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MULTI-DEGREE REDUCTION OF BÉZIER CURVES WITH CONSTRAINTS OF ENDPOINTS USING LAGRANGE MULTIPLIERS

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Abstract. In this paper, we consider multi-degree reduction of Bézier curves with continuity of any (r, s) order with respect to L_2 norm. With help of matrix theory about generalized inverses we can use Lagrange multipliers to obtain the degree reduction matrix in a very simple form as well as the degree reduced control points. Also error analysis comparing with the least squares degree reduction without constraints is given. The advantage of our method is that the relationship between the optimal multi-degree reductions with and without constraints of continuity can be derived explicitly.

1. Introduction

Given control points $\mathbf{p} = (p_0, p_1, \ldots, p_n)^t$, a degree n Bézier curve is defined by

$$p(t) = \sum_{i=0}^{n} p_i B_i^n(t), \ t \in [0, 1] \tag{1.1}$$

where $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$ is the Bernstein polynomial of degree n. The problem of degree reduction with respect to L_2 norm is to find control points $\mathbf{q} = (q_0, q_1, \ldots, q_m)^t$ which define the approximate Bézier curve

$$q(t) = \sum_{i=0}^{m} q_i B_i^m(t), \ t \in [0, 1] \tag{1.2}$$

of degree m $(m < n)$ such that the L_2 norm

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$$d_2(p,q) = \sqrt{\int_0^1 |p(t) - q(t)|^2 dt} \tag{1.3}$$

is minimized. Degree reduction of parametric curves was first proposed as an inverse problem of degree elevation, see [4], [6]. The least squares degree reduction with endpoints constraints can be found in [3].

Many authors have contributed to solve the degree reduction problem in a vector-matrix form (see, for example, [8], [11], [12], and [14]). The vector-matrix form of the least squares multi-degree reduced control points without constrains can be found in [7] and [9]. They showed that the least squares control points can be represented as

$$\mathbf{q} = (T^t T)^{-1} T^t \mathbf{p} \tag{1.4}$$

where T is the degree elevation matrix.

In [15] the multi-degree reduced control points with fixed endpoints is obtained using Lagrange multipliers. They showed that the multi-degree reduced control points with fixed endpoints can be represented in terms of the least squares degree reduced control points without constraints of endpoints continuity and the original control points. In this paper, we extend the works of [15] to the problem of multi-degree reduction of Bézier curves with continuity of any (r, s) order at the endpoints. Constructing the problem in a vector-matrix form is very similar to the works of [15]. With the help of matrix algebra about generalized inverse and the Moore-Penrose inverse of a partitioned matrix, we find the optimal multi-degree reduction matrix $R^{(r,s)}_{m\times n}$ with arbitrary order continuity with respect to L_2 norm, consequently, we can easily obtain the optimal multi-degree reduced control points $\mathbf{q}^{(r,s)}$, that is, $\mathbf{q}^{(r,s)} = R^{(r,s)}_{m\times n}\mathbf{p}$. From these results we can derive that the optimal multi-degree reduced control points can be represented in terms of the least squares degree reduced control points without constraints and the original control points. Also we derive the error of degree reduction comparing with the least squares degree reduction without constraints in a simple form.

In section 2 we introduce some important matrices used in this paper such as a degree elevation matrix and Legendre-Bernstein basis transformation matrix. Some properties of generalized inverse and the Moore-Penrose inverse of partitioned matrices are given in section 3. Matrix representation of constraints is given in section 4. In section 5, we present the optimal multi-degree reduction with (r, s) order continuity using Lagrange multipliers. Finally, an error analysis is given in section 6.

2. Preliminaries

The problem of degree reduction is an inverse process of degree elevation. For a given Bézier curve $p(t)$ with control points $\mathbf{p} = (p_0, p_1, \ldots, p_n)^t$, we have to find a Bézier curve $q(t)$ with control points $\mathbf{q} = (q_0, q_1, \ldots, q_m)^t$. The first step of degree reduction without constraints of endpoints continuity is to find control points $\mathbf{q}^{(n)}$ such that

$$\mathbf{q}^{(n)} = T_{n\times m}\mathbf{q} \tag{2.1}$$

where $T_{n\times m}$ is a degree elevation matrix whose elements are given by, see [7],

$$T_{n\times m}(i,j) = \frac{\binom{m}{j}\binom{n-m}{i-j}}{\binom{n}{i}}, \quad i = 0, 1, \ldots, n \text{ and } j = 0, 1, \ldots, m. \tag{2.2}$$

