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SOME GENERALIZATIONS OF WEAKLY M-SEMI-CONTINUOUS AND WEAKLY M-PRECONTINUOUS FUNCTIONS

Takashi Noiri* and Valeriu Popa**

Abstract. As a generalization of (i,j)-weakly m-continuous functions [43], we introduce the notion of weakly $M(i,j)$-continuous functions and obtain many characterizations and some properties of the functions. We show that the function is a unified form of some functions between m-spaces and certain kinds of weakly continuous functions in bitopological spaces.

1. Introduction

Semi-open sets, preopen sets, α-open sets and β-open sets play an important role in the researching of generalizations of continuity in topological spaces and bitopological spaces. By using these sets many authors introduced and studied various types of modifications of continuity in topological spaces and bitopological spaces. Khedr [18] and the present authors [42], [46], [51] introduced and studied weakly semi-continuous functions and weakly precontinuous functions in bitopological spaces. Irresolute functions in bitopological spaces was defined by Mukherjee [35]. Khedr and Noiri introduced and studied in [21], [22] the notions of quasi-irresolute functions and almost s-continuous functions which are generalizations of weakly continuous functions between topological spaces due to Levine [23].

In [47]-[50], the present authors introduced and investigated the notions of minimal structures, m-spaces, m-continuous functions, M-conti nuous functions, weakly m-continuous functions and weakly M-continuous functions. Recently, in [39], [41] and other papers the present authors

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Correspondence should be addressed to Takashi NOIRI, t.noiri@nifty.com.

reduced the study of some forms of continuity between bitological spaces to the study of m-continuity and M-continuity between m-spaces.

Also Min and Kim [29]-[34] introduced and studied the notions of m-semi-open, m-preopen, α-m-open, β-m-open sets and M-semi-continuity, m-semi-continuity, $m\alpha$-continuity. Quite recently, the notions of weakly M-semi-continuous functions and weakly M-precontinuous functions have been introduced in [33] and [34], respectively. And also these notions are introduced and studied in [8], [52] and other papers.

Quite recently, the first author [38] introduced the notion of bi-m-spaces which are called biminimal structure spaces in [5]. Some propeties of biminimal structure spaces are studied in [4]-[6] and other papers. The purpose of this paper is to introduce and investigate the notion of weakly $M(i,j)$-continuous functions. This function is a generalization of weak M-semicontinuty [33], weak M-precontinuity [34], (i,j)-weak m-continuity [43], (i,j)-weak quasi continuity [18], (i,j)-weak precontinuity [42], (i,j)-quasi irresoluteness [21] and (i,j)-almost s-continuity [22].

2. Preliminaries

Let (X,τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. We recall some generalized open sets in topological spaces.

Definition 2.1. Let (X,τ) be a topological space. A subset A of X is said to be
(1) *α-open* [37] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,
(2) *semi-open* [24] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$,
(3) *preopen* [27] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$,
(4) *β-open* [1] or *semi-preopen* [3] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$.

The family of all α-open (resp. semi-open, preopen, β-open) sets in (X,τ) is denoted by $\alpha(X)$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\beta(X)$).

Definition 2.2. Let (X,τ) be a topological space. A subset A of X is said to be *α-closed* [28] (resp. *semi-closed* [9], *preclosed* [27], *β-closed* [1]) if the complement of A is α-open (resp. semi-open, preopen, β-open).

Definition 2.3. Let (X,τ) be a topological space and A a subset of X. The intersection of all α-closed (resp. semi-closed, preclosed,

β-closed) sets of X containing A is called the *α-closure* [28] (resp. *semi-closure* [9], *preclosure* [10], *β-closure* [2]) of A and is denoted by $\alpha\mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, ${}_\beta\mathrm{Cl}(A)$).

DEFINITION 2.4. Let (X, τ) be a topological space and A a subset of X. The union of all α-open (resp. semi-open, preopen, β-open) sets of X contained in A is called the *α-interior* [28] (resp. *semi-interior* [9], *preinterior* [10], *β-interior* [2]) of A and is denoted by $\alpha\mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, ${}_\beta\mathrm{Int}(A)$).

3. Minimal structures and bi-m-spaces

DEFINITION 3.1. Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subfamily m_X of $\mathcal{P}(X)$ is called a *minimal structure* (briefly *m-structure*) on X [47], [48] if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X), we denote a nonempty set X with an m-structure m_X on X and call it an m-space. Each member of m_X is said to be *m_X-open* (briefly m-open) and the complement of an m_X-open set is said to be *m_X-closed* (briefly m-closed).

REMARK 3.2. Let (X, τ) be a topological space. The families τ, $\alpha(X)$, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$ and $\beta(X)$ are all minimal structures on X.

DEFINITION 3.3. Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [26] as follows:
(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X \setminus F \in m_X\}$,
(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m_X\}$.

REMARK 3.4. Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$), then we have
(1) $\mathrm{mCl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_\beta\mathrm{Cl}(A)$),
(2) $\mathrm{mInt}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_\beta\mathrm{Int}(A)$).

LEMMA 3.5. (Maki et al. [26]). *Let X be a nonempty set and m_X a minimal structure on X. For subsets A and B of X, the following properties hold:*

(1) $\mathrm{mCl}(X \setminus A) = X \setminus \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X \setminus A) = X \setminus \mathrm{mCl}(A)$,
(2) *If* $(X \setminus A) \in m_X$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$, *then* $\mathrm{mInt}(A) = A$,
(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,
(4) *If* $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,

(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,
(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

LEMMA 3.6. (Popa and Noiri [47]). *Let (X, m_X) be an m-space and A a subset of X. Then $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for each $U \in m_X$ containing x.*

DEFINITION 3.7. An m-structure m_X on a nonempty set X is said to have *property $\mathcal{B}$* [26] if the union of any family of subsets belonging to m_X belongs to m_X.

REMARK 3.8. If (X, τ) is a topological space, then the m-structures SO(X), PO(X), $\alpha(X)$ and $\beta(X)$ have property $\mathcal{B}$.

LEMMA 3.9. (Popa and Noiri [49]). *Let X be a nonempty set and m_X an m-structure on X satisfying property $\mathcal{B}$. For a subset A of X, the following properties hold:*
(1) $A \in m_X$ *if and only if* $\mathrm{mInt}(A) = A$,
(2) A *is m_X-closed if and only if* $\mathrm{mCl}(A) = A$,
(3) $\mathrm{mInt}(A) \in m_X$ *and* $\mathrm{mCl}(A)$ *is m_X-closed.*

DEFINITION 3.10. Let (X, m_X) be an m-space. A subset A of X is said to be
(1) *m-α-open* [30] if $A \subset \mathrm{mInt}(\mathrm{mCl}(\mathrm{mInt}(A)))$,
(2) *m-semi-open* [29] if $A \subset \mathrm{mCl}(\mathrm{mInt}(A))$,
(3) *m-preopen* [31] if $A \subset \mathrm{mInt}(\mathrm{mCl}(A))$,
(4) *m-β-open* [6] if $A \subset \mathrm{mCl}(\mathrm{mInt}(\mathrm{mCl}(A)))$.

The family of all m-α-open (resp. m-semi-open, m-preopen, m-β-open) sets in (X, m_X) is denoted by m$\alpha(X)$ (resp. mSO(X), mPO(X), m$\beta(X)$).

REMARK 3.11. Similar definitions of m-semi-open sets, m-preopen sets, m-α-open sets, m-β-open sets are provided in [8] and [52].

Let (X, m_X) be an m-space. We denote by mIT(X) the family of all m-structures on X determined by iterating operators mInt and mCl ([44], [45]). However, in this paper, by mIT(X) we denote m$\alpha(X)$, mSO(X), mPO(X) or m$\beta(X)$.

