

A REMARK ON FANO MANIFOLDS OF PICARD NUMBER ONE AND INDEX GREATER THAN ONE

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ABSTRACT. This note is a remark on simple observation that the degree bound of Fano $(n+1)$ -folds of Picard number one and index greater than one can be related with that of Fano n -folds of Picard number one and index one.

1. The observation

A Fano n -fold is a smooth projective variety X whose anticanonical divisor $-K_X$ is ample. The index of X is the largest integer m such that $-K_X = mH$ for some divisor H . The number $(-K_X)^n$ is called the degree and the divisor H (respectively the linear system $|H|$) is called the fundamental divisor (respectively the fundamental linear system) of the Fano n -fold X . Note that the indexes and degrees are invariant under deformation. If X has Picard number one, it is said to be of *the first species*. Iskovskikh classified Fano 3-folds of the first species in [1], where he assumed:

“The fundamental linear system $|H|$ contains a smooth surface”

This assumption was proved later by Shokurov ([2, 3]).

According to the Iskovskikh–Mori–Mukai classification of Fano 3-folds, there are 17 deformation families of Fano 3-folds of the first species and ten of them have index one. Among those ten families, five families contain Fano 3-folds that are cyclic covers over another Fano 3-fold. For example, consider the family of Fano 3-folds of the first species with index one and degree 10. Generically they are smooth sections of the Grassmannian $\text{Gr}(2, 5)$ embedded by Plücker by a subspace of codimension two and quadric. But some members of the family are double covers over smooth sections of the Grassmannian $\text{Gr}(2, 5)$ embedded by Plücker

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by a subspace of codimension three, branched along surfaces of degree two. The base varieties are Fano 3-folds of the first species with index two and degree 40. Let us say that a family of Fano n -folds of first species is of *cyclic type* if it contain Fano n -folds that are cyclic covers over another Fano n -fold, branched along smooth divisors.

In this note, we relate the bound of degrees of Fano $(n + 1)$ -fold of the first species with index greater than one with that of degrees of Fano n -fold of the first species with index one under an assumption:

“Families of Fano n -folds of the first species with index greater than one contain Fano n -folds such that the linear system $|(r - 1)H|$ contains a smooth divisor”.

This assumption is somewhat weaker than the original assumption of Iskovskih and true for dimension two, three ([2, 3]) and four ([4, 5]). Actually the fundamental divisors of Fano n -folds of index greater than one are very ample at least for dimension $n \leq 4$. Moreover Fano n -fold of index $n + 1, n, n - 1, n - 2$ are all classified ([7], [8], [5], [9]) and one can easily check that the assumption is true for those Fano n -folds.

Let

$b(n, r) = \max\{\text{degrees of Fano } n\text{-folds of the first species with index } r\},$

$b(n) = \max\{b(n, r) | r \in \mathbb{N}\}$ and

$\tilde{b}(n) = \max\{\text{degrees of Fano } n\text{-folds of cyclic type with index one}\}.$

It is known that $b(n) \leq \left(\frac{n^2+4n+3}{4}\right)^n$ ([6]).

If the fundamental divisor H is very ample (which is true for most of Fano 3-folds), the dimension of the fundamental linear system $|H|$ is the dimension of the projective space into which X is embedded. The number is invariant under deformation. Hence let

$c(n, r) = \max\{\text{dimensions of the fundamental linear systems of Fano } n\text{-folds of the first species with index } r\}.$

THEOREM 1.1. *Under the assumption,*

- (1) $b(n + 1, r) \leq \frac{r^{n+1}}{r-1} b(n, 1)$ for $r \geq 2$ and $n \geq 3$.
- (2) $c(n + 1, 2) \leq c(n, 1) + 1$ for $n \geq 3$.
- (3) $c(n + 1, r) \leq c(n, 1)$ for $r \geq 3$ and $n \geq 3$.
- (4) $\tilde{b}(n + 1) \leq 2b(n, 1)$ for $n \geq 3$.