The L_2 norm of the Bézier curve $p(t)$ with control points $\mathbf{p}$ can be expressed as a matrix notation as follows, see [7],

$$||p||_2^2 = \int_0^1 \left| \sum_{i=0}^{n} p_i B_i^n(t) \right|^2 dt = \mathbf{p}^t Q_n \mathbf{p} \tag{2.3}$$

where Q_n is defined by

$$Q_n(i,j) = \frac{1}{2n+1} \frac{\binom{n}{i}\binom{n}{j}}{\binom{2n}{i+j}} \quad i, j = 0, 1, \ldots, n. \tag{2.4}$$

In [5] we can find useful results about Legendre-Bernstein basis transformations. Consider a polynomial $P_n(t)$ of degree n, expressed in the degree n Bernstein and Legendre basis on $t \in [0, 1]$:

$$P_n(t) = \sum_{j=0}^{n} c_j B_j^n(t) = \sum_{k=0}^{n} d_k L_k(t). \tag{2.5}$$

We may express this in a vector-matrix form as

$$\mathbf{c} = M_n \mathbf{d} \tag{2.6}$$

where $\mathbf{c} = (c_0, c_1, \ldots, c_n)^t$ and $\mathbf{d} = (d_0, d_1, \ldots, d_n)^t$. Then the basis transformation matrix M_n and its inverse M_n^{-1} are given by

$$(2.7)\qquad M_n(j,k) = \frac{\sqrt{2k+1}}{\binom{n}{j}} \sum_{i=\max(0,j+k-n)}^{\min(j,k)} (-1)^{k+i} \binom{k}{i}\binom{k}{i}\binom{n-k}{j-i}$$

and

(2.8)

$$M_n^{-1}(j,k) = \frac{\sqrt{2j+1}}{n+j+1} \frac{1}{\binom{n+j}{n}} \sum_{i=0}^{j} (-1)^{j+i} \binom{j}{i}\binom{k+i}{k}\binom{n-k+j-i}{n-k}.$$

The relationship between the degree elevation matrix $T_{n\times m}$ and the Legendre-Bernstein basis transformation matrices can be found in [8].

LEMMA 2.1. *The followings are true.*

(a) $T^t_{n\times m} Q_n T_{n\times m} = Q_m$
(b) $Q_m^{-1} = M_m M_m^t$
(c) $Q_m^{-1} T^t_{n\times m} Q_n = M_m I_{m\times n} M_n^{-1}$
(d) $T_{n\times m} = M_n I_{n\times m} M_m^{-1}$

where $I_{m\times n}$ *is an* $(m+1)\times(n+1)$ *matrix whose elements are defined by*

$$(2.9)\qquad I_{m\times n}(i,j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

3. Some properties of generalized inverse

In order to solve the degree reduction using Lagrange multipliers in vector-matrix form, we need to introduce a concept of a generalized inverse as well as an inverse of a partitioned matrix.

An $m\times n$ matrix G is called a generalized inverse of an $n\times m$ matrix A if $AGA = A$ and is denoted by $G = A^-$. Clearly, if A^{-1} exists, then $A^- = A^{-1}$ is the unique generalized inverse of A. But if not, there exist infinitely many generalized inverses. If we impose some conditions, we can get a unique generalized inverse, namely the Moore-Penrose inverse. The definition of the Moore-Penrose inverse is as follows:

DEFINITION 3.1. For any matrix A, there exists a unique matrix G satisfying the following four conditions:

(i) $AGA = A$, (ii) $GAG = G$,
(iii) AG is symmetric, (iv) GA is symmetric,

and is denoted by $G = A^\dagger$.

Useful properties of generalized inverse and the Moore-Penrose inverse can be found in [1], [10], and [13]. As an application of a generalized inverse, we can obtain general solutions to a consistent linear system $A\mathbf{x} = \mathbf{b}$.

LEMMA 3.2. *For a given linear system* $A\mathbf{x} = \mathbf{b}$, *if the system is consistent, then* $\mathbf{x} = A^{-}\mathbf{b}$ *is a solution. Furthermore general solutions are given by*

$$\mathbf{x} = A^{-}\mathbf{b} + (I - A^{-}A)\mathbf{z} \quad \text{for any vector } \mathbf{z}$$

where I *is an identity matrix.*

We introduce some results about generalized inverses of partitioned matrices of the form $[U \;\; V]$ and $\begin{pmatrix} S & L^t \\ L & 0 \end{pmatrix}$.

