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STOCHASTIC DIFFERENTIAL EQUATION FOR WHITE NOISE FUNCTIONALS

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Abstract. Within white noise approach, we study the existence and uniqueness of the solution of an initial value problem for generalized white noise functionals, and then as a corollary we discuss the linear stochastic differential equation associated with a convolution of white noise functionals.

1. Introduction

The stochastic calculus has been developing extensively with applications to a wide range of research areas with random phenomena. The randomness is represented by deterministic terms in the white noise theory initiated by Hida [4] and so stochastic analysis is translated into an infinite dimensional calculus. The white noise approach to stochastic calculus has been studied by many authors, see e.g., [5, 6, 15], and references cited therein.

On the other hand, convolution products play important roles in many areas, infinite dimensional (harmonic) analysis, signal analysis and quantum mechanics, etc, and so, in general, there are several kinds of convolution products. In the white noise theory, convolution products have been studied by several authors, e.g., [15, 19, 9, 8, 9, 10], and references cited therein. Recently, in [11], the author introduced a new type of convolution product to give a unifying definition of well-known convolutions in the white noise theory.

Motivated by the study in [1], we are interested in the study of linear stochastic differential equations associated with the convolution introduced in [11]. We first study the existence and uniqueness of the solution

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of an initial value problem for generalized white noise functionals, and then as a corollary we examine the existence and uniqueness of the solution of the linear stochastic differential equation associated with the convolution.

This paper is organized as follows: In section 2, we briefly review the basic notions of white noise functionals (see [2, 12]), which are necessary for our study. In Section 3, we introduce the convolution, introduced in [11], of generalized white noise functionals. In Section 4, we study basic properties of generalized stochastic processes and examine the existence and uniqueness of the solution of an initial value problem for generalized white noise functionals.

2. White noise functionals

Let H be a (complex) separable Hilbert space with the norm $|\cdot|_0$ and let A be a positive self-adjoint operator on H such that $\|A^{-1}\|_{\mathrm{OP}} < 1$ and $\|A^{-1}\|_{\mathrm{HS}}^2 < \infty$. Then by the standard construction from H and A (see [15, 16]), we have a Gelfand triple $E \subset H \subset E^*$, where E^* is the strong dual space of the nuclear space E. The canonical bilinear form on $E^* \times E$ is denoted by $\langle \cdot, \cdot \rangle$. By the Bochner-Minlos theorem, there exists a Gaussian measure μ on $E^*_{\mathbb{R}}$ (subspace of E^* consisting of real elements) such that its characteristic function is given by

$$\int_{E^*_{\mathbb{R}}} \exp\{i\langle x, \xi\rangle\} d\mu(x) = \exp\left\{-\frac{1}{2}|\xi|_0^2\right\}, \qquad \xi \in E_{\mathbb{R}}.$$

Then $(E^*_{\mathbb{R}}, \mu)$ is called the *white noise space* or *Gaussian space*. We denote by $L^2(E^*_{\mathbb{R}}, \mu)$ the complex Hilbert space of all μ-square integrable functions on $E^*_{\mathbb{R}}$. Then the celebrated Wiener-Itô-Segal isomorphism gives the unitary isomorphism between $L^2(E^*_{\mathbb{R}}, \mu)$ and the Boson Fock space $\Gamma(H)$ which is uniquely determined by the correspondence:

$$\phi_\xi(x) = e^{\langle x,\xi\rangle - \frac{1}{2}\langle \xi,\xi\rangle} \quad \leftrightarrow \quad \phi_\xi = \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \cdots, \frac{\xi^{\otimes n}}{n!}, \cdots\right),$$

where ϕ_ξ is called an *exponential vector* (or *coherent state*) associated with $\xi \in H$. Here the (Boson) Fock space $\Gamma(H)$ over H is defined by

$$\Gamma(H) = \left\{\phi = (f_n)_{n=0}^\infty \,;\, f_n \in H^{\widehat{\otimes} n},\ \|\phi\|^2 = \sum_{n=0}^\infty n!\, |f_n|_0^2 < \infty\right\},$$

where $H^{\widehat{\otimes} n}$ is the n-fold symmetric tensor product of H and $H^{\widehat{\otimes} 0} = \mathbb{C}$.

