurrent centrifuge. This type of GC has been considered firstly in Cohen [1], where the separative power of this type of GC has been estimated as

$$\delta U = 0.166 \times 2\pi\rho DL \left(\frac{\Delta M V^2}{2RT} \right)^2, \quad (7)$$

which is only 66% less than the maximal possible separative power given by Eq. (1), and follow to V^4 dependence of δU . Nevertheless, the flow field assumed in Cohen [1] was rather artificial. Therefore it is reasonable to reconsider this model once more.

In this work for the first time we give an analytical equation for the separative power of an optimised concurrent gas centrifuge for an arbitrary binary mixture of isotopes. In contrast to the results mentioned above, we show that in the case of uniform axial velocity of the working gas, the optimised separative power is proportional to V^2 which agrees well with the empirical equation given by Eq. (3). This result forces us to assume that in spite of difference in gas flow field in the counter-current and concurrent centrifuges, the dependence of the optimised (or maximal) separative power on the parameters of the centrifuge is universal and does not depend of the design of the centrifuge.

It is necessary to stress that we discuss here the separative power of GC optimised on all parameters which can be controlled by a designer. The separative power is the function of a lot of parameters $\delta U(V, L, T, a, \alpha_1, \alpha_2, \dots)$, where the series of parameters α_i includes, for example, pressure at the wall of the rotor, F , θ , variation of temperature along the rotor δT and many others. Optimisation of the GC is reduced to a search for the maximum of this function at the variation of all the parameters α_i . Such a search is performed for every series of V, L, T , and a . Therefore, the separative power of the optimised GC depends only on the limited set of the parameters V, L, T , and a . Such a formulation of the problem carries additional difficulties in the solution of the problem because it is necessary not only to calculate the separative power of the GC, but additionally to optimise (to find maximal value) in relation to all possible parameters at fixed V, L, a and T .

In conclusion of this section, it is worth also mentioning that the process of the isotope separation and its efficiency is interesting also in application to liquids which can be subjected to the impact of the centrifugal field achieving $10^6 g$ at the temperature up to 500°C [8]. Therefore it is important to define a general equation, defining the separative power of an

optimised gas centrifuge for an arbitrary isotope mixture, not only uranium isotopes.

The paper is organised as follows. In Section 2, we present the scheme of the concurrent centrifuge, basic equations and assumptions. In Section 3 the solution is described in details. In Section 4 the optimised separative power is calculated and, finally, we discuss the solution in Section 5.

2. Isotope separation in the concurrent centrifuge

2.1. Hydrodynamics

The first model of the concurrent centrifuge has been considered by Cohen [1]. In this work we consider another version of the concurrent centrifuge. The difference between these two models is demonstrated in Fig. 1. In the Cohen scheme the gas enters the GC in two streams at different radiuses at one end of the rotor and flows axially to the other end, where the streams are removed separately. The rest of the gas is in rigid body-rotation and does not move along the axis. In our scheme the working gas UF_6 is loaded into the GC with feed flux F from the top (inlet) with the velocity v_z independent of the radius r . The gas is in corotation with the rotor and uniformly moves to the other end of the rotor. The enriched gas and depleted gas are separated by an additional coaxial cylinder of radius smaller than the radius of the rotor. This is the crucial difference between our scheme of the GC and the scheme explored by Cohen [1], which explains the difference in our results. It is necessary to stress that the scheme of Cohen is practically impossible to realise in reality. It is difficult to imagine how to provide two-stream flow of the gas through the rest of the gas corotating with the rotor in real GCs. For this reason the Cohen scheme of the concurrent centrifuge is interesting only from a methodical point of view. As far as our scheme is concerned, there is no problem in realisation of the gas flow along the rotor with more or less uniform axial velocity. This flow can easily be realised in an experiment.

