

BOUNDEDNESS IN FUNCTIONAL DIFFERENTIAL SYSTEMS VIA t_∞ -SIMILARITY

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ABSTRACT. In this paper, we show that the solutions to perturbed functional differential system

$$y' = f(t, y) + \int_{t_0}^t g(s, y(s), Ty(s))ds,$$

have a bounded properties. To show the bounded properties, we impose conditions on the perturbed part $\int_{t_0}^t g(s, y(s), Ty(s))ds$ and on the fundamental matrix of the unperturbed system $y' = f(t, y)$ using the notion of t_∞ -similarity.

1. Introduction

One of the most useful methods available for studying the qualitative properties of the solutions of a nonlinear system of differential equations involves the use of the variation of constants formula and a integral inequality. This gives an intergral equation satisfied by the solutions of the nonlinear system. Using these methods, we investigate bounds for solutions of the nonlinear differential systems further allowing more general perturbations that were previously allowed using the notions of t_∞ -similarity and h -stability.

The notion of h -stability (hS) was introduced by Pinto [17,18] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called h -systems. Pachpatte[15,16] investigated the stability, boundedness, and the asymptotic behavior of the solutions of perturbed nonlinear systems under some suitable conditions on the perturbation term g and on the

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operator T . Choi and Ryu [4] and Choi, Koo and Ryu [5] investigated bounds of solutions for nonlinear perturbed systems. Also, Goo [7,8,9] and Goo et al. [3,11] studied the boundedness of solutions for the perturbed differential systems.

2. Preliminaries

We are interested in the relations between the solutions of the unperturbed nonlinear nonautonomous differential system

$$(2.1) \quad x'(t) = f(t, x(t)), \quad x(t_0) = x_0,$$

and the solutions of the perturbed differential system of (2.1) including an operator T such that

$$(2.2) \quad y' = f(t, y) + \int_{t_0}^t g(s, y(s), Ty(s))ds, \quad y(t_0) = y_0,$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $g \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$, $f(t, 0) = 0$, $g(t, 0, 0) = 0$, and $T : C(\mathbb{R}^+, \mathbb{R}^n) \rightarrow C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operator and \mathbb{R}^n is an n -dimensional Euclidean space. We always assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$. The symbol $|\cdot|$ will be used to denote any convenient vector norm in \mathbb{R}^n .

Let $x(t, t_0, x_0)$ denote the unique solution of (2.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (2.1) and around $x(t)$, respectively,

$$(2.3) \quad v'(t) = f_x(t, 0)v(t), \quad v(t_0) = v_0$$

and

$$(2.4) \quad z'(t) = f_x(t, x(t, t_0, x_0))z(t), \quad z(t_0) = z_0.$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (2.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.3).

We recall the notion of h -stability [17].

DEFINITION 2.1. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called an (hS) h -stable if there exist a constant $c \geq 1$, and a positive bounded continuous function h on \mathbb{R}^+ such that

$$|x(t)| \leq c|x_0| h(t) h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0| < \delta$ (here $h(t)^{-1} = \frac{1}{h(t)}$).

Let \mathcal{M} denote the set of all $n \times n$ continuous matrices $A(t)$ defined on \mathbb{R}^+ and \mathcal{N} be the subset of \mathcal{M} consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞ -similarity in \mathcal{M} was introduced by Conti [6].

DEFINITION 2.2. A matrix $A(t) \in \mathcal{M}$ is t_∞ -similar to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over \mathbb{R}^+ , i.e.,

$$\int_0^\infty |F(t)| dt < \infty$$

such that

$$(2.5) \quad \dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t)$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞ -similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on \mathbb{R}^+ , and it preserves some stability concepts [6, 12].

We give some related properties that we need in the sequel.

LEMMA 2.3. [18] *The linear system*

$$(2.6) \quad x' = A(t)x, \quad x(t_0) = x_0,$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h -system (respectively h -stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function h defined on \mathbb{R}^+ such that

$$(2.7) \quad |\phi(t, t_0)| \leq c h(t) h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (2.6).