REMARK 3.12. (1) It easily follows from Lemma 3.5(3)-(4) that m$\alpha(X)$, mSO(X), mPO(X) and m$\beta(X)$ are minimal structures with property $\mathcal{B}$. They are also shown in Theorem 3.5 of [29], Theorem 3.4 of [31] and Theorem 3.4 of [30].

(2) Let (X, m_X) be an m-space and mIT(X) an iterate structure on X. If mIT(X) = mSO(X) (resp. mPO(X), m$\alpha(X)$, m$\beta(X)$), then we obtain the following definitions provided in [29] (resp. [31], [30]):
mITCl(A) = msCl(A) (resp. mpCl(A), mαCl(A), mβCl(A)),
mITInt(A) = msInt(A) (resp. mpInt(A), mαInt(A), mβInt(A)).

In Theorem 4.2 of [50] and Theorems 7.4, 8.3 and 8.4 of [40], the authors used first m-spaces with two minimal structures. The first author [38] called a bi-m-space a nonempty set with two minimal structures on X. Recently, Boonpok [5] has renamed bi-m-spaces as biminimal structure spaces. In [4], the author studied some forms of continuity between two biminimal structure spaces.

Throughout the present paper, (X, τ_1, τ_2) (resp. (X, m_1, m_2)) denotes a bitopological space (resp. bi-m-space). Let (X, τ) be a topological space and A be a subset of X. Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X. The closure of A and the interior of A with respect to τ_i are denoted by iCl(A) and iInt(A), respectively, for $i = 1, 2$. Similarly, we denote the m_X-closure of A and the m_X-interior of A with respect to m_i are denoted by $\mathrm{m}_X^i\mathrm{Cl}(A)$ and $\mathrm{m}_X^i\mathrm{Int}(A)$, respectively, for $i = 1, 2$.

Remark 3.13. A bitopological space is a particular case of a bi-m-space.

Let (X, m_X) be an m-space and (Y, σ_1, σ_2) be a bitopological space. In [43], the authors introduced and studied a form of weakly continuous functions for a function $f : (X, m_X) \to (Y, \sigma_1, \sigma_2)$.

Definition 3.14. A function $f : (X, m_X) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-weakly m-continuous* [43] at $x \in X$ if for each $V \in \sigma_i$ containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset j\mathrm{Cl}(V)$. The function f is said to be *(i, j)-weakly m-continuous* if it has this property at each point $x \in X$.

Recently, as weak forms of M-continuity [47], some functions between m-spaces are introduced and studied as follows:

Definition 3.15. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *weakly M-continuous* [49] $x \in X$ if for each $V \in m_Y$ containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset \mathrm{mCl}(V)$. The function f is said to be *weakly M-continuous* if it has this property at each point $x \in X$.

Definition 3.16. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *weakly M-semicontinuous* [33] at $x \in X$ if for each $V \in m_Y$ containing

$f(x)$, there exists $U \in \text{mSO}(X)$ containing x such that $f(U) \subset \text{msCl}(V)$. The function f is said to be *weakly M-semicontinuous* if it has this property at each point $x \in X$.

Definition 3.17. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *weakly M-precontinuous* [34] at $x \in X$ if for each $V \in m_Y$ containing $f(x)$, there exists $U \in \text{mPO}(X)$ containing x such that $f(U) \subset \text{mpCl}(V)$. The function f is said to be *weakly M-precontinuous* if the function f has this property at each point $x \in X$.

Now we introduce a new function which is a generalization of the above four functions.

Definition 3.18. A function $f : (X, m_X) \to (Y, m_Y^1, m_Y^2)$ is said to be *weakly M(i, j)-continuous* at $x \in X$ if for each $V \in m_Y^i$ containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset \text{m}_\text{Y}^\text{j}\text{Cl}(V)$. The function f is said to be *weakly $M(i, j)$-continuous* if the function f has this property at each point $x \in X$.

Remark 3.19.
(1) If we set $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$ in Definition 3.18, then we obtain the definition of (i, j)-weak m-continuity (Definition 3.14).
(2) If we set $m_Y = m_Y^1 = m_Y^2$ in Definition 3.18, then we obtain the definition of weak M-continuity (Definition 3.15).
(3) If we set $m_X = \text{mSO}(X)$, $m_Y^1 = m_Y$ and $m_Y^2 = \text{mSO}(Y)$ in Definition 3.18, then a function $f : (X, m_X) \to (Y, m_Y)$ is weakly M-semicontinuous if and only if $f : (X, \text{mSO}(X)) \to (Y, m_Y, \text{mSO}(Y))$ is weakly $M(1, 2)$-continuous.
(4) If we set $m_X = \text{mPO}(X)$, $m_Y^1 = m_Y$ and $m_Y^2 = \text{mPO}(Y)$ in Definition 3.18, then a function $f : (X, m_X) \to (Y, m_Y)$ is weakly M-precontinuous if and only if $f : (X, \text{mPO}(X)) \to (Y, m_Y, \text{mPO}(Y))$ is weakly $M(1, 2)$-continuous.

4. Characterizations of weak $M(i, j)$-continuity

Theorem 4.1. *For a function $f : (X, m_X) \to (Y, m_Y^1, m_Y^2)$, the following properties are equivalent:*
(1) *f is weakly $M(i, j)$-continuous at $x \in X$;*
(2) *for every $V \in m_Y^i$ containing $f(x)$, $x \in \text{m}_\text{X}\text{Int}(f^{-1}(\text{m}_\text{Y}^\text{j}\text{Cl}(V)))$;*
(3) *for every m_Y^i-closed set F of Y such that $x \in \text{m}_\text{X}\text{Cl}(f^{-1}(\text{m}_\text{Y}^\text{j}\text{Int}(F)))$, $x \in f^{-1}(F)$.*

Proof. (1) ⇒ (2): Let $V \in m^i_Y$ containing $f(x)$. Then, by (1) there exists $U \in m_X$ containing x such that $f(U) \subset \mathrm{m^j_Y Cl}(V)$. Thus $x \in U \subset f^{-1}(\mathrm{m^j_Y Cl}(V))$ and hence $x \in \mathrm{m_X Int}(f^{-1}(\mathrm{m^j_Y Cl}(V)))$.

(2) ⇒ (3): Let F be any m^i_Y-closed set of Y. Suppose that $x \notin f^{-1}(F)$. Then $Y \setminus F \in m^i_Y$ and $x \in X \setminus f^{-1}(F) = f^{-1}(Y \setminus F)$. By (2) and Lemma 3.5, $x \in \mathrm{m_X Int}(f^{-1}(\mathrm{m^j_Y Cl}(Y \setminus F))) = \mathrm{m_X Int}(f^{-1}(Y \setminus \mathrm{m^j_Y Int}(F))) = X \setminus \mathrm{m_X Cl}(f^{-1}(\mathrm{m^j_Y Int}(F)))$. Hence $x \notin \mathrm{m_X Cl}(f^{-1}(\mathrm{m^j_Y Int}(F)))$.