We assume that the pressure p and density ρ correspond to the hydrodynamic equilibrium in the radial direction. They are

$$p = p_w \exp \left\{ \frac{\gamma V^2}{2C^2} \left[\left(\frac{r}{a} \right)^2 - 1 \right] \right\}, \quad (8)$$

and

$$\rho = \rho_w \exp \left\{ \frac{\gamma V^2}{2C^2} \left[\left(\frac{r}{a} \right)^2 - 1 \right] \right\}, \quad (9)$$

where p_w and ρ_w are the pressure and density at the wall of the rotor, $\gamma = 1.067$ is the adiabatic index of the working gas and $C = \sqrt{\gamma RT/\mu}$ is the sound velocity, while $\mu = 0.352$ kg/mol is the molecular weight of UF_6 . The output of the gas occurs at the bottom of the rotor (outlet). To simplify the hydrodynamics we assume that the product P and waste W fluxes are separated by a concentric tube with radius r_* , which provides the specified ratio of the product flux to the feed flux $\theta = P/F$. In this case the flow lines are the straight lines parallel to the axis. Here we neglect second order effects affecting the

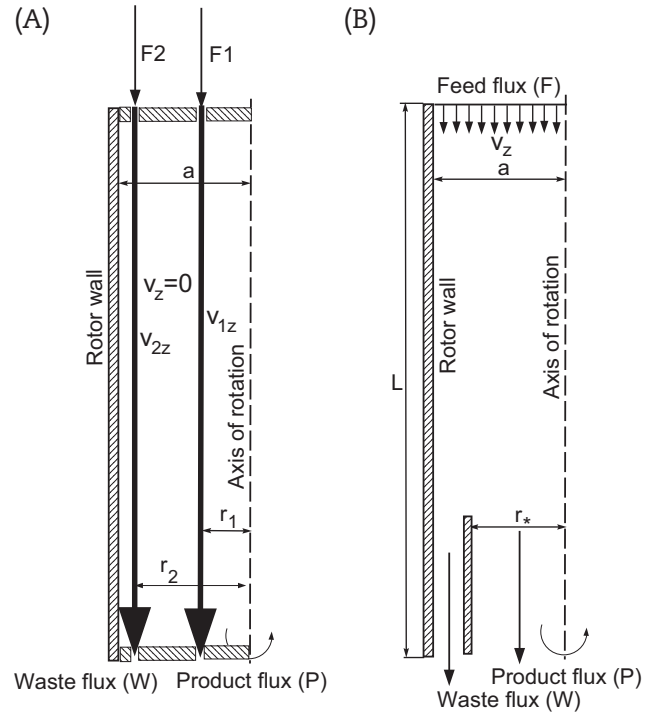


Fig. 1 – Schemes of the concurrent centrifuge: (A) used by Cohen [1] and (B) used in this work. In the Cohen scheme, the gas corotating with the rotor enters the rotor in the form of two flows, F_1 and F_2 . They go through the rotating gas, which has no axial velocity, in the form of two cylindrical flows. In our model, the corotating with the rotor gas enters the rotor at the top end (inlet) with uniform axial velocity. The product and waste fluxes are extracted at the bottom (outlet). They are separated by a concentric thin cylinder with radius r_* .

velocity of the gas due to the viscous stresses and heat conduction. Therefore, v_z is constant everywhere in the GC.

F is connected with the parameters of the gas as follows:

$$F = \int_0^a \rho v_z 2\pi r dr = \frac{2\pi\rho_w v_z a^2 C^2}{\gamma V^2} \left[1 - \exp\left(\frac{-\gamma V^2}{2C^2}\right) \right]. \quad (10)$$

The product flux P is defined as

$$P = \int_0^{r_*} \rho v_z 2\pi r dr = \frac{2\pi\rho_w v_z a^2 C^2}{\gamma V^2} \left[\exp\left\{ \frac{\gamma V^2}{2C^2} \left[\left(\frac{r_*}{a} \right)^2 - 1 \right] \right\} - \exp\left(\frac{-\gamma V^2}{2C^2}\right) \right]. \quad (11)$$