Proof. Let Z be a Fano $(n + 1)$ -fold of the first species with index $r > 1$. By the assumption, the linear system $|(r - 1)H|$ contains a smooth divisor X . By adjunction theorem and Lefschetz hyperplane theorem,

X is a Fano n -fold of the first species with index one. Note

$$(-K_X)^n = (r - 1)H^{n+1} = (r - 1) \left(-\frac{1}{r}K_Z \right)^{n+1}.$$

Therefore we have

$$(-K_Z)^{n+1} = \frac{r^{n+1}}{r - 1}(-K_X)^n \leq \frac{r^{n+1}}{r - 1} b(n, 1),$$

which leads to the first inequality. Taking the cohomology of the sheaf sequence

$$0 \rightarrow \mathcal{O}_Z((2 - r)H) \rightarrow \mathcal{O}_Z(H) \rightarrow \mathcal{O}_X(H|_X) \rightarrow 0,$$

we obtain an exact sequence

$$0 \rightarrow H^0(\mathcal{O}_Z((2-r)H)) \rightarrow H^0(\mathcal{O}_Z(H)) \rightarrow H^0(\mathcal{O}_X(H|_X)) \rightarrow H^1(\mathcal{O}_Z((2-r)H)).$$

By the Lefschetz hyperplane theorem, $H|_X$ is the fundamental divisor of X . If $r = 2$, then $h^0(\mathcal{O}_Z(H)) = h^0(\mathcal{O}_X(H|_X)) + 1$, which leads to the second inequality. If $r > 2$, then $H^0(\mathcal{O}_Z((2 - r)H)) = 0$ and $H^1(\mathcal{O}_Z((2 - r)H)) = 0$, where the latter holds due to the Serre duality and Kodaira vanishing theorem. Therefore $h^0(\mathcal{O}_Z(H)) = h^0(\mathcal{O}_X(H|_X))$. So we have the third inequality.

Let \mathcal{F} be a family Fano $(n + 1)$ -fold that is of cyclic type with index one. Then it contains a Fano $(n + 1)$ -fold W that is an m -cover over a Fano $(n + 1)$ -fold V . Let $\phi : W \rightarrow V$ be the covering map and S be the branch locus. Then S is linearly equivalent to $-mkH$ for some positive integer k , where H is the fundamental divisor of V . Note $K_W = \phi^*((m - 1)kH + K_V) = ((m - 1)k - r_V)\phi^*(H)$, where r_V be the index of V . Note that $r_V \geq 2$. Since W has index one, we have $((m - 1)k - r_V) = -1$. Note

$$m - 1 \leq (m - 1)k = r_V - 1.$$

So $m \leq r_V$, i.e. $\frac{m}{r_V} \leq 1$. Note that $\frac{r_V}{r_V - 1} \leq 2$.

Hence

$$\begin{aligned} (-K_W)^{n+1} &= (-(m - 1)k + r_V)^{n+1}(\phi^*H)^{n+1} \\ &= mH^{n+1} \\ &= \frac{m(-K_V)^{n+1}}{r_V^{n+1}} \\ &\leq \frac{(-K_V)^{n+1}}{r_V^n} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{b(n+1, r_V)}{r_V^n} \\
&\leq \frac{r_V}{r_V-1} b(n, 1) \quad (\because \text{inequality (1)}) \\
&\leq 2b(n, 1),
\end{aligned}$$

which leads to the fourth inequality. \square

It is known that $b(4) = 5^4$ ([10]) and $b(5) \leq 9^5$ ([11]). So under the assumption, we have some bounds for Fano 5-folds and 6-folds:

- (1) $b(5, r) \leq 5^4 \cdot \frac{r^6}{r-1}$ for $r \geq 2$,
- (2) $\tilde{b}(5) \leq 2 \cdot 5^4$,
- (3) $b(6, r) \leq 9^5 \cdot \frac{r^7}{r-1}$ for $r \geq 2$,
- (4) $\tilde{b}(6) \leq 2 \cdot 9^5$.

By the Lefschetz hyperplane theorem and the adjunction formula, more data from Fano n -fold of the first species with index one can be transferred to the Fano $(n+1)$ -folds of the first species with index greater than one, such as some of Hodge numbers, cupproduct structure of Chern classes and integral cohomology classes.

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