Let $\alpha = \{\alpha(n)\}_{n=0}^{\infty}$ be a sequence of positive numbers satisfying the following five conditions:

(A1) $\alpha(0) = 1 \le \alpha(1) \le \alpha(2) \le \cdots$;

(A2) the generating function $G_\alpha(t) = \displaystyle\sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} t^n$ has an infinite radius of convergence;

(A3) the power series $\widetilde{G}_\alpha(t) = \displaystyle\sum_{n=0}^{\infty} \frac{n^{2n}}{n!\alpha(n)} \left\{ \inf_{s>0} \frac{G_\alpha(s)}{s^n} \right\} t^n$ has a positive radius of convergence;

(A4) there exists a constant $C_{1\alpha} > 0$ such that for any n, m,

$$\alpha(n)\alpha(m) \le C_{1\alpha}^{n+m} \alpha(n+m);$$

(A5) there exists a constant $C_{2\alpha} > 0$ such that for any n, m,

$$\alpha(n+m) \le C_{2\alpha}^{n+m} \alpha(n)\alpha(m).$$

There are some typical examples of such sequences, e.g., $\alpha(n) \equiv 1$, $\alpha(n) = (n!)^\beta$ $(0 \le \beta < 1)$ and an interesting example of the weighted sequence α is given by the k-th order Bell numbers $\{B_k(n)\}$ defined by

$$G_{\mathrm{Bell}(k)}(t) = \frac{\overbrace{\exp(\exp(\cdots(\exp}^{k-\text{times}} t)\cdots))}{\exp(\exp(\cdots(\exp 0)\cdots))} = \sum_{n=0}^{\infty} \frac{B_k(n)}{n!} t^n.$$

We review some properties of the generating function G_α whose proofs are straightforward.

LEMMA 2.1. *Let $\alpha = \{\alpha(n)\}$ be a positive sequence satisfying (A1) and (A2), and $G_\alpha(t)$ the generating function defined therein. Then,*

1. *$G_\alpha(0) = 1$ and $G_\alpha(s) \le G_\alpha(t)$ for $0 \le s \le t$;*
2. *$e^s G_\alpha(t) \le G_\alpha(s+t)$ and $e^t \le G_\alpha(t)$ for $s, t \ge 0$;*
3. *$c[G_\alpha(t) - 1] \le G_\alpha(ct) - 1$ for any $c \ge 1$ and $t \ge 0$.*

LEMMA 2.2. *Let $\alpha = \{\alpha(n)\}$ be a positive sequence and $G_\alpha(t)$ the generating function defined therein. If α satisfies conditions (A1), (A2) and (A4), then*

$$G_\alpha(s)G_\alpha(t) \le G_\alpha(C_{1\alpha}(s+t)), \qquad s, t \ge 0.$$

If conditions (A1), (A2) and (A5) are fulfilled, then

$$G_\alpha(s+t) \le G_\alpha(C_{2\alpha}s)G_\alpha(C_{2\alpha}t), \qquad s, t \ge 0.$$

Given such a positive sequence α, we define a weighted Fock space:

$$\Gamma_\alpha(E_p) = \left\{ \phi = (f_n)_{n=0}^\infty \,;\, f_n \in E_p^{\widehat{\otimes} n},\ \|\phi\|_p^2 \equiv \sum_{n=0}^\infty n!\,\alpha(n)\,|f_n|_p^2 < \infty \right\},$$

where $E_p = \left\{ \xi \in H \,;\, |\xi|_p \equiv |A^p \xi|_0 < \infty \right\}$. We then define

$$\mathcal{W}_\alpha = \operatorname*{proj\,lim}_{p\to\infty} \Gamma_\alpha(E_p).$$

It is easily proved that $\mathcal{W}_\alpha$ is a nuclear space whose topology is given by the family of norms:

$$\|\phi\|_{p,+}^2 = \sum_{n=0}^\infty n!\,\alpha(n)\,|f_n|_p^2, \qquad \phi = (f_n), \quad p \ge 0.$$

Then we obtain a Gelfand triple:

$$\mathcal{W}_\alpha \subset \Gamma(H_{\mathbb{C}}) \cong L^2(E^*, \mu) \subset \mathcal{W}_\alpha^* \tag{2.1}$$

which is called the Cochran-Kuo-Sengupta space [2] associated with α. If there is no danger of confusion, we simply set $\mathcal{W} = \mathcal{W}_\alpha$. The canonical bilinear form on $\mathcal{W}^* \times \mathcal{W}$ is denoted by $\langle\langle \cdot, \cdot \rangle\rangle$. Then