Hereafter we neglect $\exp(-\gamma V^2/2C^2)$ because its value is close to between 10^{-11} and 10^{-14} for typical parameters of GC. Then, radius r_* is defined by the equation

$$\theta = \exp\left\{ \frac{\gamma V^2}{2C^2} \left[\left(\frac{r_*}{a} \right)^2 - 1 \right] \right\} \quad (12)$$

2.2. Separative power

Concentration, c , in the feed flux equals to natural concentration of uranium, $c_0 = 7.114 \times 10^{-3}$. We assume that c does

not depend on r in the feed flux. c increases near the rotational axis and reduces at the wall, upon motion of the gas along the rotor. The separative power of the GC is defined as follows

$$\delta U = PG(c_p) + WG(c_w) - FG(c_f), \quad (13)$$

where $G(c) = (2c - 1)\ln[c/(1 - c)]$ is the separative potential introduced by Fuchs and Peierls [9]. Generalisation of this equation (see Borisevich et al. [6]) to the case of the nonuniform flow through the feed inlet, product and waste outlets gives

$$\delta U = \int G(c)\rho v dS. \quad (14)$$

In our case the integration over surface S which covers all the boundaries of the working volume gives

$$\delta U = 2\pi \int_0^a \rho v_z [G(c) - G(c_0)] r dr, \quad (15)$$

where $G(c)$ is taken at the outlet.

3. Solution of the problem

3.1. Basic equations and assumptions

The equation defining diffusion of the mixture of uranium isotopes in gaseous UF_6 is as follows:

$$\frac{\partial \rho c}{\partial t} + \frac{\partial j_k}{\partial x_k} = 0, \quad (16)$$

where the components J_k , of the flux of UF_6 with the light uranium isotope in the cylindrical system of coordinates, are given by the equations [10].

$$J_z = \rho v_z c - \rho D \frac{\partial c}{\partial z}, \quad (17)$$

and

$$J_r = \rho v_r c - \rho D \left[\frac{\partial c}{\partial r} + \frac{\Delta M}{M} c(1 - c) \frac{\partial \ln p}{\partial r} \right], \quad (18)$$

where v_r is the radial component of the gas velocity, c is the concentration of UF_6 with light isotope U235. In the case under consideration $v_r = 0$, p should be taken from Eq. (6). The natural concentration c_0 satisfies the condition $c_0 \ll 1$. Concentration does not change strongly in the concurrent GC. Therefore, the condition $c_0 \ll 1$ is valid everywhere and we can consider the term $(1 - c)$ as a constant equal to $(1 - c_0)$. Product ρD does not depend on pressure. Therefore, we consider it as a constant.

It is worth pointing out here that we use hydrodynamical equations for transport of binary mixture of isotopes which is valid only if the path length of molecules is much less than the radius of the rotor. This condition is not fulfilled in the central part of the rotor where the density of the gas is extremely small. Nevertheless, the violation of the hydrodynamic approximation can be neglected because this region gives an exponentially small contribution to the

separative power [see Eq. (15)]. Therefore below we formally assume that the hydrodynamic approximation is valid everywhere.

Substitution of the components of J into Eq. (16) gives the following equation for the steady-state diffusion

$$\frac{\partial \rho v_z c}{\partial z} - \rho D \frac{\partial^2 c}{\partial z^2} - \rho D \frac{\partial}{\partial r} \left(\frac{r \partial c}{\partial r} + \delta \frac{\gamma V^2}{C^2} \frac{r^2}{a^2} c \right) = 0, \quad (19)$$

where $\delta = (\Delta M/M)(1 - c_0)$.