(3) ⇒ (1): Let V be any m^i_Y-open set containing $f(x)$. Then $x \notin f^{-1}(Y \setminus V)$ and $Y \setminus V$ is m^i_Y-closed. By (3), $x \notin \mathrm{m_X Cl}(f^{-1}(\mathrm{m^j_Y Int}(Y \setminus V))) = \mathrm{m_X Cl}(f^{-1}(Y \setminus \mathrm{m^j_Y Cl}(V))) = \mathrm{m_X Cl}(X \setminus f^{-1}(\mathrm{m^j_Y Cl}(V))) = X \setminus \mathrm{m_X Int}(f^{-1}(\mathrm{m^j_Y Cl}(V)))$. Therefore, there exists $U \in m_X$ containing x such that $U \subset f^{-1}(\mathrm{m^j_Y Cl}(V))$; hence $f(U) \subset \mathrm{m^j_Y Cl}(V)$). □

Corollary 4.2. *For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:*

1. *f is weakly M-semicontinuous at $x \in X$;*
2. *for every $V \in m_Y$ containing f(x), $x \in \mathrm{msInt}(f^{-1}(\mathrm{msCl}(V)))$;*
3. *for every m_Y-closed set F of Y such that $x \in \mathrm{msCl}(f^{-1}(\mathrm{msInt}(F)))$, $x \in f^{-1}(F)$.*

Remark 4.3. By Remark 3.19, we can obtain a quite similar charactrizations of weak M-precontinuity from Theorem 4.1.

Definition 4.4. A subset B of a bi-m-space (Y, m^1_Y, m^2_Y) is said to be *m_{ij}-regular closed* [4] if $B = \mathrm{m^i_Y Cl}(\mathrm{m^j_Y Int}(B))$.

Theorem 4.5. *For a function $f : (X, m_X) \to (Y, m^1_Y, m^2_Y)$, where m^1_Y and m^2_Y have property $\mathcal{B}$, the following properties are equivalent:*

1. *f is weakly $M(i,j)$-continuous at $x \in X$;*
2. *for every subset B of Y with $x \in \mathrm{m_X Cl}(f^{-1}(\mathrm{m^j_Y Int}(\mathrm{m^i_Y Cl}(B))))$, $x \in f^{-1}(\mathrm{m^i_Y Cl}(B))$;*
3. *for every m_{ij}-regular closed set F of Y such that $x \in \mathrm{m_X Cl}(f^{-1}(\mathrm{m^j_Y Int}(F)))$, $x \in f^{-1}(F)$;*
4. *for every m^j_Y-open set V of Y with $x \in \mathrm{m_X Cl}(f^{-1}(V))$, $x \in f^{-1}(\mathrm{m^i_Y Cl}(V))$.*

Proof. (1) ⇒ (2): Let B be any subset of Y with $x \in \mathrm{m_X Cl}(f^{-1}(\mathrm{m^j_Y Int}(\mathrm{m^i_Y Cl}(B))))$. Since m^i_Y has property $\mathcal{B}$, by Lemma 3.9, $\mathrm{m^i_Y Cl}(B)$ is m^i_Y-closed. Then, by Theorem 4.1,

$x \in \mathrm{m_X Cl}(f^{-1}(\mathrm{m^j_Y Int}(\mathrm{m^i_Y Cl}(B))))$ implies $x \in f^{-1}(\mathrm{m^i_Y Cl}(B))$.

(2) ⇒ (3): Let F be any m_{ij}-regular closed set of Y such that $x \in \mathrm{m_X Cl}(f^{-1}(m^j_Y Int(F)))$. By (2), $x \in f^{-1}(\mathrm{m^i_Y Cl}(\mathrm{m^j_Y Int}(B))) = f^{-1}(F)$.

(3) ⇒ (4): Let V be any m^j_Y-open set of Y with $x \in \mathrm{m_X Cl}(f^{-1}(V))$. Then $\mathrm{m^i_Y Cl}(V) = \mathrm{m^i_Y Cl}(\mathrm{m^j_Y Int}(V))$ and $\mathrm{m^i_Y Cl}(V)$ is m_{ij}-regular closed. By assumption, $x \in \mathrm{m_X Cl}(f^{-1}(V)) = \mathrm{m_X Cl}(f^{-1}(\mathrm{m^j_Y Int}(V)))$
$\subset \mathrm{m_X Cl}(f^{-1}(\mathrm{m^j_Y Int}(\mathrm{m^i_Y Cl}(V))))$. By (3), $x \in f^{-1}(\mathrm{m^i_Y Cl}(V))$.

(4) ⇒ (1): Let V be any m^i_Y-open set of Y containing $f(x)$. Since m^j_Y has property $\mathcal{B}$, $\mathrm{m^j_Y Cl}(V)$ is m^j_Y-closed and $Y \setminus \mathrm{m^j_Y Cl}(V)$ is m^j_Y-open. Suppose that $x \notin \mathrm{m_X Int}(f^{-1}(\mathrm{m^j_Y Cl}(V)))$. Then
$x \in X \setminus \mathrm{m_X Int}(f^{-1}(\mathrm{m^j_Y Cl}(V))) = \mathrm{m_X Cl}(X \setminus f^{-1}(\mathrm{m^j_Y Cl}(V))$
$= \mathrm{m_X Cl}(f^{-1}(Y \setminus \mathrm{m^j_Y Cl}(V)))$. Since $Y \setminus \mathrm{m^j_Y Cl}(V)$ is m^j_Y-open, by (4) $x \in f^{-1}(\mathrm{m^i_Y Cl}(Y \setminus \mathrm{m^j_Y Cl}(V))) = f^{-1}(Y \setminus \mathrm{m^i_Y Int}(\mathrm{m^j_Y Cl}(V))) \subset f^{-1}(Y \setminus \mathrm{m^i_Y Int}(V)) = f^{-1}(Y \setminus V)) = X \setminus f^{-1}(V)$. Hence $x \notin f^{-1}(V)$ and $f(x) \notin V$. This is a contradiction. Therefore, we obtain that $x \in \mathrm{m_X Int}(f^{-1}(\mathrm{m^j_Y Cl}(V)))$. By Theorem 4.1, f is weakly $M(i,j)$-continuous at x. □

Corollary 4.6. *For a function $f : (X, m_X) \to (Y, m_Y)$, where m_Y has property $\mathcal{B}$, the following properties are equivalent:*

(1) *f is weakly M-semicontinuous at $x \in X$;*
(2) *for every subset B of Y with $x \in \mathrm{msCl}(f^{-1}(\mathrm{msInt}(\mathrm{mCl}(B))))$, $x \in f^{-1}(\mathrm{mCl}(B))$;*
(3) *for every m_{ij}-regular closed set F of Y such that $x \in \mathrm{msCl}(f^{-1}(\mathrm{msInt}(F)))$, $x \in f^{-1}(F)$;*
(4) *for every ms-open set V of Y with $x \in \mathrm{msCl}(f^{-1}(V))$, $x \in f^{-1}(\mathrm{mCl}(V))$.*

Remark 4.7. By Remark 3.19, we can obtain a quite similar charactrizations of weak M-precontinuity from Theorem 4.5.

For a function $f : (X, m_X) \to (Y, m^1_Y, m^2_Y)$, we define $D_{M(i,j)}(f)$ as follows:

$D_{M(i,j)}(f) = \{x \in X : f$ is not weakly $M(i,j)$-continuous at $x\}$.

Theorem 4.8. *For a function $f : (X, m_X) \to (Y, m^1_Y, m^2_Y)$, the following properties hold:*

$$D_{M(i,j)}(f) = \bigcup_{G \in m^i_Y} \{f^{-1}(G) \setminus \mathrm{m_X Int}(f^{-1}(\mathrm{m^j_Y Cl}(G)))\}$$
$$= \bigcup_{F \in \mathcal{F}} \{\mathrm{m_X Cl}(f^{-1}(\mathrm{m^j_Y Int}(F)) \setminus f^{-1}(F)\},$$

where $\mathcal{F}$ is the family of m^i_Y-closed sets of Y.

