OPTIMAL CONSUMPTION, PORTFOLIO, AND LIFE INSURANCE WITH BORROWING CONSTRAINT AND RISK AVERSION CHANGE

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ABSTRACT. This paper investigates an optimal consumption, portfolio, and life insurance strategies of a family when there is a borrowing constraint and risk aversion change at the time of death of the breadwinner. A CRRA utility is employed and by using the dynamic programming method, we obtain analytic expressions for the optimal strategies.

1. Introduction

Seminal works on continuous time consumption and portfolio optimization problems by Merton [7, 8] have been extended to various directions. It is very natural to consider entering into life insurance contracts in order to hedge mortality risk of a wage earner, so we may regard life insurance purchase as a part of a wage earner's portfolio. Contributions in this strand of literature include Richard [10] and Pliska and Ye [9]. Although in Pliska and Ye [9], Richard [10]'s assumption of a fixed lifetime was relaxed, investment in risky asset was not considered. A number of literature explored generalizations and extensions of Pliska and Ye [9]: see for example, Huang and Milevsky [2], Kwak et al. [6], and Shen and Wei [13].

A wage earner can borrow against future labor income and invest in the financial market and consume to derive utility. In the real world situation, however, it is allowed to borrow against only the partial amount of the whole future income stream or not even allowed to borrow any of

the future income stream. Optimal consumption, portfolio and retirement problems of a wage earner who faces borrowing constraints were actively investigated: see for example, He and Pagès [3] and Dybvig and Liu [1].

In this paper, we obtain an analytic expression of an optimal consumption, portfolio, and life insurance strategies of a family with a borrowing constraint and with risk aversion change at the time of death of the breadwinner. Kwak et al. [5] explored the optimal retirement problem when there is a risk aversion change at retirement using martingale and duality method. Recently, Jang and Lee [4] investigated a similar problem of an economic agent who faces with borrowing constraints. Our study is differenet from the above two works, in that the former considers life insurance against the loss of wage earner's income caused by mortality and the time of income decrement is a random time, whereas the latter deals with the trade-off between labor income and disutility from labor, and the reduction of income results from discretionary retirement and the retirement time is the optimal stopping time. We use the dynamic programming method and the relevant Bellman equation is transformed into a linear differential equation.

2. The model

We consider a family with a breadwinner whose labor income is the only source of non-financial income. We view the family as an economic agent and assume that the family invest in the financial market and the insurance market. The financial market consists of the usual two classes of asset: the one is the riskless asset B_t and the other is the risky asset S_t. The price processes are as follows.

$$dB_t/B_t = r, \ dS_t/S_t = \mu dt + \sigma dW_t,$$

where r is the risk free rate, μ is the constant rate of return of the risky asset, and σ is the constant volatility (the standard deviation of return) of the risky asset. W_t is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$

Let us assume that the family is infinitely lived and denote by τ the random lifetime of the breadwinner. We suppose that τ is independent of $\mathcal{F}$ and has the constant mortality intensity λ such that

$$P(\tau < t) = 1 - e^{-\lambda t}.$$

The breadwinner is an wage earner and the wage is the only non-financial income, so the family needs to buy a life insurance on the breadwinner. If we denote by p_t the premium rate at time t, the fair compensation L_t from the life insurance contract at time t is given by

$$L_t = \frac{p_t}{\lambda}.$$

Let π_t and $c_t \geq 0$ be the amount of money invested in the risky asset at time t and the consumption rate at time t, respectively. We assume that

$$\int_0^t \pi_s^2 ds < \infty, \quad \int_0^t c_s ds < \infty \text{ for all } t \geq 0 \text{ a.s.,}$$

and $\mathbf{c} \{c_t\}_{t\geq 0}$ and $\boldsymbol{\pi} \{\pi_t\}_{t\geq 0}$ are adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, which is the $\mathbb{P}$-augmentation of the filtration generated by the standard Brownian motion $\{W_t\}_{t\geq 0}$. Breadwinner's labor income is assumed to be constant and denoted by I. Then the family's wealth level X_t at time t follows the differential equation

(2.1)
$$dX_t = [rX_t + \pi_t(\mu - r) - c_t - p_t + I] \, dt + \pi_t \sigma dW_t, \text{ for } 0 \leq t < \tau,$$