Let us introduce new variables $t = \exp\{\gamma V^2/4C^2[(r/a)^2 - 1]\}$, $\bar{z} = z/a$ and express concentration c as

$$c = c_0 t^{-\delta} Y(t, \bar{z}). \quad (20)$$

In these variables, density $\rho = \rho_w t^2$. For Y we obtain the following equation using these variables

$$t \frac{\partial}{\partial t} t \frac{\partial Y}{\partial t} - 4t^2 \frac{\rho_w v_z a}{\rho D} \left(\frac{C^2}{\gamma V^2} \right)^2 \frac{\partial Y}{\partial \bar{z}} + 4 \left(\frac{C^2}{\gamma V^2} \right)^2 \frac{\partial^2 Y}{\partial \bar{z}^2} - \delta^2 Y = 0. \quad (21)$$

Here we neglect variation of $(r/a)^2$. This is reasonable because the geometrical scale Δr , on which density, pressure and t vary along the radius, is small compared with a . Indeed,

$$\frac{\Delta r}{a} = \frac{C^2}{\gamma V^2}. \quad (22)$$

For UF_6 typical $C = 86$ m/s, while $V > 600$ m/s. This means, that $\left(\frac{\Delta r}{a}\right) < 2 \times 10^{-2}$.

Let us express ρ_w in Eq. (21) through F using Eq. (8). Eq. (21) therefore takes the form

$$t \frac{\partial}{\partial t} t \frac{\partial Y}{\partial t} - t^2 \frac{2F}{\pi a \rho D} \left(\frac{C^2}{\gamma V^2} \right) \frac{\partial Y}{\partial \bar{z}} + 4 \left(\frac{C^2}{\gamma V^2} \right)^2 \frac{\partial^2 Y}{\partial \bar{z}^2} - \delta^2 Y = 0. \quad (23)$$

Here it is convenient to replace variable \bar{z} on variable ξ according to the equation

$$\xi = \frac{\gamma V^2}{2C^2} \bar{z}. \quad (24)$$

Then we obtain

$$t \frac{\partial}{\partial t} t \frac{\partial Y}{\partial t} - t^2 \frac{F}{\pi a \rho D} \frac{\partial Y}{\partial \xi} + \frac{\partial^2 Y}{\partial \xi^2} - \delta^2 Y = 0. \quad (25)$$

This equation can be simplified. The third term describing diffusion along the axial direction can be neglected provided that it is much smaller than the second term describing convection along the axial direction. This condition holds if

$$\frac{F}{\pi a \rho D} \gg \frac{1}{\xi_L}, \quad (26)$$

where $\xi_L = (2C^2/\gamma V^2)(L/a)$. We will subsequently see [see Eq. (63)] that for the optimised GC this condition always takes place.

The basic equation for the concentration takes the following form in this case

$$t \frac{\partial}{\partial t} t \frac{\partial Y}{\partial t} - t^2 \frac{F}{\pi a \rho D} \frac{\partial Y}{\partial \xi} - \delta^2 Y = 0. \quad (27)$$

3.2. Boundary conditions

The boundary condition at the inlet (feed flux) follows to the distribution

$$Y(t, 0) = t^\delta, \tag{28}$$

which corresponds to the uniform on radius concentration.

No boundary conditions should be specified at the outlet end of the rotor because we neglect the axial diffusion and the concentration is advected to this boundary from the computational domain. The boundary condition at the wall of the rotor corresponds to zero flux of the concentration

$$\frac{\partial c}{\partial r} + \delta c \frac{\partial \ln p}{\partial r} = 0. \tag{29}$$

Substitution of the pressure distribution [Eq. (6)] into this equation gives the following boundary condition for function Y

$$\frac{\gamma V^2}{2C^2} \frac{r}{a^2} t^{-\delta} \left(t \frac{\partial Y}{\partial t} + \delta Y \right) = 0. \tag{30}$$

At the wall of the rotor ($t = 1$) this gives the following boundary condition

$$t \frac{\partial Y}{\partial t} + \delta Y = 0. \tag{31}$$

We formally extend the variation of t from 1 at the wall of the rotor to 0, because t is of the order 10^{-11} to 10^{-14} at the axis of rotation. We also neglect the variation of r/a assuming that this parameter equals 1. Therefore, the boundary condition at $t = 0$ should be

$$t^{-\delta} \left(t \frac{\partial Y}{\partial t} + \delta Y \right) = 0. \tag{32}$$