Proof. We show only the first equality because the proof of the other is similar to the first one. Let $x \in D_{M(i,j)}(f)$. By Theorem 4.1, there exists $V \in m^i_Y$ such that $f(x) \in V$ and $x \notin \mathrm{m_X Int}(f^{-1}(\mathrm{m^j_Y Cl}(V)))$. Therefore, we have $x \in f^{-1}(V) \setminus \mathrm{m_X Int}(f^{-1}(\mathrm{m^j_Y Cl}(V))) \subset \bigcup_{G \in m^i_Y} \{f^{-1}(G) \setminus \mathrm{m_X Int}(f^{-1}(\mathrm{m^j_Y Cl}(G)))\}$. Conversely, let $x \in \bigcup_{G \in m^i_Y} \{f^{-1}(G) \setminus \mathrm{m_X Int}(f^{-1}(\mathrm{m^j_Y Cl}(G)))\}$. There exists $V \in m^i_Y$ such that $x \in f^{-1}(V) \setminus \mathrm{m_X Int}(f^{-1}(\mathrm{m^j_Y Cl}(V)))$. By Theorem 4.1, $x \in D_{M(i,j)}(f)$. □

For a function $f : (X, m_X) \to (Y, m_Y)$, we define $D_{Ms}(f)$ as follows:

$$D_{Ms}(f) = \{x \in X : f \text{ is not weakly } M\text{-semicontinuous at } x\}.$$

Then by Remark 3.19 and Theorem 4.8 we obtain the following corollary.

Corollary 4.9. *For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties hold:*

$$\begin{aligned} D_{Ms}(f) &= \bigcup_{G \in m_Y} \{f^{-1}(G) \setminus \mathrm{msInt}(f^{-1}(\mathrm{msCl}(G)))\} \\ &= \bigcup_{F \in \mathcal{F}} \{\mathrm{msCl}(f^{-1}(\mathrm{msInt}(F)) \setminus f^{-1}(F)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of m_Y-closed sets of Y.

Remark 4.10. By Remark 3.19, we can obtain a quite similar results of weak M-precontinuity from Theorem 4.8.

Theorem 4.11. *For a function $f : (X, m_X) \to (Y, m^1_Y, m^2_Y)$, where m^1_Y and m^2_Y have property $\mathcal{B}$, the following properties hold:*

$$\begin{aligned} D_{M(i,j)}(f) &= \bigcup_{B \in \mathcal{P}(Y)} \{\mathrm{m_X Cl}(f^{-1}(\mathrm{m^j_Y Int}(\mathrm{m^i_Y Cl}(B)))) \setminus f^{-1}(\mathrm{m^i_Y Cl}(B))\} \\ &= \bigcup_{G \in m^j_Y} \{\mathrm{m_X Cl}(f^{-1}(G)) \setminus f^{-1}(\mathrm{m^i_Y Cl}(G))\} \\ &= \bigcup_{F \in \mathcal{F}} \{\mathrm{m_X Cl}(f^{-1}(\mathrm{m^j_Y Int}(F))) \setminus f^{-1}(F)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of m_{ij} regular closed sets of Y.

Proof. The proof is similar to that of Theorem 4.8. □

Corollary 4.12. *For a function $f : (X, m_X) \to (Y, m_Y)$, where m_Y has property $\mathcal{B}$, the following properties hold:*

$$\begin{aligned} D_{Ms}(f) &= \bigcup_{B \in \mathcal{P}(Y)} \{\mathrm{msCl}(f^{-1}(\mathrm{msInt}(\mathrm{mCl}(B)))) \setminus f^{-1}(\mathrm{mCl}(B))\} \\ &= \bigcup_{G \in \mathrm{mSO}(Y)} \{\mathrm{msCl}(f^{-1}(G)) \setminus f^{-1}(\mathrm{mCl}(G))\} \\ &= \bigcup_{F \in \mathcal{F}} \{\mathrm{msCl}(f^{-1}(\mathrm{msInt}(F))) \setminus f^{-1}(F)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of m_{ij} regular closed sets of Y.

Remark 4.13. By Remark 3.19, we can obtain a quite similar results of weak M-precontinuity from Theorem 4.11.

THEOREM 4.14. *For a function $f : (X, m_X) \to (Y, m_Y^1, m_Y^2)$, the following properties are equivalent:*

(1) *f is weakly $M(i,j)$-continuous;*

(2) $f^{-1}(V) \subset \mathrm{m_X Int}(f^{-1}(\mathrm{m_Y^j Cl}(V)))$ *for every m_Y^i-open set V of Y;*

(3) $\mathrm{m_X Cl}(f^{-1}(\mathrm{m_Y^j Int}(F))) \subset f^{-1}(F)$ *for every m_Y^i-closed set F of Y.*

Proof. (1) $\Rightarrow$ (2): Let V be any m_Y^i-open set and $x \in f^{-1}(V)$. Then, by Theorem 4.1 $x \in \mathrm{m_X Int}(f^{-1}(\mathrm{m_Y^j Cl}(V)))$ and hence $f^{-1}(V) \subset \mathrm{m_X Int}(f^{-1}(\mathrm{m_Y^j Cl}(V)))$.

(2) $\Rightarrow$ (3): Let F be any m_Y^i-closed set of Y. Suppose that $x \notin f^{-1}(F)$. Then $Y \setminus F \in m_Y^i$ is and $x \in X \setminus f^{-1}(F) = f^{-1}(Y \setminus F)$. By (2) and Lemma 3.5, $x \in \mathrm{m_X Int}(f^{-1}(\mathrm{m_Y^j Cl}(Y \setminus F))) = \mathrm{m_X Int}(f^{-1}(Y \setminus \mathrm{m_Y^j Int}(F))) = X \setminus \mathrm{m_X Cl}(f^{-1}(\mathrm{m_Y^j Int}(F)))$. Therefore, $x \notin \mathrm{m_X Cl}(f^{-1}(\mathrm{m_Y^j Int}(F)))$ and hence $\mathrm{m_X Cl}(f^{-1}(\mathrm{m_Y^j Int}(F))) \subset f^{-1}(F)$.

(3) $\Rightarrow$ (1): Let V be any m_Y^i-open set and $x \in f^{-1}(V)$. Then $x \notin f^{-1}(Y \setminus V)$ and $Y \setminus V$ is m_Y^i-closed. By (3), $x \notin \mathrm{m_X Cl}(f^{-1}(\mathrm{m_Y^j Int}(Y \setminus V))) = \mathrm{m_X Cl}(f^{-1}(Y \setminus \mathrm{m_Y^j Cl}(V))) = \mathrm{m_X Cl}(X \setminus f^{-1}(\mathrm{m_Y^j Cl}(V))) = X \setminus \mathrm{m_X Int}(f^{-1}(\mathrm{m_Y^j Cl}(V)))$. Therefore, $x \in \mathrm{m_X Int}(f^{-1}(\mathrm{m_Y^j Cl}(V)))$. By Theorem 4.1, f is weakly $M(i,j)$-continuous. □

THEOREM 4.15. *For a function $f : (X, m_X) \to (Y, m_Y^1, m_Y^2)$, where m_Y^1 and m_Y^2 have property $\mathcal{B}$, the following properties are equivalent:*

(1) *f is weakly $M(i,j)$-continuous;*

(2) *for every subset B of Y,*

$$\mathrm{m_X Cl}(f^{-1}(\mathrm{m_Y^j Int}(\mathrm{m_Y^i Cl}(B)))) \subset f^{-1}(\mathrm{m_Y^i Cl}(B));$$

(3) *for every m_{ij}-regular closed set F of Y,*

$$\mathrm{m_X Cl}(f^{-1}(\mathrm{m_Y^j Int}(F))) \subset f^{-1}(F);$$

(4) *for every m_Y^j-open set V of Y,* $\mathrm{m_X Cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{m_Y^i Cl}(V))$.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Suppose $x \in \mathrm{m_X Cl}(f^{-1}(\mathrm{m_Y^j Int}(\mathrm{m_Y^i Cl}(B))))$. By Theorem 4.5, $x \in f^{-1}(\mathrm{m_Y^i Cl}(B))$. Hence $\mathrm{m_X Cl}(f^{-1}(\mathrm{m_Y^j Int}(\mathrm{m_Y^i Cl}(B)))) \subset f^{-1}(\mathrm{m_Y^i Cl}(B))$.