LEMMA 3.3. ([10]) *Let* $A = [U \;\; V]$. *If* $VK^{\dagger}V = V$, *then the Moore-Penrose inverse of* A *is given by*

$$A^{\dagger} = \begin{pmatrix} U^{\dagger} - U^{\dagger}VK^{\dagger} \\ K^{\dagger} \end{pmatrix}$$

where

$$K = (I - UU^{\dagger})V.$$

LEMMA 3.4. ([10]) *Let* S *be a* $k \times k$ *positive semidefinite matrix and* L *be any* $q \times k$ *matrix. If the row space of* L *is contained in the row space of* S, *then*

$$\begin{pmatrix} S & L^t \\ L & 0 \end{pmatrix}^{-} = \begin{pmatrix} S^{-} - S^{-}L^tW^{-}LS^{-} & S^{-}L^tW^{-} \\ W^{-}LS^{-} & -W^{-} \end{pmatrix}$$

where $W = LS^{-}L^t$.

4. Matrix representation of constraints

Given a degree n Bézier curve $p(t)$, an optimal multi-degree reduction with endpoints continuity of (r, s) order with respect to L_2 norm is to find a degree m $(m < n-1)$ Bézier curve $q(t)$ such that L_2 norm $d_2(p, q)$ is minimized where

$$\frac{d^i q(0)}{dt^i} = \frac{d^i p(0)}{dt^i}, \; i = 0, 1, \ldots, r; \tag{4.1}$$

and

$$\frac{d^j q(1)}{dt^j} = \frac{d^j p(1)}{dt^j}, \; j = 0, 1, \ldots, s. \tag{4.2}$$

Assume that m is sufficiently large so that $r + s < m < n - 1$ and $0 \leq r, s \leq n-m$ as in the works [2]. The matrix forms of the constraints (4.1) and (4.2) can be found in [14] as follows:

$$\begin{pmatrix} b_{0,0}^{(m,n)} & & & \\ b_{0,1}^{(m,n)} & b_{1,1}^{(m,n)} & & \\ \vdots & & \ddots & \\ b_{0,r}^{(m,n)} & b_{1,r}^{(m,n)} & \cdots & b_{r,r}^{(m,n)} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_r \end{pmatrix} = \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{pmatrix} \tag{4.3}$$

and

$$\begin{pmatrix} b_{m-s,n-s}^{(m,n)} & \cdots & b_{m-1,n-s}^{(m,n)} & b_{m,n-s}^{(m,n)} \\ & \ddots & & \vdots \\ & & b_{m-1,n-1}^{(m,n)} & b_{m,n-1}^{(m,n)} \\ & & & b_{m,n}^{(m,n)} \end{pmatrix} \begin{pmatrix} q_{m-s} \\ \vdots \\ q_{m-1} \\ q_m \end{pmatrix} = \begin{pmatrix} p_{n-s} \\ \vdots \\ p_{n-1} \\ p_n \end{pmatrix} \tag{4.4}$$

where

$$b_{i,j}^{(m,n)} = \frac{\binom{m}{i}\binom{n-m}{j-i}}{\binom{n}{j}}. \tag{4.5}$$

Let us define a $(k+1) \times (k+1)$ matrix $I_k^{(r,s)}$ as follows

$$I_k^{(r,s)} = \begin{pmatrix} I_{r+1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{s+1} \end{pmatrix}$$

so that

$$I_m^{(r,s)}\mathbf{q} = (q_0, ..., q_r, 0, \ldots, 0, q_{m-s}, ..., q_m)^t$$

and

$$I_n^{(r,s)}\mathbf{p} = (p_0, ..., p_r, 0, \ldots, 0, p_{n-s}, ..., p_n)^t.$$

Now we can combine two equations (4.3) and (4.4) as a single equation in a vector-matrix form

$$C\mathbf{q} = I_n^{(r,s)}\mathbf{p} \tag{4.6}$$

where

$$C = \begin{pmatrix} L & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & U \end{pmatrix} \tag{4.7}$$

where L and U are the coefficient matrices in Eqs. (4.3) and (4.4), respectively. Note that two matrices L and U are both triangular matrices, hence they are invertible. It is easy to see that the Moore-Penrose inverse of C as follows:

$$C^{\dagger} = \begin{pmatrix} L^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & U^{-1} \end{pmatrix} \tag{4.8}$$

The explicit forms of L^{-1} and U^{-1} can be found in [14],

$$L^{-1}_{jk} = \frac{\binom{n}{k}}{\binom{m}{j}} a_{j-k}, \qquad j = 0, 1, \ldots, r; k = 0, 1, \ldots, j, \tag{4.9}$$

and

$$U^{-1}_{jk} = \frac{\binom{n}{k}}{\binom{m}{j}} a_{k-j}, \qquad j = 0, 1, \ldots, s; k = j, j+1, \ldots, s \tag{4.10}$$

where $\{a_l\}$ is a sequence of constants defined by $a_0 = 1$ and

$$a_l = -\sum_{i=0}^{l-1} \binom{n-m}{l-i} a_i, \; l = 1, 2, \ldots. \tag{4.11}$$

Lemma 4.1. $C^{\dagger} T_{n\times m} = I_m^{(r,s)}$.