3.3. Solution

We can expand the solution of Eq. (27) on functions $\exp(-k\xi)$ in a sum of series as follows

$$Y(t, \xi) = \sum \exp(-k_i \xi) S_i(\xi), \tag{33}$$

where functions S_i satisfy the equation

$$\frac{\partial}{\partial t} t \frac{\partial S_i}{\partial t} + \left(\frac{Fk_i}{\pi a \rho D} - \frac{\delta^2}{t^2} \right) S_i = 0. \tag{34}$$

The general solution of this equation can be expressed as a linear combination of Bessel functions $J_\delta(\lambda t)$ and $J_{-\delta}(\lambda t)$

$$S_i = E J_\delta(\lambda_i t) + B J_{-\delta}(\lambda_i t), \tag{35}$$

where E and B are some constants and $\lambda_i = \sqrt{(Fk_i)/(\pi a \rho D)}$. To define these constants, let us firstly consider the boundary condition at $t = 0$. According to Abramowitz and Stegun [11], the functions $J_\delta(\lambda_i t)$ and $J_{-\delta}(\lambda_i t)$ behave near the point $t = 0$ as

$$\begin{aligned} J_\delta(\lambda_i t) &= \frac{1}{\Gamma(\delta + 1)} \left(\frac{\lambda_i t}{2} \right)^\delta, \quad J_{-\delta}(\lambda_i t) \\ &= \frac{1}{\Gamma(1 - \delta)} \left(\frac{\lambda_i t}{2} \right)^{-\delta} \left(1 - \frac{(\lambda_i t)^2 \Gamma(1 - \delta)}{4\Gamma(2 - \delta)} \right). \end{aligned} \tag{36}$$

Thus, one of the functions is regular in this point, while another diverges. Substitution of the function J_δ in the limit $t \rightarrow 0$ (see Abramowitz and Stegun [11]) into the boundary condition [Eq. (32)] gives the expression

$$\frac{2\delta}{\Gamma(1 + \delta)} \left(\frac{\lambda_i}{2} \right)^2 \tag{37}$$

which does not equal 0. Substitution of $J_{-\delta}$ in the limit $t \rightarrow 0$ into Eq. (32) gives the expression

$$\frac{2\lambda_i^{2-\delta} t^{2-2\delta}}{2^{2-\delta} \Gamma(2 - \delta)}, \tag{38}$$

which goes to 0 at $t \rightarrow 0$. Thus, $S_i = B J_{-\delta}(\lambda_i t)$. The boundary condition at the wall of the rotor gives

$$\lambda_i \delta \frac{J_{-\delta}(\lambda_i)}{\delta \lambda_i} + \delta J_{-\delta}(\lambda_i) = 0. \tag{39}$$

It follows from the properties of Bessel functions, that [11].

$$t \frac{\partial J_{-\delta}(t)}{\partial t} + \delta J_{-\delta}(t) = -J_{1-\delta}(t). \tag{40}$$

Therefore, the boundary condition at the rotor wall gives us the equation defining the eigenvalue of the problem

$$J_{1-\delta}(\lambda_i) = 0. \tag{41}$$

The four first eigenvalues are $\lambda_0 = 0$, $\lambda_1 = 3.82$, $\lambda_2 = 7$, and $\lambda_3 = 10.15$ at $\delta = 3 \times [1 - (7.114 \times 10^{-3})]/352$.