(2) $\Rightarrow$ (3): Let F be any m_{ij}-regular closed set of Y. By (2), $\mathrm{m_X Cl}(f^{-1}(\mathrm{m_Y^j Int}(F))) = \mathrm{m_X}(f^{-1}(\mathrm{m_Y^j Int}(\mathrm{m_Y^i Cl}(\mathrm{m_y^j Int}(F)))))$ $\subset f^{-1}(\mathrm{m_Y^i Cl}(\mathrm{m_Y^j Int}(F))) = f^{-1}(F)$.

(3) $\Rightarrow$ (4): Let V be any m_Y^j-open set of Y. Then $\mathrm{m_Y^i Cl}(V) = \mathrm{m_Y^i Cl}(\mathrm{m_Y^j Int}(V))$ and $\mathrm{m_Y^i Cl}(V)$ is m_{ij}-regular closed. By (3),

$\mathrm{m_X Cl}(f^{-1}(V)) = \mathrm{m_X Cl}(f^{-1}(\mathrm{m^j_Y Int}(V))) \subset \mathrm{m_X Cl}(f^{-1}(\mathrm{m^j_Y Int}(\mathrm{m^i_Y Cl}(V))))$
$\subset f^{-1}(\mathrm{m^i_Y Cl}(V))$.

(4) $\Rightarrow$ (1): Let V be any m^i_Y-open set of Y. Since m^j_Y has property $\mathcal{B}$, $\mathrm{m^j_Y Cl}(V)$ is m^j_Y-closed and $Y \setminus \mathrm{m^j_Y Cl}(V)$ is m^j_Y-open. By (4), $\mathrm{m_X Cl}(f^{-1}(Y \setminus \mathrm{m^j_Y Cl}(V))) \subset f^{-1}(\mathrm{m^i_Y Cl}(Y \setminus \mathrm{m^j_Y Cl}(V))) \subset X \setminus f^{-1}(V)$. Therefore, we obtain $f^{-1}(V) \subset \mathrm{m_X Int}(f^{-1}(\mathrm{m^j_Y Cl}(V)))$ and by Theorem 4.14 f is weakly $M(i,j)$-continuous. □

Remark 4.16. (1) By Theorems 4.14 and 4.15, we obtain the results from Theorems 3.3, 3.5 and 3.6 of [33].

(2) By Theorems 4.14 and 4.15, we obtain the results from Theorems 3.3 and 3.5 of [34].

(3) Let $m^1_Y = \sigma_1$ and $m^2_Y = \sigma_2$, then by Theorems 4.14 and 4.15 we obtain Theorem 3.1 of [43].

Definition 4.17. Let (X, m^1_X, m^2_X) be a bi-m-space and A a subset of X. A point x of X is called an *m_{ij}-θ-adherent* point of A if $A \cap \mathrm{m^j_X Cl}(U) \neq \emptyset$ for every m_i-open set U containing x.

The set of all m_{ij}-θ-adherent points of A is called the *m_{ij}-θ-closure* of A and is denoted by $m_{ij}\mathrm{Cl}_\theta(A)$. If $A = m_{ij}\mathrm{Cl}_\theta(A)$, then A is said to be m_{ij}-θ-closed. A subset A of X is said to be *m_{ij}-θ-open* if $X \setminus A$ is m_{ij}-θ-closed.

Lemma 4.18. *Let (X, m^1_X, m^2_X) be a bi-m-space, where m^i_X has property $\mathcal{B}$. Then $m_{ij}\mathrm{Cl}_\theta(A)$ is m^i_X-closed for each subset A of X.*

Proof. Let $x \in X \setminus m_{ij}\mathrm{Cl}_\theta(A)$. Then $x \notin m_{ij}\mathrm{Cl}_\theta(A)$. Hence there exists $U_x \in m^i_X$ containing x such that $\mathrm{m^j_X Cl}(U_x) \cap A = \emptyset$. Then $U_x \cap A = \emptyset$ which implies that $U_x \cap m_{ij}\mathrm{Cl}_\theta(A) = \emptyset$. Indeed, suppose that $U_x \cap m_{ij}\mathrm{Cl}_\theta(A) \neq \emptyset$. Then, there exists $y \in U_x \cap m_{ij}\mathrm{Cl}_\theta(A)$. Therefore, $y \in U_x$ and $y \in m_{ij}\mathrm{Cl}_\theta(A)$. Therefore, we have $\mathrm{m^j_X Cl}(U_x) \cap A \neq \emptyset$. This is a contradiction. Hence $x \in U_x \subset X \setminus m_{ij}\mathrm{Cl}_\theta(A)$. Since m^i_X has property $\mathcal{B}$, $X \setminus m_{ij}\mathrm{Cl}_\theta(A) = \cup U_x \in m^i_X$. It follows that $m_{ij}\mathrm{Cl}_\theta(A)$ is m^i_X-closed. □

Lemma 4.19. *Let (X, m^1_X, m^2_X) be a bi-m-space. If U is an m^i_X-open set, then $m_{ji}\mathrm{Cl}_\theta(U) = \mathrm{m^j_X Cl}(U)$.*

Proof. Suppose that $x \notin m_{ji}\mathrm{Cl}_\theta(U)$. Then there exists $V \in m^j_X$ containing x such that $\mathrm{m^i_X Cl}(V) \cap U = \emptyset$; hence $V \cap U = \emptyset$. Therefore, $V \cap \mathrm{m^j_X Cl}(U) = \emptyset$ and hence $x \notin \mathrm{m^j_X Cl}(U)$. Therefore, $m_{ji}\mathrm{Cl}_\theta(U) \supset$

$\mathrm{m_X^j Cl}(U)$. Conversely, suppose that $x \notin \mathrm{m_X^j Cl}(U)$. Then there exists $V \in m_X^j$ containing x such that $V \cap U = \emptyset$. Since $U \in m_X^i$, $U \cap \mathrm{m_X^i Cl}(V) = \emptyset$ and hence $x \notin m_{ji}\mathrm{Cl}_\theta(U)$. Hence $m_{ji}\mathrm{Cl}_\theta(U) \subset \mathrm{m_X^j Cl}(U)$. Hence $m_{ji}\mathrm{Cl}_\theta(U) = \mathrm{m_X^j Cl}(U)$. □

THEOREM 4.20. *For a function $f : (X, m_X) \to (Y, m_Y^1, m_Y^2)$, where m_Y^1 and m_Y^2 have property $\mathcal{B}$, the following properties are equivalent:*

(1) *f is weakly $M(i,j)$-continuous;*
(2) *$f(\mathrm{m_X Cl}(A)) \subset m_{ij}\mathrm{Cl}_\theta(f(A))$ for every subset A of X;*
(3) *$\mathrm{m_X Cl}(f^{-1}(B)) \subset f^{-1}(m_{ij}\mathrm{Cl}_\theta(B))$ for every subset B of Y;*
(4) *$\mathrm{m_X Cl}(f^{-1}(\mathrm{m_Y^j Int}(m_{ij}\mathrm{Cl}_\theta(B)))) \subset f^{-1}(m_{ij}\mathrm{Cl}_\theta(B))$ for every subset B of Y.*

Proof. (1) ⇒ (2): Suppose that f is weakly $M(i,j)$-continuous. Let A be any subset of X, $x \in \mathrm{m_X Cl}(A)$ and V be an m_Y^i-open set of Y containing $f(x)$. Then, there exists an m_X-open set U containing x such that $f(U) \subset \mathrm{m_Y^j Cl}(V)$. Since $x \in \mathrm{m_X Cl}(A)$, by Lemma 3.6 we obtain $U \cap A \neq \emptyset$ and hence $\emptyset \neq f(U) \cap f(A) \subset \mathrm{m_Y^j Cl}(V) \cap f(A)$. Therefore, we obtain $f(x) \in m_{ij}\mathrm{Cl}_\theta(f(A))$ and hence $f(\mathrm{m_X Cl}(A)) \subset m_{ij}\mathrm{Cl}_\theta(f(A))$.