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^\infty n!\, \langle F_n, f_n \rangle, \quad \Phi = (F_n) \in \mathcal{W}^*, \quad \phi = (f_n) \in \mathcal{W},$$

and it holds that

$$|\langle\langle \Phi, \phi \rangle\rangle| \le \|\Phi\|_{-p,-}\, \|\phi\|_{p,+}, \quad \|\Phi\|_{-p,-}^2 = \sum_{n=0}^\infty \frac{n!}{\alpha(n)}\, |F_n|_{-p}^2 .$$

Example 2.3. The Gelfand triples given as in (2.1) with $\alpha(n) \equiv 1$ and $\alpha(n) = (n!)^\beta$ $(0 \le \beta < 1)$ are called the *Hida–Kubo–Takenaka space* [14] and *Kondratiev–Streit space* [13], respectively.

Proposition 2.4. *Let α, β be sequences of positive numbers. If $\alpha \prec \beta$ (if and only if $\alpha(n) \le \beta(n)$ for all $n \in \mathbb{N}$), then it holds that*

$$\mathcal{W}_\beta \subset \mathcal{W}_\alpha \subset \Gamma(H) \subset \mathcal{W}_\alpha^* \subset \mathcal{W}_\beta^*, \tag{2.2}$$

where all inclusions are continuous.

Proof. The proof is straightforward. □

From the fact that $\{\phi_\xi \,;\, \xi \in E\}$ spans a dense subspace of $\mathcal{W}$, an element $\Phi \in \mathcal{W}^*$ is uniquely determined by the *S-transform* $S\Phi$ of Φ which is a function on E defined by

$$S\Phi(\xi) = \langle\langle \Phi, \phi_\xi \rangle\rangle, \qquad \xi \in E.$$

THEOREM 2.5 ([2]). *A function $F : E \to \mathbb{C}$ is the S-transform of a white noise functional $\Phi \in \mathcal{W}^*$, i.e., $F = S\Phi$, if and only if the following two conditions are satisfied:*

(S1) *for any $\xi, \eta \in E$, the function $z \mapsto F(z\xi + \eta)$ is entire holomorphic on $\mathbb{C}$;*

(S2) *there exist constant numbers $C \geq 0$ and $p \geq 0$ such that*

$$|F(\xi)|^2 \leq C G_\alpha(|\xi|_p^2) \qquad \xi \in E.$$

3. Convolutions of generalized white noise functionals

For each given $\Phi, \Psi \in \mathcal{W}^*$, by applying Theorem 2.5, we prove that there exists a unique white noise functional in $\mathcal{W}^*$, denoted by $\Phi \diamond \Psi$ and called the *Wick product*, such that

$$S(\Phi \diamond \Psi)(\xi) = S(\Phi)(\xi) S(\Psi)(\xi), \quad \xi \in E.$$

Let $U, V \in \mathcal{L}(E^*, E^*)$. For each given $\mathbf{F} \in \mathcal{W}^*$, we define a convolution $*_{U,V;\mathbf{F}}$ (see [11]) of generalized white noise functionals by

$$\Phi *_{U,V;\mathbf{F}} \Psi = \Gamma(U)\Phi \diamond \Gamma(V)\Psi \diamond \mathbf{F}, \qquad \Phi, \Psi \in \mathcal{W}^*,$$

where $\Gamma(U) \in \mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$ (the space of all continuous linear operators from $\mathcal{W}^*$ into itself) is the second quantization of U defined by

$$\Gamma(U)\Phi = (U^{\otimes n} F_n), \qquad \Phi = (F_n) \in \mathcal{W}^*.$$

EXAMPLE 3.1 ([11]). (1) For any $\Phi, \Psi \in \mathcal{W}^*$ and the vacuum vector ϕ_0, we have

$$\Phi *_{U,V;\phi_0} \Psi = \Gamma(U)\Phi \diamond \Gamma(V)\Psi \diamond \phi_0 = \Gamma(U)\Phi \diamond \Gamma(V)\Psi = \Phi *^l_{U^*,V^*} \Psi,$$

where U^* is the adjoint operator of the given linear operator U with respect to the canonical bilinear form $\langle \cdot, \cdot \rangle$. The convolution $*^l_{U^*,V^*}$ has been studied in [10].