(2.2)
$$dX_t = [rX_t + \pi_t(\mu - r) - c_t] \, dt + \pi_t \sigma dW_t, \text{ for } \tau \leq t,$$

and it holds that

$$X_\tau = X_{\tau_-} + L_\tau.$$

In this paper, we assume that the family derives utility from consumption and the utility function is of CRRA (constant relative risk aversion) type, and the coefficient of relative risk aversion of the family is constant over the family's lifetime but makes a one-time change at the death time of the breadwinner: if γ_t is the coefficient of relative risk aversion at time t, $\gamma_t = \gamma_1$ for $0 \leq t < \tau$, and $\gamma_t = \gamma_2$ for $\tau \leq t$. So the utility function $u(c_t)$ of the family at time t is given by

$$u(c_t) = \frac{c^{1-\gamma_1}}{1 - \gamma_1}, \text{ for } 0 \leq t < \tau, \text{ and } u(c_t) = \frac{c^{1-\gamma_2}}{1 - \gamma_2}, \text{ for } \tau \leq t,$$

with the coefficient of relative risk aversion satisfying $\gamma_1, \gamma_2 > 0$, $\gamma_1 \neq 1, \gamma_2 \neq 1$.

3. The optimization problems and the solutions

To guarantees the well-definedness of the optimization problem, we make the following assumption.

ASSUMPTION 3.1. Throughout this paper, we assume that

$$r + \lambda + \frac{\rho - r}{\gamma_1} + \frac{\gamma_1 - 1}{2\gamma_1^2}\theta^2 > 0, \ K_2 r + \frac{\rho - r}{\gamma_2} + \frac{\gamma_2 - 1}{2\gamma_2^2}\theta^2 > 0,$$

where $\theta(\mu - r)/\sigma$ is called the market price of risk.

DEFINITION 3.2. Let m_+ and m_- are positive and negative root to the equation

$$\frac{1}{2}\theta^2 m^2 + \left(r - \rho - \frac{1}{2}\theta^2\right)m - (r + \lambda) = 0,$$

respectively. Then it follows that $m_- < 0 < 1 < m_+$.

For $\tau \le t$, we call a pair of control $(\mathbf{c}^p, \boldsymbol{\pi}^p)$ *admissible* at initial capital $X_0 = x > 0$, if $X_t > 0$ and $c_t \ge 0$. After death of the breadwinner, the optimization problem of the family is to find the following value function

$$(3.1) \qquad v_p(x) \max_{(\mathbf{c^P}, \boldsymbol{\pi}^p) \in \widetilde{\mathcal{A}}(x)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\frac{c_t^{1-\gamma_2}}{1 - \gamma_2}dt\right],$$

subject to the dynamic budget constraint (2.2), where $\widetilde{\mathcal{A}}(x)$ is the set of all admissible pairs at x.

PROPOSITION 3.3. *The value function v_p is given by*

$$v_p(x) = \frac{x^{1-\gamma_2}}{K_2^{\gamma_2}(1 - \gamma_2)},$$

and the optimal consumption and portfolio processes $(c^{p,*}, \pi^{p,*})$ are as follows

$$c_t^{p,*} = K_2 X_t, \ \pi_t^{p,*} = \frac{\theta}{\sigma\gamma_2}X_t.$$

Proof. See Merton [7, 8]. □

Without any borrowing constraint, the family can borrow against the breadwinner's labor income, so a negative wealth level is allowed. More specifically, $-I/(r + \lambda) \le X_t$, i.e., borrowing up to the whole expected discounted (considering the mortality risk) income stream is possible. In this paper, however, we impose a borrowing constraint, i.e., the family cannot borrow against any of the future income stream: $X_t > 0$. In this sense, for $0 \le t < \tau$, we call a triple of control $(\mathbf{c}, \boldsymbol{\pi}, \mathbf{p})$ *admissible* at initial capital $X_0 = x > 0$, if $X_t > 0$ and $c_t \ge 0$ for all $0 \le t < \tau$. Once we obtained the value function after death of the breadwinner, we

can define the optimization problem while the breadwinner is alive as follows.

$$(3.2) \qquad V(x) \max_{(\mathbf{c},\boldsymbol{\pi},\mathbf{p})\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\tau e^{-\rho t}\frac{c_t^{1-\gamma_1}}{1-\gamma_1}dt + e^{-\rho\tau}v_p(X_\tau)\right],$$

subject to the dynamic budget constraint (2.1). By the dynamic programming principle, the relevant Bellman equation of the value function $V(x)$ is given by

$$(\rho+\lambda)V(x) = \max_\pi\left\{(\mu-r)\pi V'(x)+\frac{1}{2}\sigma^2\pi^2 V''(x)\right\} + \max_c\left\{\frac{c^{1-\gamma_1}}{1-\gamma_1}-cV'(x)\right\}$$

$$(3.3) \qquad + \max_p\left\{\lambda v_p\left(x+\frac{p}{\lambda}\right)-pV'(x)\right\}+(rx+I)V'(x).$$