(2) ⇒ (3): Let B be any subset of Y. Then $f(\mathrm{m_X Cl}(f^{-1}(B))) \subset m_{ij}\mathrm{Cl}_\theta(f(f^{-1}(B))) \subset m_{ij}\mathrm{Cl}_\theta(B)$ and hence
$\mathrm{m_X Cl}(f^{-1}(B)) \subset f^{-1}(m_{ij}\mathrm{Cl}_\theta(B))$.

(3) ⇒ (4): Let B be any subset of Y. Then, by Lemma 4.18 $m_{ij}\mathrm{Cl}_\theta(B)$ is m_Y^i-closed in Y and by using Lemmas 4.18 and 4.19 we obtain
$\mathrm{m_X Cl}(f^{-1}(\mathrm{m_Y^j Int}(m_{ij}\mathrm{Cl}_\theta(B)))) \subset f^{-1}(m_{ij}\mathrm{Cl}_\theta(\mathrm{m_Y^j Int}(m_{ij}\mathrm{Cl}_\theta(B))))$
$= f^{-1}(\mathrm{m_Y^i Cl}(\mathrm{m_Y^j Int}(m_{ij}\mathrm{Cl}_\theta(B)))) \subset f^{-1}(\mathrm{m_Y^i Cl}(m_{ij}\mathrm{Cl}_\theta(B)))$
$= f^{-1}(m_{ij}\mathrm{Cl}_\theta(B))$.

(4) ⇒ (1): Let V be any m_Y^j-open set of Y. Then by Lemma 4.19, $V \subset \mathrm{m_Y^j Int}(\mathrm{m_Y^i Cl}(V)) = \mathrm{m_Y^j Int}(m_{ij}\mathrm{Cl}_\theta(V))$ and we have
$\mathrm{m_X Cl}(f^{-1}(V)) \subset \mathrm{m_X Cl}(f^{-1}(\mathrm{m_Y^j Int}(m_{ij}\mathrm{Cl}_\theta(V)))) \subset f^{-1}(m_{ij}\mathrm{Cl}_\theta(V))$
$= f^{-1}(\mathrm{m_Y^i Cl}(V))$.

Thus we obtain $\mathrm{m_X Cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{m_Y^i Cl}(V))$. It follows from Theorem 4.15 that f is weakly $M(i,j)$-continuous. □

DEFINITION 4.21. Let (X, m_X) be an m-space and A be a subset of X. The *m_X-frontier* of A [43], $\mathrm{m_X Fr}(A)$, is defined as follows: $\mathrm{m_X Fr}(A) = \mathrm{m_X Cl}(A) \cap \mathrm{m_X Cl}(X \setminus A) = \mathrm{m_X Cl}(A) \setminus \mathrm{m_X Int}(A)$.

THEOREM 4.22. *Let (X, m_X) be an m-space and (Y, m_Y^1, m_Y^2) a bi-m-space. The set of all points x of X at which a function $f : (X, m_X) \to$*

(Y, m^1_Y, m^2_Y) is not weakly $M(i,j)$-continuous is identical with the union of all m_X-frontiers of the inverse images of the m^j_Y-closure of m^i_Y-open sets of Y contining f(x).

Proof. Let x be a point of X at which f is not weakly $M(i,j)$-continuous. Then, there exists a m^i_Y-open set V of Y containing $f(x)$ such that $U \cap (X \setminus f^{-1}(\mathrm{m^j_Y Cl}(V))) \neq \emptyset$ for every m_X-open set U of X containing x. By Lemma 3.6, $x \in \mathrm{m_X Cl}(X \setminus f^{-1}(\mathrm{m^j_Y Cl}(V)))$. Since $x \in f^{-1}(\mathrm{m^j_Y Cl}(V))$, $x \in \mathrm{m_X Cl}(f^{-1}(\mathrm{m^j_Y Cl}(V)))$ and $x \in \mathrm{m_X Fr}(f^{-1}(\mathrm{m^j_Y Cl}(V)))$.

Conversely, if f is weakly $M(i,j)$-continuous at x, then for each m^i_Y-open set V of Y containing $f(x)$, there exists an m_X-open set U containing x such that $f(U) \subset \mathrm{m^j_Y Cl}(V)$ and hence $x \in U \subset f^{-1}(\mathrm{m^j_Y Cl}(V))$. Therefore, we obtain that $x \in \mathrm{m_X Int}(f^{-1}(\mathrm{m^j_Y Cl}(V)))$ and hence $x \notin \mathrm{m_X Fr}(f^{-1}(\mathrm{m^j_Y Cl}(V)))$. □

COROLLARY 4.23. *The set of all points x of X at which a function $f : (X, m_X) \to (Y, m_Y)$ is not weakly M-semicontinuous is identical with the union of all $mSO(X)$-frontiers of the inverse images of the m_Y-semi-closure of m_Y-open sets of Y contining f(x).*

5. Weak $M(i,j)$-continuity and M-continuity

DEFINITION 5.1. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *M-continuous* [47] at $x \in X$ if for each $V \in m_Y$ containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$. The function f is said to be M-continuous if it has this property at each $x \in X$.

DEFINITION 5.2. A function $f : (X, m_X) \to (Y, m^1_Y, m^2_Y)$ is said to be *M-i-continuous* if $f : (X, m_X) \to (Y, m^i_Y)$ is M-continuous.

LEMMA 5.3. *For a function $f : (X, m_X) \to (Y, m^1_Y, m^2_Y)$, the following properties are equivalent:*
(1) *f is M-i-continuous;*
(2) $f^{-1}(V) = \mathrm{m_X Int}(f^{-1}(V))$ *for every m^i_Y-open set V of Y;*
(3) $f^{-1}(F) = \mathrm{m_X Cl}(f^{-1}(F))$ *for every m^i_Y-closed set F of Y.*

Proof. The proof follows from Definition 5.2 and Theorem 3.1 of [47]. □

DEFINITION 5.4. A bi-m-space (X, m^1_X, m^2_X) is said to be *m_{ij}-regular* if for each $x \in X$ and each m^i_X-open set U containing x, there exists an m^i_X-open set V such that $x \in V \subset \mathrm{m^j_X Cl}(V) \subset U$.