(2) For any $\Phi, \Psi \in \mathcal{W}^*$, we have

$$\Phi *_{U,V;\mathbf{F}} \Psi = \Gamma(U)\Phi \diamond \Gamma(V)\Psi \diamond \mathbf{F} = \Gamma(U)\Phi *_{\mathbf{F}} \Gamma(V)\Psi.$$

The convolution $*_{\mathbf{F}}$ has been studied in [8].

(3) For any $\Phi, \Psi \in \mathcal{W}^*$, we have

$$\Phi *_{\frac{1}{\sqrt{2}}I, -\frac{1}{\sqrt{2}}I;\phi_0} \Psi = \Gamma\left(\frac{1}{\sqrt{2}}I\right)\Phi \diamond \Gamma\left(-\frac{1}{\sqrt{2}}I\right)\Psi.$$

The convolution $*_{\frac{1}{\sqrt{2}}I, -\frac{1}{\sqrt{2}}I;\phi_0}$ has been studied in [20] (see also [9, 10]) and it is called the *Yeh convolution*.

(4) For any $\Phi, \Psi \in \mathcal{W}^*$, we have

$$\Phi *_{I,I;\phi_0} \Psi = \Phi \diamond \Psi.$$

The convolution has been studied by Obata and Ouerdiane in [19].

LEMMA 3.2. *Let $F \in \mathcal{W}^*$ and $U, V \in \mathcal{L}(E^*, E^*)$ be given. If $U = V$, then it holds that*

$$\Phi *_{U,V;\mathbf{F}} \Psi = \Psi *_{U,V;\mathbf{F}} \Phi$$

for any $\Phi, \Psi \in \mathcal{W}^*$.

Proof. The proof is straightforward. □

LEMMA 3.3. *Let $F \in \mathcal{W}^*$ and $U, V \in \mathcal{L}(E^*, E^*)$ be given. Suppose that $UV = VU$, $U^2 = U$, $V^2 = V$ and $\Gamma(U)F = \Gamma(V)F$. Then it holds that*

$$(\Phi_1 *_{U,V;\mathbf{F}} \Phi_2) *_{U,V;\mathbf{F}} \Phi_3 = \Phi_1 *_{U,V;\mathbf{F}} (\Phi_2 *_{U,V;\mathbf{F}} \Phi_3)$$

for any $\Phi_1, \Phi_2, \Phi_3 \in \mathcal{W}^*$.

Proof. The proof is straightforward. □

In general, the convolution $*_{U,V;\mathbf{F}}$ is not commutative and not associative.

4. Stochastic differential equations

In general, a one-parameter family $\{\Phi_t\}_{t\in[0,T]} \subset \mathcal{W}^*$ of generalized white noise functionals is called a *generalized stochastic process*. In this paper, motivated by the study of [17] (see also [12]), a generalized stochastic process is always assumed to be continuous, i.e., the map $[0, T] \ni t \to \Phi_t \in \mathcal{W}^*$ is continuous.

LEMMA 4.1. *A function $t \mapsto \Phi_t \in \mathcal{W}^*$ defined on an interval is continuous if and only if for any t_0 in the interval, there exist $K \geq 0$, $p \geq 0$ and an open neighborhood U_0 of t_0 such that*

$$|S\Phi_t(\xi)|^2 \leq K G_\alpha(|\xi|_p^2), \qquad \xi \in E, \quad t \in U_0,$$

and

$$\lim_{t\to t_0} S\Phi_t(\xi) = S\Phi_{t_0}(\xi), \qquad \xi \in E.$$

Proof. The proof is a modification of the argument used in the proof of Theorem 1.8 in [18] (see also Lemma 5 in [12]). □

EXAMPLE 4.2. Let $H = L^2(\mathbb{R}, dt)$. For each $\zeta \in H$, there exists a sequence $\{\zeta_n\}_{n=1}^\infty \subset E$ such that $\{\zeta_n\}_{n=1}^\infty$ converges to ζ in H. Note that for each $n \in \mathbb{N}$, $X_{\zeta_n} = \langle \cdot, \zeta_n \rangle$ is a Gaussian random variable defined on $E_{\mathbb{R}}^*$ and $\{X_{\zeta_n}\}_{n=1}^\infty$ is a Cauchy sequence in $L^2(E_{\mathbb{R}}^*, \mu)$. We define the random variable $X_\zeta = \lim_{n\to\infty} X_{\zeta_n}$ in $L^2(E_{\mathbb{R}}^*, \mu)$. Then X_ζ is a Gaussian random variable with mean 0 and variance $|\zeta|_0^2$. For each $t \geq 0$, put $B_t = X_{\mathbf{1}_{[0,t]}}$. Then $\{B_t\}_{t\geq 0}$ is called a *realization of Brownian motion* and by applying Lemma 4.1, we can see that the map $[0, \infty) \ni t \to B_t \in \mathcal{W}_\alpha^*$ is continuous, where $\alpha \equiv 1$.