The eigenfunction at $\lambda = 0$ can be defined directly from Eq. (34) as follows:

$$S_0(t) = B_0 t^{-\delta}. \tag{42}$$

Finally, the solution takes the form

$$Y(t, \xi) = B_0 t^{-\delta} + \sum_{i=1}^{i=\infty} B_i \exp(-k_i \xi) J_{-\delta}(\lambda_i t). \tag{43}$$

Coefficients B_i in the expansion of the solution into the sum of the series [Eq. (33)] are defined from the boundary condition at the inlet $\xi = 0$. This gives

$$t^\delta = B_0 t^{-\delta} + \sum_{i=1}^{i=\infty} B_i J_{-\delta}(\lambda_i t). \tag{44}$$

Eigenfunctions $J_{-\delta}(\lambda_i t)$ are orthogonal to each other and to $t^{-\delta}$ with weight function t . Therefore, multiplication of this equation on $t^{1-\delta}$ and integration over t gives the following value of B_0

$$B_0 = 1 - \delta. \tag{45}$$

Coefficients B_i are defined from the equations

$$B_i = \frac{\int_0^1 t^\delta J_{-\delta}(\lambda_i t) t dt}{\int_0^1 J_{-\delta}^2(\lambda_i t) t dt}. \tag{46}$$

It can be shown that for the specified boundary conditions at $t = 0$ and at $t = 1$ we have

$$\int_0^1 t^\delta J_{-\delta}(\lambda_i t) dt = \frac{1}{\lambda_i^{2+\delta}} \left[\frac{2^{1+\delta}}{\Gamma(-\delta)} - \lambda_i^{1+\delta} J_{-(1+\delta)}(\lambda_i) \right], \quad (47)$$

and

$$\int_0^1 J_{-\delta}^2(\lambda_i t) dt = \frac{1}{2J_{-\delta}^2}(\lambda_i). \quad (48)$$

4. Optimised separation power

It follows from Eqs. (8) and (10) that

$$2\pi\rho v_z r dr = 2F t dt. \quad (49)$$

Therefore, the separative power [Eq. (15)] can be presented as

$$\delta U = F \int_0^1 [G(c) - G(c_0)] 2t dt, \quad (50)$$

where the integration is performed over the outlet. It is convenient to express k_i as

$$k_i = \frac{\pi a \rho D}{F} \lambda_i^2. \quad (51)$$

In this case the solution can be presented as

$$c(t, \xi) = c_0(1 - \delta)t^{-2\delta} \left[1 + \sum_1^{\infty} \frac{B_i}{(1 - \delta)} t^\delta \exp\left(-\frac{\pi a \rho D}{F} \lambda_i^2 \xi\right) J_{-\delta}(\lambda_i t) \right]. \quad (52)$$

Let us introduce new variable χ as follows

$$F = \pi a \rho D \lambda_1^2 \xi \chi. \quad (53)$$

Then the solution takes the form

$$c(t, \chi) = c_0(1 - \delta)t^{-2\delta} \left[1 + \sum_1^{\infty} \frac{B_i}{(1 - \delta)} t^\delta \exp\left(-\frac{\lambda_i^2}{\lambda_1^2 \chi}\right) J_{-\delta}(\lambda_i t) \right], \quad (54)$$

and the separative power becomes

$$\delta U = \pi a \rho D \lambda_1^2 \xi \chi \int_0^1 [G(c) - G(c_0)] 2t dt. \quad (55)$$

Substitution of Eq. (24) into this equation at $z = L$ gives the following expression for δU

$$\delta U = \frac{\pi}{8} \left(\frac{\Delta M}{M} \right)^2 \rho D \lambda_1^2 \gamma \frac{V^2}{C^2} L \Phi(\delta, c_0, \chi), \quad (56)$$

where function Φ is

$$\Phi(\delta, c_0, \chi) = 4\chi \left(\frac{M}{\Delta M} \right)^2 \int_0^1 [G(c) - G(c_0)] 2t dt. \quad (57)$$

This function depends on three parameters: δ, c_0 and χ . It follows from Fig. 2 that the dependence of Φ on δ and c_0 can be neglected at $\delta \ll 1$ and $c_0 \ll 1$. Variation of these parameters in a rather wide range results in variation of the maximum of the