Lemma 5.5. *If A bi-m-space* (X, m_X^1, m_X^2) *is* m_{ij}*-regular, then* $m_{ij}\mathrm{Cl}_\theta(F) = F$ *for every* m_X^i*-closed set F of X.*

Proof. Let F be any m_X^i-closed set of X and $x \in m_{ij}\mathrm{Cl}_\theta(F)$, then $\mathrm{m_X^j Cl}(U) \cap F \neq \emptyset$ for every m_X^i-open set U containing x. Since X is m_{ij}-regular, there exists an m_X^i-open set V such that $x \in V \subset \mathrm{m_X^j Cl}(V) \subset U$. Since $x \in V \in m_X^i$, $\mathrm{m_X^j Cl}(V) \cap F \neq \emptyset$. This implies that $U \cap F \neq \emptyset$ and hence $x \in \mathrm{m_X^i Cl}(F)$. Then, we have $F \subset m_{ij}\mathrm{Cl}_\theta(F) \subset \mathrm{m_X^i Cl}(F) = F$. □

Theorem 5.6. *Let* (Y, m_Y^1, m_Y^2) *be an* m_{ij}*-regular bi-m-space, where* m_Y^1 *and* m_Y^2 *have property* $\mathcal{B}$*. For a function* $f : (X, m_X) \to (Y, m_Y^1, m_Y^2)$*, the following properties are equivalent:*

(1) *f is M-i-continuous;*
(2) $f^{-1}(m_{ij}\mathrm{Cl}_\theta(B)) = \mathrm{m_X Cl}(f^{-1}(m_{ij}\mathrm{Cl}_\theta(B)))$ *for every subset B of Y;*
(3) *f is weakly* $M(i, j)$*-continuous;*
(4) $f^{-1}(F) = \mathrm{m_X Cl}(f^{-1}(F))$ *for every* m_{ij}*-θ-closed set F of Y;*
(5) $f^{-1}(V) = \mathrm{m_X Int}(f^{-1}(V))$ *for every* m_{ij}*-θ-open set V of Y.*

Proof. (1) ⇒ (2): Let B be any subset of Y. By Lemma 4.18, $m_{ij}\mathrm{Cl}_\theta(B)$ is m_Y^i-closed in Y. It follows from Lemma 5.5 that $f^{-1}(m_{ij}\mathrm{Cl}_\theta(B)) = \mathrm{m_X Cl}(f^{-1}(m_{ij}\mathrm{Cl}_\theta(B)))$.

(2) ⇒ (3): Let B be any subset of Y. Then by (2) and Lemma 3.5 we have

$$\mathrm{m_X Cl}(f^{-1}(B)) \subset \mathrm{m_X Cl}(f^{-1}(m_{ij}\mathrm{Cl}_\theta(B))) = f^{-1}(m_{ij}\mathrm{Cl}_\theta(B)).$$

By Theorem 4.20, f is weakly $M(i, j)$-continuous.

(3) ⇒ (4): Let F be any m_{ij}-θ-closed set of Y. Then by Theorem 4.20, $\mathrm{m_X Cl}(f^{-1}(F)) \subset f^{-1}(m_{ij}\mathrm{Cl}_\theta(F)) = f^{-1}(F)$. By Lemma 3.5, $f^{-1}(F) = \mathrm{m_X Cl}(f^{-1}(F))$.

(4) ⇒ (5): Let V be any m_{ij}-θ-open set of Y. Then by (4), $X \setminus f^{-1}(V) = f^{-1}(Y \setminus V) = \mathrm{m_X Cl}(f^{-1}(Y \setminus V)) = X \setminus \mathrm{m_X Int}(f^{-1}(V))$. Hence $f^{-1}(V) = \mathrm{m_X Int}(f^{-1}(V))$.

(5) ⇒ (1): Since (Y, m_Y^1, m_Y^2) is m_{ij}-regular, by Lemma 5.5 $m_{ij}\mathrm{Cl}_\theta(B) = B$ for every m_Y^i-closed set B of Y and hence every m_Y^i-open set is m_{ij}-θ-open. Therefore, by Lemma 5.3, f is M-i-continuous. □

6. Minimal structures in bitopological spaces

Definition 6.1. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) (i,j)-*semi-open* [25] if $A \subset j\mathrm{Cl}(i\mathrm{Int}(A))$, where $i \neq j$, i, $j = 1$, 2,
(2) (i,j)-*preopen* [14] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(A))$, where $i \neq j$, i, $j = 1$, 2,
(3) (i,j)-α-*open* [15] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(i\mathrm{Int}(A)))$, where $i \neq j$, i, $j = 1$, 2,
(4) (i,j)-*semi-preopen* (briefly (i,j)-*sp-open*) [20] if there exists an (i,j)-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j, i, j = 1, 2$.

The family of all (i,j)-semi-open (resp. (i,j)-preopen, (i,j)-α-open, (i,j)-*sp*-open) sets of (X, τ_1, τ_2) is denoted by $(i,j)\mathrm{SO}(X)$ (resp. $(i,j)\mathrm{PO}(X)$, $(i,j)\alpha(X)$, $(i,j)\mathrm{SPO}(X)$).

Remark 6.2. Let (X, τ_1, τ_2) be a bitopological space. Then $(i,j)\mathrm{SO}(X)$, $(i,j)\mathrm{PO}(X)$, $(i,j)\alpha(X)$ and $(i,j)\mathrm{SPO}(X)$ are all m-structures on X.

In the following, we denote by $m_{ij}(X)$ a minimal structure on X determined by τ_1 and τ_2 as in Definition 6.1. If $m_{ij}(X) = (i,j)\mathrm{SO}(X)$ (resp. $(i,j)\mathrm{PO}(X)$, $(i,j)\alpha(X)$, $(i,j)\mathrm{SPO}(X)$), by Definition 3.3 for a subset A of X we have

$m_{ij}\mathrm{Cl}(A) = (i,j)\mathrm{sCl}(A)$ [25] (resp. $(i,j)\mathrm{pCl}(A)$ [20], $(i,j)\alpha\mathrm{Cl}(A)$ [36], $(i,j)\mathrm{spCl}(A)$) [20],

$m_{ij}\mathrm{Int}(A) = (i,j)\mathrm{sInt}(A)$ (resp.$(i,j)\mathrm{pInt}(A)$, $(i,j)\alpha\mathrm{Int}(A)$, $(i,j)\mathrm{spInt}(A)$).

Remark 6.3. Let (X, τ_1, τ_2) be a bitopological space. Then the families $(i,j)\mathrm{SO}(X)$, $(i,j)\mathrm{PO}(X)$, $(i,j)\alpha(X)$ and $(i,j)\mathrm{SPO}(X)$ are all m-structures on X satisfying property $\mathcal{B}$ by Theorem 2 of [25] (resp. Theorem 4.2 of [16] or Theorem 3.2 of [20], Theorem 5 of [36], Theorem 3.2 of [20]).

Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. A point x of X is called an (i,j)-*semi-θ-adherent point* [21] of A, if $A \cap (j,i)\mathrm{sCl}(U) \neq \emptyset$ for every (i,j)-semi-open set U containing x. The set of all (i,j)-semi-θ-adherent points of A is called the (i,j)-*semi-θ-closure* of A and is denoted by $(i,j)\mathrm{sCl}_\theta(A)$. If $A = (i,j)\mathrm{sCl}_\theta(A)$, then A is said to be (i,j)-*semi-θ-closed.* A subset A of X is said to be (i,j)-*semi-θ-open* if $X \setminus A$ is (i,j)-semi-θ-closed. A bitopological space (X, τ_1, τ_2) is said to be *(i, j)-semi-regular* [21] if for each (i,j)-semi-open set G and each $x \in G$, there exists an (i,j)-semi-open set U such that $x \in U \subset (j,i)\mathrm{sCl}(U) \subset G$.