We need Alekseev formula to compare between the solutions of (2.1) and the solutions of perturbed nonlinear system

$$(2.8) \quad y' = f(t, y) + g(t, y), \quad y(t_0) = y_0,$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (2.8) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

LEMMA 2.4. [2] Let x and y be a solution of (2.1) and (2.8), respectively. If $y_0 \in \mathbb{R}^n$, then for all $t \geq t_0$ such that $x(t, t_0, y_0) \in \mathbb{R}^n$, $y(t, t_0, y_0) \in \mathbb{R}^n$,

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^t \Phi(t, s, y(s)) g(s, y(s)) ds.$$

THEOREM 2.5. [4] If the zero solution of (2.1) is hS, then the zero solution of (2.3) is hS.

THEOREM 2.6. [5] Suppose that $f_x(t, 0)$ is t_∞ -similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (2.3) is hS, then the solution $z = 0$ of (2.4) is hS.

LEMMA 2.7. (Bihari – type inequality) Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u . Suppose that, for some $c > 0$,

$$u(t) \leq c + \int_{t_0}^t \lambda(s)w(u(s))ds, \quad t \geq t_0 \geq 0.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t \lambda(s)ds \right], \quad t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^u \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$ and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t \lambda(s)ds \in \text{dom} W^{-1} \right\}.$$

LEMMA 2.8. [10] Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,

$$\begin{aligned} u(t) \leq c &+ \int_{t_0}^t \lambda_1(s)u(s)ds + \int_{t_0}^t \lambda_2(s)w(u(s))ds + \int_{t_0}^t \lambda_3(s) \int_{t_0}^s (\lambda_4(\tau)u(\tau) \\ &+ \lambda_5(\tau) \int_{t_0}^\tau \lambda_6(r)u(r)dr)d\tau ds + \int_{t_0}^t \lambda_7(s) \int_{t_0}^s \lambda_8(\tau)w(u(\tau))d\tau ds. \end{aligned}$$

Then

$$\begin{aligned} u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s (\lambda_4(\tau) + \lambda_5(\tau) \int_{t_0}^\tau \lambda_6(r)dr)d\tau \right. \\ \left. + \lambda_7(s) \int_{t_0}^s \lambda_8(\tau)d\tau)ds \right], \end{aligned}$$

where $t_0 \leq t < b_1$, W, W^{-1} are the same functions as in Lemma 2.7, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s (\lambda_4(\tau) + \lambda_5(\tau) \int_{t_0}^\tau \lambda_6(r) dr) d\tau + \lambda_7(s) \int_{t_0}^s \lambda_8(\tau) d\tau) ds \in \text{dom } W^{-1} \right\}.$$

For the proof we need the following corollaries.

COROLLARY 2.9. *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,*

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s) \int_{t_0}^s (\lambda_2(\tau) u(\tau) + \lambda_3(\tau) \int_{t_0}^\tau \lambda_4(r) u(r) dr) d\tau ds + \int_{t_0}^t \lambda_5(s) \int_{t_0}^s \lambda_6(\tau) w(u(\tau)) d\tau ds.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t (\lambda_1(s) \int_{t_0}^s (\lambda_2(\tau) + \lambda_3(\tau) \int_{t_0}^\tau \lambda_4(r) dr) d\tau + \lambda_5(s) \int_{t_0}^s \lambda_6(\tau) d\tau) ds \right],$$

where $t_0 \leq t < b_1$, W, W^{-1} are the same functions as in Lemma 2.7, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) \int_{t_0}^s (\lambda_2(\tau) + \lambda_3(\tau) \int_{t_0}^\tau \lambda_4(r) dr) d\tau + \lambda_5(s) \int_{t_0}^s \lambda_6(\tau) d\tau) ds \in \text{dom } W^{-1} \right\}.$$

COROLLARY 2.10. *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,*

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s) u(s) ds + \int_{t_0}^t \lambda_2(s) w(u(s)) ds + \int_{t_0}^t \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) u(\tau) d\tau ds.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) d\tau) ds \right],$$

where $t_0 \leq t < b_1$, W, W^{-1} are the same functions as in Lemma 2.7, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) d\tau) ds \in \text{dom } W^{-1} \right\}.$$