Proof. Comparing explicit elements of two matrices C and $T_{n\times m}$ we can see that the first $(r+1)$ rows and the last $(s+1)$ rows of C and $T_{n\times m}$ are the same. Hence we have the results. □

5. Degree reduction using Lagrange multipliers

The problem of degree reduction with endpoints continuity of higher order with respect to L_2 norm can be restated in a vector-matrix form similar to the statement in [15]. The object functions to be minimized are the same, but the constraints used in this problem are given Eqs. (4.1) and (4.2), or equivalently, in Eq. (4.6). Multiplying $C^{\dagger}$ to both sides of Eq. (4.6) we have

$$I_m^{(r,s)} \mathbf{q} = C^{\dagger} \mathbf{p}. \tag{5.1}$$

For the simplicity of computation we use constraints given in Eq. (5.1) instead of Eq. (4.6). Hence the problem of multi-degree reduction with continuity of (r, s) order with respect to L_2 norm can be re-stated as follows:

Find the control points $\mathbf{q}$ such that

$$\textbf{Minimize } d_2^2(p,q) = (\mathbf{p} - T_{n\times m}\mathbf{q})^t Q_n (\mathbf{p} - T_{n\times m}\mathbf{q})$$

$$\textbf{Subject to } I_m^{(r,s)}\mathbf{q} = C^\dagger \mathbf{p}.$$

Since

$$d_2^2(p,q) = \mathbf{p}^t Q_n \mathbf{p} - 2\mathbf{q}^t T_{n\times m}^t Q_n \mathbf{p} + \mathbf{q}^t T_{n\times m}^t Q_n T_{n\times m}\mathbf{q}, \tag{5.2}$$

by differentiating $d_2^2(p,q)$ and $I_m^{(r,s)}\mathbf{q}$ by $\mathbf{q}^t$, respectively, we have

$$\frac{\partial}{\partial \mathbf{q}^t}(d_2^2(p,q)) = -2T_{n\times m}^t Q_n \mathbf{p} + 2T_{n\times m}^t Q_n T_{n\times m}\mathbf{q}$$
$$\frac{\partial}{\partial \mathbf{q}^t}(I_m^{(r,s)}\mathbf{q}) = I_m^{(r,s)}.$$

Hence an introduction of a vector of Lagrange multipliers λ leads to equations

$$\begin{aligned} T_{n\times m}^t Q_n T_{n\times m}\mathbf{q} + I_m^{(r,s)}\lambda &= T_{n\times m}^t Q_n \mathbf{p} \\ I_m^{(r,s)}\mathbf{q} &= C^\dagger \mathbf{p}. \end{aligned}$$

Since $T_{n\times m}^t Q_n T_{n\times m} = Q_m$ by Lemma 2.1, we have equations in a vector-matrix form as

$$\begin{pmatrix} Q_m & I_m^{(r,s)} \\ I_m^{(r,s)} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \lambda \end{pmatrix} = \begin{pmatrix} T_{n\times m}^t Q_n \mathbf{p} \\ C^\dagger \mathbf{p} \end{pmatrix}. \tag{5.3}$$

To solve this linear system we have investigate some properties of a matrix $W = I_m^{(r,s)} Q_m^{-1} I_m^{(r,s)}$. These can be found in the following lemma.

Lemma 5.1. *Let* $W = I_m^{(r,s)} Q_m^{-1} I_m^{(r,s)}$. *Then the followings are true.*

(a) $I_m^{(r,s)} W^\dagger = W^\dagger I_m^{(r,s)} = W^\dagger$

(b) $I_m^{(r,s)} Q_m^{-1} W^\dagger = W^\dagger Q_m^{-1} I_m^{(r,s)} = I_m^{(r,s)}$

Proof. The proof of (a) is very easy and is omitted. Since

$$I_m^{(r,s)} Q_m^{-1} W^\dagger = I_m^{(r,s)} Q_m^{-1} I_m^{(r,s)} W^\dagger = WW^\dagger = I_m^{(r,s)},$$

we have $I_m^{(r,s)} Q_m^{-1} W^\dagger = I_m^{(r,s)}$. Also three matrices $I_m^{(r,s)}$, Q_m^{-1}, and $W^\dagger$ are symmetric, we have $W^\dagger Q_m^{-1} I_m^{(r,s)} = I_m^{(r,s)}$. □

Now we can solve the problem of Lagrange multipliers using Lemma 5.1. Note that Q_m is positive definite, therefore the inverse of Q_m exists and we have following result.