THEOREM 3.4. *Suppose that a strictly increasing and strictly concave function $v(x)$ solves the Bellman equation (3.3) and the control $(\hat{c}^*,\hat{\pi}^*,\hat{p}^*)$ determined by*

$$(\hat{c}^*,\hat{\pi}^*,\hat{p}^*) = \arg\max_{(c,\pi,p)}\left\{(\mu-r)\pi v'(x)+\frac{1}{2}\sigma^2\pi^2 v''(x)+\frac{c^{1-\gamma_1}}{1-\gamma_1}-cv'(x)\right.$$

$$\left.+ \lambda v_p\left(x+\frac{p}{\lambda}\right)-pv'(x)\right\}$$

is admissible at x. Then, $V(x)=v(x)$ and $(\hat{c}^*,\hat{\pi}^*,\hat{p}^*)$ is an optimal consumption, portfolio, and life insurance strategies to the optimization problem (3.2).

Proof. This verification theorem can be proved along similar lines to the proof of Theorem 3.1 in Sotomayor and Cadenillas [11], hence we omit it. □

THEOREM 3.5. *Let*

$$h(z)\left(m_- - \frac{1}{\gamma_1}\right)\left(\frac{\gamma_1}{1-\gamma_1 m_+}-\frac{\gamma_1}{1-\gamma_1 m_-}\right)e^{\frac{1}{\gamma_1}z}$$

$$+\left(m_- - \frac{1}{\gamma_2}\right)\frac{\lambda}{K_2}\left(\frac{\gamma_2}{1-\gamma_2 m_+}-\frac{\gamma_2}{1-\gamma_2 m_-}\right)e^{\frac{1}{\gamma_2}z}+m_-\left(\frac{1}{m_+}-\frac{1}{m_-}\right)I=0,$$

and α be the solution to the equation $h(z)=0$. Define

$$X(z)Be^{m_- z}-\frac{2}{\theta^2(m_+-m_-)}\left\{\left(\frac{\gamma_1}{1-\gamma_1 m_+}-\frac{\gamma_1}{1-\gamma_1 m_-}\right)e^{\frac{1}{\gamma_1}z}\right.$$

$$\left.+\frac{\lambda}{K_2}\left(\frac{\gamma_2}{1-\gamma_2 m_+}-\frac{\gamma_2}{1-\gamma_2 m_-}\right)e^{\frac{1}{\gamma_2}z}+\left(\frac{1}{m_+}-\frac{1}{m_-}\right)I\right\},$$

where

$$B = \frac{2}{\theta^2(m_+ - m_-)m_-} \left\{ \left(\frac{1}{1 - \gamma_1 m_+} - \frac{1}{1 - \gamma_1 m_-} \right) e^{\left(-m_- + \frac{1}{\gamma_1}\right)\alpha} \right.$$
$$\left. + \frac{\lambda}{K_2} \left(\frac{1}{1 - \gamma_2 m_+} - \frac{1}{1 - \gamma_2 m_-} \right) e^{\left(-m_- + \frac{1}{\gamma_2}\right)\alpha} \right\}.$$

Then the value function $V(x)$ is given by

$$V(x) = \frac{1}{\rho + \lambda} e^{-\xi}\{(r + \lambda)x + I\} + \frac{1}{2} \frac{\theta^2}{\rho + \lambda} e^{-\xi} X'(\xi)$$
$$+ \frac{1}{\rho + \lambda} \frac{\gamma_1}{1 - \gamma_1} e^{-\xi\left(1 - \frac{1}{\gamma_1}\right)} + \frac{1}{\rho + \lambda} \frac{\lambda}{K_2} \frac{\gamma_2}{1 - \gamma_2} e^{-\xi\left(1 - \frac{1}{\gamma_2}\right)}$$

where ξ solves the algebraic equation $x = X(\xi)$.

Proof. The first order conditions (FOCs) of the Bellman equation (3.3) are
(3.4)
$$\pi^* = -\frac{\mu - r}{\sigma^2} \frac{V'(x)}{V''(x)}, \quad c^* = (V'(x))^{-\frac{1}{\gamma_1}}, \quad p^* = \lambda(K_2^{\gamma_2} V'(x))^{-\frac{1}{\gamma_2}} - \lambda x,$$

and the Bellman equation (3.3) can be rewritten as

(3.5)
$$(\rho + \lambda)V(x) = \{(r + \lambda)x + I\}V'(x) - \frac{1}{2}\theta^2 \frac{(V'(x))^2}{V''(x)} + \frac{\gamma_1}{1 - \gamma_1}(V'(x))^{1 - \frac{1}{\gamma_1}}$$
$$+ \frac{\lambda}{K_2} \frac{\gamma_2}{1 - \gamma_2}(V'(x))^{1 - \frac{1}{\gamma_2}}.$$