COROLLARY 2.11. Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s)u(s)ds + \int_{t_0}^t \lambda_2(s)w(u(s))ds \\ + \int_{t_0}^t \lambda_3(s) \int_{t_0}^s \lambda_4(\tau)w(u(\tau))d\tau ds.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s \lambda_4(\tau)d\tau) ds \right],$$

where $t_0 \leq t < b_1$, W, W^{-1} are the same functions as in Lemma 2.7, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s \lambda_4(\tau)d\tau) ds \in \text{dom} W^{-1} \right\}.$$

LEMMA 2.12. [9] Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s)w(u(s))ds + \int_{t_0}^t \lambda_2(s) \int_{t_0}^s (\lambda_3(\tau)u(\tau) + \lambda_4(\tau)w(u(\tau))) \\ + \lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r)w(u(r))dr) d\tau ds + \int_{t_0}^t \lambda_7(s) \int_{t_0}^s \lambda_8(\tau)w(u(\tau))d\tau ds.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau)) \\ + \lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r)dr) d\tau + \lambda_7(s) \int_{t_0}^s \lambda_8(\tau)d\tau) ds \right],$$

where $t_0 \leq t < b_1$, W, W^{-1} are the same functions as in Lemma 2.7, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau)) \\ + \lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r)dr) d\tau + \lambda_7(s) \int_{t_0}^s \lambda_8(\tau)d\tau) ds \in \text{dom} W^{-1} \right\}.$$

We prepare the following corollary for the proof.

COROLLARY 2.13. *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,*

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s) \int_{t_0}^s (\lambda_2(\tau)u(\tau) + \lambda_3(\tau)w(u(\tau)) + \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(r)w(u(r))dr)d\tau ds.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t \lambda_1(s) \int_{t_0}^s (\lambda_2(\tau) + \lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(r)dr)d\tau ds \right],$$

where $t_0 \leq t < b_1$, W, W^{-1} are the same functions as in Lemma 2.7, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t \lambda_1(s) \int_{t_0}^s (\lambda_2(\tau) + \lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(r)dr)d\tau ds \in \text{dom}W^{-1} \right\}.$$

3. Main results

In this section, we investigate boundedness for solutions of perturbed functional differential systems using the notions of t_∞ -similarity and h -stability.

To obtain the bounded result, the following assumptions are needed:

- (H1) $w(u)$ is nondecreasing function in u such that $u \leq w(u)$ and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$.
- (H2) The solution $x = 0$ of (2.1) is hS with the increasing function h .
- (H3) $f_x(t, 0)$ is t_∞ -similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$.

THEOREM 3.1. *Let $a, b, c, k \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (2.2) satisfies*

$$(3.1) \quad \begin{aligned} |g(t, y, Ty)| &\leq a(t)|y(t)| + b(t)w(|y(t)|) + |Ty(t)|, \\ |Ty(t)| &\leq c(t) \int_{t_0}^t k(s)|y(s)|ds \end{aligned}$$

where $a, b, c, k, w \in L^1(\mathbb{R}^+)$, $w \in C((0, \infty))$, T is a continuous operator. Then, any solution $y(t) = y(t, t_0, y_0)$ of (2.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t \int_{t_0}^s \left(a(\tau) + b(\tau) + c(\tau) \int_{t_0}^{\tau} k(r)dr \right) d\tau ds \right],$$

where W, W^{-1} are the same functions as in Lemma 2.7, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^t \int_{t_0}^s \left(a(\tau) + b(\tau) + c(\tau) \int_{t_0}^{\tau} k(r)dr \right) d\tau ds \in \text{dom}W^{-1} \right\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. By (H2) and Theorem 2.5, the solution $v = 0$ of (2.3) is hS. Therefore, because of (H3), by Theorem 2.6, the solution $z = 0$ of (2.4) is hS. Applying the nonlinear variation of constants formula Lemma 2.4, (H2), together with (3.1), we have

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))| \int_{t_0}^s |g(\tau, y(\tau), Ty(\tau))| d\tau ds \\ &\leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^t c_2h(t)h(s)^{-1} \int_{t_0}^s \left(a(\tau)|y(\tau)| \right. \\ &\quad \left. + b(\tau)w(|y(\tau)|) + c(\tau) \int_{t_0}^{\tau} k(r)|y(r)|dr \right) d\tau ds. \end{aligned}$$