THEOREM 5.2. *Let A be a matrix defined by*

$$A = \begin{pmatrix} Q_m & I_m^{(r,s)} \\ I_m^{(r,s)} & 0 \end{pmatrix}.$$

Then we have

$$A^\dagger = \begin{pmatrix} Q_m^{-1} - Q_m^{-1}W^\dagger Q_m^{-1} & Q_m^{-1}W^\dagger \\ W^\dagger Q_m^{-1} & -W^\dagger \end{pmatrix}$$

where $W = I_m^{(r,s)} Q_m^{-1} I_m^{(r,s)}$.

Proof. Let

$$G = \begin{pmatrix} Q_m^{-1} - Q_m^{-1}W^\dagger Q_m^{-1} & Q_m^{-1}W^\dagger \\ W^\dagger Q_m^{-1} & -W^\dagger \end{pmatrix}.$$

Then using the facts in lemma 5.1, we have

$$\begin{aligned} AG &= \begin{pmatrix} Q_m & I_m^{(r,s)} \\ I_m^{(r,s)} & 0 \end{pmatrix} \begin{pmatrix} Q_m^{-1} - Q_m^{-1}W^\dagger Q_m^{-1} & Q_m^{-1}W^\dagger \\ W^\dagger Q_m^{-1} & -W^\dagger \end{pmatrix} \\ &= \begin{pmatrix} I_{m+1} - W^\dagger Q_m^{-1} + I_m^{(r,s)} W^\dagger Q_m^{-1} & W^\dagger - W^\dagger \\ I_m^{(r,s)} Q_m^{-1} - I_m^{(r,s)} Q_m^{-1} W^\dagger Q_m^{-1} & I_m^{(r,s)} Q_m^{-1} W^\dagger \end{pmatrix} \\ &= \begin{pmatrix} I_{m+1} & 0 \\ 0 & I_m^{(r,s)} \end{pmatrix}. \end{aligned}$$

Similarly we have $GA = \begin{pmatrix} I_{m+1} & 0 \\ 0 & I_m^{(r,s)} \end{pmatrix}$. Hence AG and GA are symmetric and

$$AGA = A, \quad GAG = G.$$

Therefore G is the Moore-Penrose inverse of A. □

By Lemma 3.2, the general solutions to Eq. (5.3) is given by

$$\begin{pmatrix} \mathbf{q} \\ \lambda \end{pmatrix} = A^\dagger \begin{pmatrix} T_{n\times m}^t Q_n \mathbf{p} \\ C^\dagger \mathbf{p} \end{pmatrix} + (I - A^\dagger A) \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} \tag{5.4}$$

for arbitrary vectors $\mathbf{z}_1$ and $\mathbf{z}_2$. Since $A^\dagger A = \begin{pmatrix} I_{m+1} & 0 \\ 0 & I_m^{(r,s)} \end{pmatrix}$, we have

$$\mathbf{q} = Q_m^{-1} T_{n\times m}^t Q_n \mathbf{p} - Q_m^{-1} W^\dagger (Q_m^{-1} T_{n\times m}^t Q_n - C^\dagger)\mathbf{p}, \tag{5.5}$$

which does not involve $\mathbf{z}_i$'s and is therefore unique. Hence the optimal degree reduction matrix $R_{m\times n}^{(r,s)}$ with endpoints continuity of (r, s) order is given by

$$R^{(r,s)}_{m\times n} = Q_m^{-1}T^t_{n\times m}Q_n - Q_m^{-1}W^\dagger(Q_m^{-1}T^t_{n\times m}Q_n - C^\dagger), \tag{5.6}$$

which is independent from the original control points $\mathbf{p}$.

THEOREM 5.3. *The optimal multi-degree reduction matrix $R^{(r,s)}_{m\times n}$ of Bézier curves with endpoints continuity of (r, s) order with respect to L_2 norm is given by*

$$R^{(r,s)}_{m\times n} = Q_m^{-1}T^t_{n\times m}Q_n - Q_m^{-1}W^\dagger(Q_m^{-1}T^t_{n\times m}Q_n - C^\dagger),$$

where $W = I_m^{(r,s)}Q_m^{-1}I_m^{(r,s)}$.