Along similar lines to the proof of Proposition 2.1 in Zariphopoulou [14], we see that $V(x)$ is a strictly concave function. So, $-\ln V'(x)$ is strictly increasing, and there exists an inverse function $X(\cdot)$ such that

$$-\ln V'(X(z)) = z,$$

with the identities

(3.6) $$V'(X(z)) = e^{-z}, \quad V''(X(z)) = -\frac{e^{-z}}{X'(z)}.$$

This transformation can be found in Sethi et al. [12]. Then, (3.5) can be written as

$$(3.7) \quad (\rho + \lambda)V(X(z)) = \{(r + \lambda)X(z) + I\}e^{-z} + \frac{1}{2}\theta^2 e^{-z}X'(z)$$

$$+ \frac{\gamma_1}{1 - \gamma_1}e^{-z\left(1 - \frac{1}{\gamma_1}\right)} + \frac{\lambda}{K_2}\frac{\gamma_2}{1 - \gamma_2}e^{-z\left(1 - \frac{1}{\gamma_2}\right)}$$

By differentiating (3.7) with respect to z, we obtain the following linear differential equation

(3.8)
$$\frac{1}{2}\theta^2 X''(z) + \left(r - \rho - \frac{1}{2}\theta^2\right)X'(z) - (r + \lambda)X(z) = I - e^{\frac{z}{\gamma_1}} - \frac{\lambda}{K_2}e^{\frac{z}{\gamma_2}}.$$

If we try a homogenous solution e^{mz}, the general solution to the equation (3.8) is given by

$$X(z) = Ae^{m_+ z} + Be^{m_- z} + X_p(z),$$

for some constants A and B and

$$X_p(z) = -\frac{2}{\theta^2(m_+ - m_-)}\left\{\left(\frac{\gamma_1}{1 - \gamma_1 m_+} - \frac{\gamma_1}{1 - \gamma_1 m_-}\right)e^{\frac{1}{\gamma_1}z}\right.$$

$$\left. + \frac{\lambda}{K_2}\left(\frac{\gamma_2}{1 - \gamma_2 m_+} - \frac{\gamma_2}{1 - \gamma_2 m_-}\right)e^{\frac{1}{\gamma_2}z} + \left(\frac{1}{m_+} - \frac{1}{m_-}\right)I\right\}.$$

We apply no blow up condition and set $A = 0$. Due to the borrowing constraint $X_t > 0$, there exists $\tilde{z}$ such that

$$(3.9) \quad X(\tilde{z}) = 0 : \quad Be^{m_- \tilde{z}} + X_p(\tilde{z}) = 0.$$

From no risky investment condition at the wealth level 0, we have, from (3.4) and (3.6),

$$(3.10) \quad X'(\tilde{z}) = 0 : \quad m_- Be^{m_- \tilde{z}} + X'_p(\tilde{z}) = 0.$$

Equating (3.9) and (3.10) yields

$$\left(m_- - \frac{1}{\gamma_1}\right)\left(\frac{\gamma_1}{1 - \gamma_1 m_+} - \frac{\gamma_1}{1 - \gamma_1 m_-}\right)e^{\frac{1}{\gamma_1}\tilde{z}}$$

$$+ \left(m_- - \frac{1}{\gamma_2}\right)\frac{\lambda}{K_2}\left(\frac{\gamma_2}{1 - \gamma_2 m_+} - \frac{\gamma_2}{1 - \gamma_2 m_-}\right)e^{\frac{1}{\gamma_2}\tilde{z}} + m_-\left(\frac{1}{m_+} - \frac{1}{m_-}\right)I = 0,$$

from which $\tilde{z}$ can be found and B can be found from (3.10). $\square$

From the first order conditions (FOCs) (3.4), we can summarize the optimal consumption, portfolio, and life insurance strategies as follows.

THEOREM 3.6. *The optimal strategies (c^*, π^*, p^*) are given by*

$$c_t^* = e^{\frac{\xi_t}{\gamma_1}}, \ \pi_t^* = \frac{\theta}{\sigma} X'(\xi_t), \ p_t^* = \frac{\lambda}{K_2} e^{\frac{\xi_t}{\gamma_2}} - \lambda X(\xi_t),$$

where ξ_t is the solution to the algebraic equation $X_t = X(\xi_t)$ for given wealth level X_t for $0 \leq t < \tau$.

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