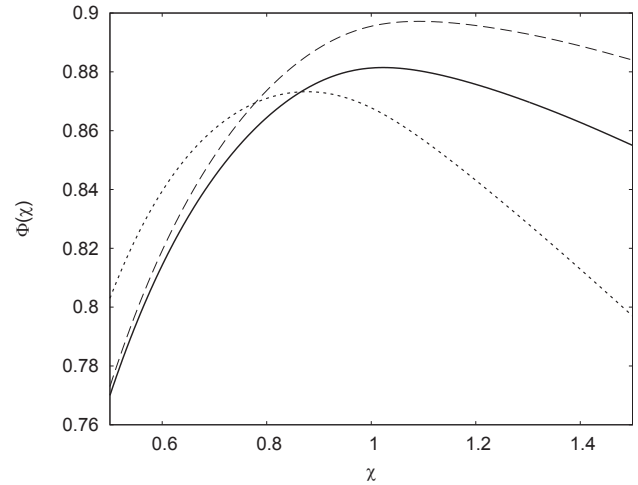


Fig. 2 – Dependence of Φ on χ at $c_0 = 7.114 \times 10^{-3}$ and $\delta = 3/352(1 - c_0)$ (solid line), at $c_0 = 7.114 \cdot 10^{-3}$ and $\delta = 0.1(1 - c_0)$ (dotted line), and at $c_0 = 0.5$ and $\delta = 3/352(1 - c_0)$ (dashed line).

function and its position only by a few percent. These parameters are specified by the properties of the working gas and are usually constant. The only variable parameter is χ . This is the dimensionless feed flux. Optimisation of the concurrent centrifuge is possible only on this flux. This has clear physical sense. At fixed centrifuge length, L , and pressure, a small feed flux results in a small velocity, v_z . Slow advection results in maximal possible separation of the isotopes. In this condition decrease of the feed flux results in decrease of δU , because of the first term in the right-hand side of Eq. (50). In the opposite case of large feed flux, velocity v_z can be so large that the separation becomes negligibly small. δU decreases because of decrease of the integral in the right-hand side of Eq. (50). Therefore, δU has to have a maximum at some feed flux, which corresponds to the optimal feed flux in the centrifuge. The dependence of Φ on χ is shown in Fig. 2. This function has a maximum equal to ~ 0.9 at $\chi = 1$.

To express the optimised separative power of the centrifuge in conventional separation work units (SWU) per year, it is necessary to multiply Eq. (56) by 1 year in seconds and by the ratio of weight of the metallic uranium over weight of the working gas equal to 238/352. After that we obtain

$$\delta U = 12.7 \left(\frac{V}{700 \text{ m/s}} \right)^2 \left(\frac{300 \text{ K}}{T} \right) \left(\frac{L}{1 \text{ m}} \right), \text{ kg} \times \frac{\text{SWU}}{\text{year}} \quad (58)$$

At the calculation of the separative power we assumed the conventional value for $\rho D = 2.3 \times 10^{-5} \text{ Pa} \cdot \text{s}$.

It is interesting to understand why the optimised separative power of the concurrent centrifuge depends on the velocity of the rotor rotation as V^2 and this does not contradict Eq. (4). It follows from Eq. (52) that the maximal value of $q - 1$ equals

$$q - 1 = t_*^{-2\delta} - 1 = 2\delta \ln \frac{1}{\theta}, \quad (59)$$

where $t_* = \theta$ [see Eq. (12)]. If we kept the separation of the fluxes θ constant, upon increase of V , the value $q - 1$ remains

constant as well. According to Eq. (12) this means that upon growth of V , the radius r of the concentric tube separating the product and waste fluxes should increase, keeping the product

$$\left(\frac{V}{C}\right)^2 \left[\left(\frac{r_*}{a}\right)^2 - 1\right]$$

constant. As a result, all the growth of δU with V is due to growth of the feed flux. This flux increases with V because the rate of the radial diffusion increases as V^2 . Therefore, it is possible to increase the mass flux proportional to V^2 to provide efficient separation. The same is valid for dependence of δU on L . q and θ do not depend on L . The feed flux increases proportionally to L because the working gas has more travelling time for radial separation.