Let $R_{m\times n} = Q_m^{-1}T^t_{m,r}Q_n$. Then we can see that the matrix $R_{m\times n}$ is the least squares degree reduction matrix without constraints, see [8], and also $R_{m\times n}$ can be represented in several expressions

$$R_{m\times n} = Q_m^{-1}T^t_{m,r}Q_n = (T^t_{m,r}Q_nT_{m,r})^{-1}T^t_{m,r}Q_n = (T^t_{m,r}T_{m,r})^{-1}T^t_{m,r}.$$

Note that the matrix $R_{m\times n}$ is the Moore-Penrose inverse of the degree elevation matrix $T_{n\times m}$, see [7]. We can show that the matrix $R^{(r,s)}_{m\times n}$ in Theorem 5.3 is also a generalized inverse of $T_{n\times m}$.

THEOREM 5.4. *The optimal multi-degree reduction matrix $R^{(r,s)}_{m\times n}$ is a generalized inverse of $T_{n\times m}$.*

Proof. By part (a) of Lemma 2.1, $T^t_{n\times m}Q_nT_{n\times m} = Q_m$, hence

$$Q_m^{-1}T^t_{n\times m}Q_nT_{n\times m} = Q_m^{-1}Q_m = I_{m+1}.$$

Also by Lemma 4.1 we have

$$I_m^{(r,s)}(Q_m^{-1}T^t_{n\times m}Q_n - C^\dagger)T_{n\times m} = (I_m^{(r,s)} - I_m^{(r,s)}) = 0.$$

Using the fact $W^\dagger = W^\dagger I_m^{(r,s)}$ in Lemma 5.1 we have $R^{(r,s)}_{m\times n}T_{n\times m} = I_{m+1}$. Therefore $T_{n\times m}R^{(r,s)}_{m\times n}T_{n\times m} = T_{n\times m}$ holds. □

We have the optimal multi-degree reduction matrix $R^{(r,s)}_{m\times n}$, however the computation seems to be very complicated. So using the results about Lengendre-Bernstein basis transformations, we simplify this result.

Note that by Lemma 2.1, $Q_m^{-1} = M_mM_m^t$. Hence $R^{(r,s)}_{m\times n}$ becomes

$$R^{(r,s)}_{m\times n} = R_{m\times n} - M_mM_m^tW^\dagger[R_{m\times n} - C^\dagger]. \tag{5.7}$$

If we let

$$M_{rs} = I_m^{(r,s)}M_m, \tag{5.8}$$

then

$$W = I_m^{(r,s)} Q_m^{-1} I_m^{(r,s)} = I_m^{(r,s)} M_m M_m^t I_m^{(r,s)} = M_{rs} M_{rs}^t.$$

Therefore we have

$$\begin{aligned} R_{m\times n}^{(r,s)} &= R_{m\times n} - M_m M_{rs}^t (M_{rs} M_{rs}^t)^\dagger [R_{m\times n} - C^\dagger] \\ &= R_{m\times n} - M_m M_{rs}^\dagger [R_{m\times n} - C^\dagger] \end{aligned}$$

because $M_{rs}^t (M_{rs} M_{rs}^t)^\dagger = M_{rs}^\dagger$.

Now we have the optimal multi-degree reduction matrix $R_{m\times n}^{(r,s)}$ in a simple form as follows.

THEOREM 5.5. *The optimal multi-degree reduction matrix $R_{m\times n}^{(r,s)}$ of Bézier curves with endpoints continuity of (r, s) order at the endpoints with respect to L_2 norm is given by*

$$R_{m\times n}^{(r,s)} = R_{m\times n} - M_m M_{rs}^\dagger \left[R_{m\times n} - C^\dagger \right].$$

The degree reduced control points $\mathbf{q}^{(r,s)}$ can be obtained from the original control points $\mathbf{p}$ as

$$\mathbf{q}^{(r,s)} = R_{m\times n}^{(r,s)} \mathbf{p}.$$

Also the least squares degree reduced control points $\tilde{\mathbf{q}}$ without constraints is given by

$$\tilde{\mathbf{q}} = R_{m\times n} \mathbf{p}. \tag{5.9}$$

Let $\mathbf{q}_c = C^\dagger \mathbf{p}$, then we get

$$\mathbf{q}^{(r,s)} = \tilde{\mathbf{q}} - M_m M_{rs}^\dagger (\tilde{\mathbf{q}} - \mathbf{q}_c). \tag{5.10}$$

THEOREM 5.6. *Let $\tilde{\mathbf{q}} = R_{m\times n}\mathbf{p}$ be the least squares degree reduced control points without constraints. Then the optimal multi-degree reduced control points with endpoints continuity of (r, s) order at the endpoints with respect to L_2 norm is given by*

$$\mathbf{q}^{(r,s)} = \tilde{\mathbf{q}} - M_m M_{rs}^\dagger (\tilde{\mathbf{q}} - \mathbf{q}_c)$$

where $\mathbf{q}_c = C^\dagger \mathbf{p}$.