By the assumptions (H1) and (H2), we have

$$\begin{aligned} |y(t)| &\leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^t c_2h(t) \int_{t_0}^s \left(a(\tau) \frac{|y(\tau)|}{h(\tau)} + b(\tau)w \left(\frac{|y(\tau)|}{h(\tau)} \right) \right. \\ &\quad \left. + c(\tau) \int_{t_0}^{\tau} k(r) \frac{|y(r)|}{h(r)} dr \right) d\tau ds. \end{aligned}$$

Putting $u(t) = |y(t)||h(t)|^{-1}$, then, by Corollary 2.9, it follows that

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t \int_{t_0}^s \left(a(\tau) + b(\tau) + c(\tau) \int_{t_0}^{\tau} k(r)dr \right) d\tau ds \right]$$

where $c = c_1|y_0|h(t_0)^{-1}$. From the above estimation, we obtain the desired result. Thus, the theorem is proved. \square

REMARK 3.2. Letting $a(t) = 0$ in Theorem 3.1, we obtain the similar result as that of Theorem 3.5 in [13].

THEOREM 3.3. *Let $a, b, c, k \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (2.2) satisfies*

$$(3.2) \quad \int_{t_0}^t |g(s, y(s), Ty(s))| ds \leq a(t)|y(t)| + b(t)w(|y(t)|) + |Ty(t)|,$$

and

$$(3.3) \quad |Ty(t)| \leq c(t) \int_{t_0}^t k(s)|y(s)| ds$$

where $a, b, c, k, w \in L^1(\mathbb{R}^+)$, $w \in C((0, \infty))$, T is a continuous operator. Then, any solution $y(t) = y(t, t_0, y_0)$ of (2.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t \left(a(s) + b(s) + c(s) \int_{t_0}^s k(\tau) d\tau \right) ds \right],$$

where $t_0 \leq t < b_1$, W, W^{-1} are the same functions as in Lemma 2.7, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^t \left(a(s) + b(s) + c(s) \int_{t_0}^s k(\tau) d\tau \right) ds \in \text{dom}W^{-1} \right\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. By the same argument as in the proof in Theorem 3.1, the solution $z = 0$ of (2.4) is hS. Using the nonlinear variation of constants formula Lemma 2.4, (H2), together with (3.2) and (3.3), we have

$$|y(t)| \leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^t c_2h(t)h(s)^{-1} \left(a(s)|y(s)| + b(s)w(|y(s)|) + c(s) \int_{t_0}^s k(\tau)|y(\tau)| d\tau \right) ds.$$

Using the assumptions (H1) and (H2), we have

$$|y(t)| \leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^t c_2h(t) \left(a(s) \frac{|y(s)|}{h(s)} + b(s)w \left(\frac{|y(s)|}{h(s)} \right) + c(s) \int_{t_0}^s k(\tau) \frac{|y(\tau)|}{h(\tau)} d\tau \right) ds.$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, by Corollary 2.10, we have

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t \left(a(s) + b(s) + c(s) \int_{t_0}^s k(\tau)d\tau \right) ds \right],$$

where $c = c_1|y_0|h(t)h(t_0)^{-1}$. The above estimation yields the desired result since the function h is bounded, and so the proof is complete. \square

REMARK 3.4. Letting $b(t) = 0$ in Theorem 3.3, we obtain the similar result as that of Theorem 3.1 in [11].

THEOREM 3.5. Let $a, b, c, k \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (2.2) satisfies

$$(3.4) \quad \begin{aligned} |g(t, y, Ty)| &\leq a(t)|y(t)| + b(t)w(|y(t)|) + |Ty(t)|, \\ |Ty(t)| &\leq c(t) \int_{t_0}^t k(s)w(|y(s)|)ds \end{aligned}$$

where $a, b, c, k, w \in L^1(\mathbb{R}^+)$, $w \in C((0, \infty))$, T is a continuous operator. Then, any solution $y(t) = y(t, t_0, y_0)$ of (2.2) is bounded on $[t_0, \infty)$ and