PROPOSITION 4.3. *Let $U \in \mathcal{L}(E^*, E^*)$ and $\{\Phi_t\}_{t\in[0,T]} \subset \mathcal{W}^*$ be a generalized stochastic process. Then $\{\Gamma(U)\Phi_t\}_{t\in[0,T]} \subset \mathcal{W}^*$ is a generalized stochastic process*

Proof. The proof is a simple application of Lemma 4.1. □

THEOREM 4.4. *Let $\{\Phi_{1,t}\}_{t\in[0,T]}, \{\Phi_{2,t}\}_{t\in[0,T]} \subset \mathcal{W}^*$ be two generalized stochastic processes. Then $\{\Phi_{1,t} \diamond \Phi_{2,t}\}_{t\in[0,T]} \subset \mathcal{W}^*$ is a generalized stochastic process.*

Proof. Let $t_0 \in [0, T]$. By assumption, there exist $K \geq 0$, $p \geq 0$ and an open neighborhood U_0 of t_0 such that

$$|S\Phi_{i,t}(\xi)|^2 \leq K G_\alpha(|\xi|_p^2), \qquad \xi \in E, \quad t \in U_0, \quad i = 1, 2.$$

Therefore, for any $\xi \in E$ and $t \in U_0$, by Lemma 2.2, we obtain that

$$\begin{aligned} |S(\Phi_{1,t} \diamond \Phi_{2,t})(\xi)| &\leq K^2 \left(G_\alpha(|\xi|_p^2)\right)^2 \leq K^2 G_\alpha(2C_{1\alpha}\rho^{2q}\, |\xi|_{p+q}^2) \\ &\leq K^2 G_\alpha(|\xi|_{p+q}^2) \end{aligned}$$

for any $q \geq 0$ with $2C_{1\alpha}\rho^{2q} \leq 1$. On the other hand, for any $\xi \in E$, we have

$$\lim_{t\to t_0} S(\Phi_{1,t} \diamond \Phi_{2,t})(\xi) = \lim_{t\to t_0} S\Phi_{1,t}(\xi) S\Phi_{2,t}(\xi) = S(\Phi_{1,t_0} \diamond \Phi_{2,t_0})(\xi).$$

Hence by Lemma 4.1, the map $t \to \Phi_{1,t} \diamond \Phi_{2,t} \in \mathcal{W}^*$ is continuous. □

COROLLARY 4.5. *Let $U, V \in \mathcal{L}(E^*, E^*)$ and $\mathbf{F} \in \mathcal{W}^*$. Let $\{\Phi_{1,t}\}_{t\in[0,T]}$ and $\{\Phi_{2,t}\}_{t\in[0,T]}$ be two generalized stochastic processes in $\mathcal{W}^*$. Then $\{\Phi_{1,t} *_{U,V;\mathbf{F}} \Phi_{2,t}\}_{t\in[0,T]} \subset \mathcal{W}^*$ is a generalized stochastic process.*

Proof. The proof is obvious from Proposition 4.3 and Theorem 4.4. □

For general theory, we now study the initial value problem:

$$\frac{d\Phi}{dt} = F(t, \Phi), \qquad \Phi|_{t=0} = \Phi_0, \quad 0 \le t \le T, \tag{4.1}$$

where $F : [0,T] \times \mathcal{W}^* \to \mathcal{W}^*$ is a continuous function and Φ_0 is a generalized white noise functional.

We need to consider two weight sequences $\alpha = \{\alpha(n)\}$ and $\omega = \{\omega(n)\}$ with conditions (A1)–(A5) and the generating functions related in such a way that

$$G_\alpha(t) = \exp \gamma\{G_\omega(t) - 1\}, \tag{4.2}$$

where $\gamma > 0$ is a certain constant. In this case, (2.2) holds.