We have simple forms of multi-degree reduction matrix and degree-reduced control points. As seen in Theorem 5.6, the optimal degree reduced control points can be represented in a very simple form. The matrix $M_{rs}^\dagger$ and the vector $\mathbf{q}_c$ depend on the order (r, s). Although we introduced the constraints matrix C and $C^\dagger$ in order to derive our results in a vector-matrix form, the elements of the vector $\mathbf{q}_c = (q_0, \ldots, q_r, 0, \ldots, q_{m-s}, \ldots, q_m)^t$ can be obtained iteratively as given in [2] as follows:

(5.11)

$$\begin{cases} q_0 = \dfrac{1}{b_{0,0}^{(m,n)}} p_0, \ q_j = \dfrac{1}{b_{j,j}^{(m,n)}} \left(p_j - \displaystyle\sum_{i=0}^{j-1} b_{i,j}^{(m,n)} q_i \right), \ j = 1, 2, \ldots, r, \\ q_m = \dfrac{1}{b_{m,n}^{(m,n)}} p_n, \\ q_{m-j} = \dfrac{1}{b_{m-j,n-j}^{(m,n)}} \left(p_{n-j} - \displaystyle\sum_{i=0}^{j-1} b_{m-i,n-j}^{(m,n)} q_{m-i} \right), \ j = 1, 2, \ldots, s. \end{cases}$$

We describe a simple algorithm to compute the Moore-Penrose inverse $M_{rs}^{\dagger}$. Since $M_{rs} = I_m^{(r,s)} M_m$, M_{rs} has the first $(r+1)$ rows and the last $(s+1)$ rows only being not zero. Let M_{rs}^0 be a matrix extracting zero rows from M_m and let

$$M_{rs}^0 = \begin{pmatrix} \mathbf{v}_0^t \\ \mathbf{v}_1^t \\ \vdots \\ \mathbf{v}_{r+s+1}^t \end{pmatrix}, \tag{5.12}$$

that is, $\mathbf{v}_i^t$ is the i-th row vector of the matrix M_{rs}^0. The Moore-Penrose inverse of M_{rs} can be obtained using Lemma 3.3 iteratively.

Step 0: Let $M_0 = \mathbf{v}_0^t$. Then $M_0^{\dagger} = \frac{1}{\mathbf{v}_0^t \mathbf{v}_0} \mathbf{v}_0$.

Step 1: For $k = 1, 2, \ldots, r+s+1$, let

$$M_k = \begin{pmatrix} M_{k-1} \\ \mathbf{v}_k \end{pmatrix}$$

and let

$$\mathbf{c}^t = \mathbf{v}_k (I - M_{k-1}^{\dagger} M_{k-1}).$$

Since $\mathbf{c}^t$ is a column vector, the Moore-Penrose inverse of $\mathbf{c}^t$ is

$$(\mathbf{c}^t)^{\dagger} = \frac{1}{\mathbf{c}^t \mathbf{c}} \mathbf{c},$$

and we have

$$M_k^{\dagger} = \begin{pmatrix} M_{k-1}^{\dagger} - (\mathbf{c}^t)^{\dagger} \mathbf{v}_k^t M_{k-1}^t & (\mathbf{c}^t)^{\dagger} \end{pmatrix}.$$

The resulting matrix is $M_{r+s+1}^{\dagger}$. Then by inserting appropriate zero columns into $M_{r+s+1}^{\dagger}$, we can obtain $M_{rs}^{\dagger}$.

6. Error analysis

In the previous section we have multi-degree reduced control points

$$\mathbf{q}^{(r,s)} = \tilde{\mathbf{q}} - M_m M_{rs}^{\dagger}(\tilde{\mathbf{q}} - \mathbf{q}_c).$$

If we let $\tilde{q}(t)$ and $q^{(r,s)}(t)$ the Bézier curves represented by the control points $\tilde{\mathbf{q}}$ and $\mathbf{q}^{(r,s)}$, respectively, then the approximation errors (squared) of $\tilde{q}(t)$ and $q^{(r,s)}(t)$ are given by

$$Err_{(2)} = d_2^2(p, \tilde{q}) = (\mathbf{p} - T_{n\times m}\tilde{\mathbf{q}})^t Q_n (\mathbf{p} - T_{n\times m}\tilde{\mathbf{q}}) \tag{6.1}$$

and

$$Err_{(r,s)} = d_2^2(p, q^{(r,s)}) = (\mathbf{p} - T_{n\times m}\mathbf{q}^{(r,s)})^t Q_n (\mathbf{p} - T_{n\times m}\mathbf{q}^{(r,s)}). \tag{6.2}$$

Using the results in the previous section, we derive the relationship between two errors (squared) $Err_{(2)}$ and $Err_{(r,s)}$.