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t \int_{t_0}^s \left(a(\tau) + b(\tau) + c(\tau) \int_{t_0}^\tau k(r)dr \right) d\tau ds \right]$$

where W, W^{-1} are the same functions as in Lemma 2.7, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^t \int_{t_0}^s \left(a(\tau) + b(\tau) + c(\tau) \int_{t_0}^\tau k(r)dr \right) d\tau ds \in \text{dom}W^{-1} \right\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. By the same argument as in the proof in Theorem 3.1, the solution $z = 0$ of (2.4) is hS. By Lemma 2.3, Lemma 2.4, (H2), together with (3.4), we obtain

$$\begin{aligned} |y(t)| &\leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^t c_2h(t)h(s)^{-1} \int_{t_0}^s \left(a(\tau)|y(\tau)| \right. \\ &\quad \left. + b(\tau)w(|y(\tau)|) + c(\tau) \int_{t_0}^\tau k(r)w(|y(r)|)dr \right) d\tau ds. \end{aligned}$$

By the assumptions (H1) and (H2), we have

$$|y(t)| \leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^t c_2h(t) \int_{t_0}^s \left(a(\tau) \frac{|y(\tau)|}{h(\tau)} + b(\tau)w\left(\frac{|y(\tau)|}{h(\tau)}\right) + c(\tau) \int_{t_0}^\tau k(r)w\left(\frac{|y(r)|}{h(r)}\right)dr \right) d\tau ds.$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, it follows from Corollary 2.13 that we have

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t \int_{t_0}^s \left(a(\tau) + b(\tau) + c(\tau) \int_{t_0}^\tau k(r)dr \right) d\tau ds \right],$$

where $c = c_1|y_0|h(t_0)^{-1}$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (2.2) is bounded on $[t_0, \infty)$, and so the proof is complete. \square

REMARK 3.6. Letting $a(t) = 0$ in Theorem 3.5, we obtain the similar result as that of Theorem 3.4 in [3].

THEOREM 3.7. Let $a, b, c, k \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (2.2) satisfies

$$(3.5) \quad \int_{t_0}^t |g(s, y(s), Ty(s))| ds \leq a(t)|y(t)| + b(t)w(|y(t)|) + |Ty(t)|,$$

and

$$(3.6) \quad |Ty(t)| \leq c(t) \int_{t_0}^t k(s)w(|y(s)|) ds$$

where $a, b, c, k, w \in L^1(\mathbb{R}^+)$, $w \in C((0, \infty))$, T is a continuous operator. Then, any solution $y(t) = y(t, t_0, y_0)$ of (2.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t \left(a(s) + b(s) + c(s) \int_{t_0}^s k(\tau) d\tau \right) ds \right],$$

where $t_0 \leq t < b_1$, W, W^{-1} are the same functions as in Lemma 2.7, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^t \left(a(s) + b(s) + c(s) \int_{t_0}^s k(\tau) d\tau \right) ds \in \text{dom}W^{-1} \right\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. By the same argument as in the proof in Theorem 3.1, the solution $z = 0$ of (2.4) is hS. Using the nonlinear

variation of constants formula Lemma 2.4, (H2), together with (3.5) and (3.6), we have

$$|y(t)| \leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^t c_2h(t)h(s)^{-1} \left(a(s)|y(s)| + b(s)w(|y(s)|) \right. \\ \left. + c(s) \int_{t_0}^s k(\tau)w(|y(\tau)|)d\tau \right) ds.$$

Using (H1) and (H2), we obtain

$$|y(t)| \leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^t c_2h(t) \left(a(s)\frac{|y(s)|}{h(s)} + b(s)w\left(\frac{|y(s)|}{h(s)}\right) \right. \\ \left. + c(s) \int_{t_0}^s k(\tau)w\left(\frac{|y(\tau)|}{h(\tau)}\right)d\tau \right) ds.$$

Letting $u(t) = |y(t)||h(t)|^{-1}$, then, by Corollary 2.11, we have

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t \left(a(s) + b(s) + c(s) \int_{t_0}^s k(\tau)d\tau \right) ds \right],$$

where $c = c_1|y_0|h(t)h(t_0)^{-1}$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (2.2) is bounded on $[t_0, \infty)$, and so the proof is complete. \square

REMARK 3.8. Letting $b(t) = 0$ in Theorem 3.7, we obtain the similar result as that of Theorem 3.1 in [3].

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