THEOREM 4.6. *Let $\alpha = \{\alpha(n)\}$ and $\omega = \{\omega(n)\}$ be two weight sequences with conditions (A1)–(A5), and assume that their generating functions are related as in (4.2). Let $F : [0,T] \times \mathcal{W}_\alpha^* \to \mathcal{W}_\alpha^*$ be a continuous function satisfying the conditions:*

(i) *there exist $p \ge 0$ and a nonnegative function $K \in L^1[0,T]$ such that*

$$|SF(s,\Phi)(\xi) - SF(s,\Psi)(\xi)|^2 \le K(s) G_\omega(|\xi|_p^2) |S\Phi(\xi) - S\Psi(\xi)|^2,$$

for all $\xi \in E$, $\Phi, \Psi \in \mathcal{W}_\omega^*$, and $s \in [0,T]$;

(ii) *there exist $p \ge 0$ and a nonnegative function $K \in L^1[0,T]$ such that*

$$|SF(s,\Phi)(\xi)|^2 \le K(s) G_\omega(|\xi|_p^2)(1 + |S\Phi(\xi)|^2),$$

for all $\xi \in E$, $\Phi \in \mathcal{W}_\omega^*$, and $s \in [0,T]$.

Then, for any $\Phi_0 \in \mathcal{W}_\omega^*$, the initial value problem (4.1) has a unique solution $\Phi_t \in \mathcal{W}_\alpha^*$, $t \in [0,T]$.

Proof. In principle, the proof is based on the standard Picard–Lindelöf method of successive approximations (see e.g., [3]) applied to the S-transforms. We define

$$\Phi_t^{(0)} = \Phi_0,$$

$$\Phi_t^{(n)} = \Phi_0 + \int_0^t F\left(s, \Phi_s^{(n-1)}\right) ds, \qquad n \ge 1.$$

Then by applying Theorem 2.5 we can see that for each $n \in \mathbb{N}$, the map $t \to \Phi_t^{(n)} \in \mathcal{W}_\omega^*$ is continuous and $\{\Phi_t^{(n)}\}$ converges to a Φ_t in $\mathcal{W}_\alpha^*$, and then we can see that Φ_t is the unique solution of the initial value problem (4.1). The details of the above sketch are simple modifications of the proof of Theorem 9 in [12] (see also Theorem 13.43 in [15]). Therefore, we skip the detailed computations. □

Let $\{L_t\}, \{M_t\} \subset \mathcal{W}_\omega^*$ be two generalized stochastic processes, where t runs over $[0, T]$. Consider the initial value problem:

$$\frac{d}{dt}\,\Phi_t = L_t *_{U,V;\mathbf{F}} \Phi_t + M_t, \qquad \Phi|_{t=0} = \Phi_0 \in \mathcal{W}_\omega^*. \tag{4.3}$$

COROLLARY 4.7. *Let $\alpha = \{\alpha(n)\}$ and $\omega = \{\omega(n)\}$ be two weight sequences with conditions (A1)–(A5), and assume that their generating functions are related as in (4.2). For each $\Phi_0 \in \mathcal{W}_\omega^*$, the initial value problem* (4.3) *has a unique solution $\Phi_t \in \mathcal{W}_\alpha^*$, $t \in [0, T]$.*

Proof. The proof is a simple application of Theorem 4.6. □

A special type of the stochastic differential equation given as in (4.3) was studied in [1]. By taking $U = V = I$ and $F = \phi_0$ the vacuum vector, the convolution $*_{U,V;\mathbf{F}}$ coincides with the Wick product $\diamond$. Hence we have the following corollary.

COROLLARY 4.8. *Let $\alpha = \{\alpha(n)\}$ and $\omega = \{\omega(n)\}$ be two weight sequences with conditions (A1)–(A5), and assume that their generating functions are related as in (4.2). For each $\Phi_0 \in \mathcal{W}_\omega^*$, the initial value problem*

$$\frac{d}{dt}\,\Phi_t = L_t \diamond \Phi_t + M_t, \qquad \Phi|_{t=0} = \Phi_0 \in \mathcal{W}_\omega^*$$

has a unique solution $\Phi_t \in \mathcal{W}_\alpha^*$, $t \in [0, T]$.

A study of a quantum analogue of the differential equation given as in (4.3) is now in progress and will be reported in a separate paper.

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