Lemma 6.1.

$$(\mathbf{p} - T_{n\times m}\tilde{\mathbf{q}})^t Q_n T_{n\times m} = 0.$$

Proof. By lemma 2.1 we have $\tilde{\mathbf{q}} = Q_m^{-1} T_{n\times m}^t Q_n \mathbf{p}$ and $T_{n\times m}^t Q_n T_{n\times m} = Q_m$. Hence we have

$$\begin{aligned}(\mathbf{p} - T_{n\times m}\tilde{\mathbf{q}})^t Q_n T_{n\times m} &= \mathbf{p}^t (I_{n+1} - Q_n T_{n\times m} Q_m^{-1} T_{n\times m}^t) Q_n T_{n\times m} \\ &= \mathbf{p}^t (Q_n T_{n\times m} - Q_n T_{n\times m}) = 0.\end{aligned}$$

□

Expanding Eq. (6.2), we have

$$\begin{aligned}Err_{(r,s)} &= (\mathbf{p} - T_{n\times m}\tilde{\mathbf{q}})^t Q_n (\mathbf{p} - T_{n\times m}\tilde{\mathbf{q}}) \\ &\quad +2(\mathbf{p} - T_{n\times m}\tilde{\mathbf{q}})^t Q_n T_{n\times m} M_m M_{rs}^{\dagger}\tilde{\mathbf{q}}_0 \\ &\quad + (\tilde{\mathbf{q}} - \mathbf{q}_c)^t (M_{rs}^{\dagger})^t M_m^t T_{n\times m}^t Q_n T_{n\times m} M_m M_{rs}^{\dagger}(\tilde{\mathbf{q}} - \mathbf{q}_c).\end{aligned} \tag{6.3}$$

By lemma 6.1, the second term of Eq. (6.3) equals to zero. Also by lemma 2.1 we have

$$T_{n\times m}^t Q_n T_{n\times m} = Q_m = (M_m^t)^{-1} M_m^{-1}$$

hence the third term of Eq. (6.3) becomes

$$\begin{aligned}&(\tilde{\mathbf{q}} - \mathbf{q}_c)^t (M_{rs}^{\dagger})^t M_m^t T_{n\times m}^t Q_n T_{n\times m} M_m M_{rs}^{\dagger}(\tilde{\mathbf{q}} - \mathbf{q}_c) \\ &= (\tilde{\mathbf{q}} - \mathbf{q}_c)^t (M_{rs}^{\dagger})^t M_m^t (M_m^t)^{-1} M_m^{-1} M_m M_{rs}^{\dagger}(\tilde{\mathbf{q}} - \mathbf{q}_c) \\ &= (\tilde{\mathbf{q}} - \mathbf{q}_c)^t (M_{rs}^{\dagger})^t M_{rs}^{\dagger}(\tilde{\mathbf{q}} - \mathbf{q}_c) \\ &= (\tilde{\mathbf{q}} - \mathbf{q}_c)^t W^{\dagger}(\tilde{\mathbf{q}} - \mathbf{q}_c).\end{aligned}$$

Note that $M_{rs}M_{rs}^t = I_m^{(r,s)}Q_m^{-1}I_m^{(r,s)} = W$ is defined in Theorem 5.2. Now we have the result

$$Err_{(r,s)} = Err_{(2)} + (\tilde{\mathbf{q}} - \mathbf{q}_c)^t W^\dagger (\tilde{\mathbf{q}} - \mathbf{q}_c). \tag{6.4}$$

It appears that two errors have close relationship.

7. Conclusions

We have derived the optimal multi-degree reduction matrix $R_{m\times n}^{(r,s)}$ with endpoints continuity of (r, s) order in Theorem 5.5 in a simple form using Lagrange multipliers. Consequently, the optimal degree reduced control points $\mathbf{q}^{(r,s)}$ is given in Theorem 5.6. As seen in Theorem 5.5 and 5.6, our results have close relationship with the least squares degree reduction. Also error analysis comparing the least squares degree reduction is given.

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