The optimised separative power does not depend on the pressure of the gas at the wall of the rotor. This is another important feature of the obtained solution. It is clear why this occurs. Everywhere density enters into the equations in combination ρv_z . Product ρD depends only on temperature, not density. Correspondingly, Eq. (52) depends on the feed flux F . Due to Eq. (10) the optimal separative power is not unique on the plane of parameters p_w, v_z . Combining Eqs. (10) and (53) at $\chi=1$ we determine that the optimal δU is constant along the line located on the plane p_w, v_z , defined by the equation

$$p_w \cdot v_z = \frac{1}{4} \frac{\rho D \lambda_1^2 L V^2}{a^2} \left(\frac{\gamma V^2}{C^2}\right). \quad (60)$$

In the case of specification of pressure the axial velocity will be

$$v_z = 60 \left(\frac{100 \text{ mmHg}}{p_w}\right) \left(\frac{6 \text{ cm}}{a}\right)^2 \left(\frac{V}{700 \text{ m/s}}\right)^4 \left(\frac{300 \text{ K}}{T}\right) \left(\frac{L}{1 \text{ m}}\right), \text{ m/s}. \quad (61)$$

Optimal feed flux equals

$$F = \pi \rho D \lambda_1^2 \frac{\gamma V^2}{2 C^2} L = 35 \left(\frac{V}{700 \text{ m/s}}\right)^2 \left(\frac{L}{1 \text{ m}}\right), \text{ g/s}. \quad (62)$$

This means that the optimal concurrent centrifuge needs a very large feed flux and provides very small coefficient of separation defined by Eq. (59).

Now we are ready to make sure that Eq. (26) is fulfilled for the optimised centrifuges. Substitution of Eq. (62) into Eq. (26) shows that our solution is valid provided that

$$\left(\lambda_1 \frac{L}{a}\right)^2 \gg 1. \quad (63)$$

This condition is fulfilled for all GC.

5. Discussion

In this work we have solved analytically the problem of diffusion of an isotope mixture in the prescribed uniform axial flow of the gas with an exponential profile of density along radius in the concurrent centrifuge. The separative power of the GC has been calculated and optimised on the basis of this

solution. The analytical equation for the optimised separative power of the gas centrifuge is obtained practically from first principles for any arbitrary binary mixture. The equation agrees well with the empirical data [compare Eqs. (3) and (58)] in the particular case of a mixture of uranium isotopes. Surprisingly, even the numerical coefficient in this equation coincides with the experimental one in the limits of uncertainties of the experimental data. Therefore, it is difficult to avoid a conclusion that Eq. (56) has a much wider region of application than was expected by us at the beginning. It is likely that the optimised separative power of the counter-current centrifuges is described by the same equation as well. In other words, we suspect that the upper limit of the separative power of the gas centrifuge is a unique function of the basic parameters and does not depend on the design of the centrifuge. At the present level of our knowledge this is still an assumption. To make a final conclusion it is necessary to perform much work on computer simulation of gas dynamics, separation, and optimisation of the counter-current centrifuges and to compare the results with Eq. (56). Nevertheless, the first results in this direction confirm that Eq. (58) gives the correct optimised separative power and even more accurate than Eq. (3). This equation allowed us recently to resolve a rather important problem. It follows from numerical simulations [12] that the specific optimised separative power reduces with the length of the rotor, while Eq. (3) predicts a constant value. Recently we performed special computational simulations and show that this reduction is connected with a growth of temperature with the length of the rotor of the optimised centrifuge [13]. According to Eq. (58) this results in the reduction of the separative power.

Conflict of interest

The authors declare no conflict of interest.

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