Definition 6.4. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be (i, j)-*weakly semi-continuous* [18] (resp. (i, j)-*weakly precontinuous* [42]) if for each $x \in X$ and each σ_i-open set V of Y containing $f(x)$, there exists an (i, j)-semi-open (resp. (i, j)-preopen) set U containing x such that $f(U) \subset j\mathrm{Cl}(V)$.

Hence, a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-weakly semi-continuous (resp. (i, j)-weakly precontinuous) if and only if a function $f : (X, (i, j)\mathrm{SO}(X)) \to (Y, \sigma_1, \sigma_2)$ (resp. $f : (X, (i, j)\mathrm{PO}(X)) \to (Y, \sigma_1, \sigma_2))$ is weakly $M(i, j)$-continuous.

Definition 6.5. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be (i, j)-*quasi irresolute* [21] (resp. (i, j)-*almost s-continuous* [22]) if for each $x \in X$ and each (i, j)-semi-open set V of Y containing $f(x)$, there exists $U \in (i, j)\mathrm{SO}(X)$ (resp. $U \in \tau_i$) containing x such that $f(U) \subset (j, i)\mathrm{sCl}(V)$.

Hence, a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-quasi irresolute (resp. (i, j)-almost s-continuous) if and only if a function $f : (X, (i, j)\mathrm{SO}(X)) \to (Y, (i, j)\mathrm{SO}(Y), (j, i)\mathrm{SO}(Y))$ (resp. $f : (X, \tau_i) \to (Y, (i, j)\mathrm{SO}(Y), (j, i)\mathrm{SO}(Y)))$ is weakly $M(i, j)$-continuous.

Therefore, by the results of Sections 4 and 5, we can obtain the results established in [18], [21], [22], [42] and [43].

Remark 6.6. Let (X, τ_1, τ_2) be a bitopological space. Then the families $(i, j)\mathrm{SO}(X)$ and $(i, j)\mathrm{PO}(X)$ have property $\mathcal{B}$.

(1) If we set $m_X = (i, j)\mathrm{SO}(X)$, $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$, then by Theorems 4.14 and 4.15 we obtain the results established in Theorems 2.1 and 2.2 in [18] and Theorems 3.1 and 3.2 in [46].
(2) If we set $m_X = (i, j)\mathrm{PO}(X)$, $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$, then by Theorems 4.14 and 4.15 we obtain the results established in Theorems 3.1 and 3.2 of [42].
(3) If we set $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$, then by Theorems 4.14 and 4.15 we obtain the results established in Theorem 3.1 of [43].
(4) If we set $m_X = (i, j)\mathrm{SO}(X)$, $m_Y^1 = (i, j)\mathrm{SO}(Y)$ and $m_Y^2 = (j, i)\mathrm{SO}(Y)$, then by Theorem 4.14 we obtain the results established in Proposition 15 (1), (4), (5) of [21] and Theorem 2.4 (1), (2) of [19].

Remark 6.7. (1) If we set $m_X = (i, j)\mathrm{SO}(X)$, $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$, then by Theorem 5.6 we obtain the results established in Theorem 3.2 in [51].

(2) If we set $m_X = (i,j)\mathrm{PO}(X)$, $m^1_Y = \sigma_1$ and $m^2_Y = \sigma_2$, then by Theorem 5.6 we obtain the results established in Theorems 3.3 of [42].
(3) If we set $m^1_Y = \sigma_1$ and $m^2_Y = \sigma_2$, then by Theorem 5.6 we obtain the results established in Theorem 4.1 of [43].
(4) If we set $m_X = m_{ij}\mathrm{SO}(X)$, $m^1_Y = (i,j)\mathrm{SO}(Y)$ and $m^2_Y = (j,i)\mathrm{SO}(Y)$, then by Theorem 5.6 we obtain the results established in Theorem 2.7 of [19].

Remark 6.8. (1) If we set $m_X = (i,j)\mathrm{SO}(X)$, $m^1_Y = \sigma_1$ and $m^2_Y = \sigma_2$, then by Theorem 4.22 we obtain the results established in Theorem 4.3 in [51].
(2) If we set $m_X = (i,j)\mathrm{PO}(X)$, $m^1_Y = \sigma_1$ and $m^2_Y = \sigma_2$, then by Theorem 4.22 we obtain the results established in Theorem 4.3 of [42].

By Theorem 4.15, we can obtain the characterizations of (i,j)-almost s-continuous functions.

Corollary 6.9. *For a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*
(1) *f is (i,j)-almost s-continuous;*
(2) *for every subset B of Y,*

$$i\mathrm{Cl}(f^{-1}((j,i)\mathrm{sInt}((i,j)\mathrm{sCl}(B)))) \subset f^{-1}((i,j)\mathrm{sCl}(B));$$

(3) *for every semi-regular closed set F of Y,*

$$i\mathrm{Cl}(f^{-1}((j,i)\mathrm{sInt}(F))) \subset f^{-1}(F);$$

(4) *for every (j,i)-semi-open set V of Y, $i\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}((i,j)\mathrm{sCl}(V))$.*

Remark 6.10. (1) If we set $m_X = (i,j)\mathrm{SO}(X)$, $m^1_Y = \sigma_1$ and $m^2_Y = \sigma_2$, then by Theorem 4.20 we obtain the results established in Theorem 3.3 in [46].
(2) If we set $m_X = (i,j)\mathrm{PO}(X)$, $m^1_Y = \sigma_1$ and $m^2_Y = \sigma_2$, then by Theorem 4.20 we obtain the results established in Theorems 3.2 of [42].
(3) If we set $m^1_Y = \sigma_1$ and $m^2_Y = \sigma_2$, then by Theorem 5.6 we obtain the results established in Theorem 4.20 of [43].
(4) If we set $m_X = m_{ij}\mathrm{SO}(X)$, $m^1_Y = (i,j)\mathrm{SO}(Y)$ and $m^2_Y = (j,i)\mathrm{SO}(Y)$, then by Theorem 4.20 we obtain the results established in Theorem 2.3 of [19].

(5) If we set $m_X = \tau_i$, $m^1_Y = (i,j)\mathrm{SO}(Y)$ and $m^2_Y = (j,i)\mathrm{SO}(Y)$, then by Theorem 4.20 we obtain new characterizations of (i,j)-almost s-continuous functions.

COROLLARY 6.11. *For a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, *the following properties are equivalent:*

(1) f *is* (i,j)*-almost* s*-continuous;*
(2) *for every subset* A *of* X, $f(i\mathrm{Cl}(A)) \subset (i,j)\mathrm{sCl}_\theta(f(A))$*;*
(3) *for every subset* B *of* Y, $i\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}((i,j)\mathrm{sCl}_\theta(B))$*;*
(4) *for every subset set* B *of* Y, $i\mathrm{Cl}(f^{-1}((j,i)\mathrm{Int}((i,j)\mathrm{sCl}_\theta(B)))) \subset f^{-1}((i,j)\mathrm{sCl}_\theta(B))$.

For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, we define $D_{(i,j)s}(f)$ as follows:

$D_{(i,j)s}(f) = \{x \in X : f$ is not (i,j)-weakly semi-continuous at $x\}$.

Then, by Theorems 4.8 and 4.11 we obtain the following corollary.

COROLLARY 6.12. *For a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, *the following properties hold:*

$$\begin{aligned} D_{(i,j)s}(f) &= \textstyle\bigcup_{G\in\sigma_i}\{f^{-1}(G) \setminus (i,j)\mathrm{sInt}(f^{-1}(j\mathrm{Cl}(G)))\} \\ &= \textstyle\bigcup_{F\in\mathcal{F}} \{(i,j)\mathrm{sCl}(f^{-1}(j\mathrm{Int}(F)) \setminus f^{-1}(F)\}, \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)} \{(i,j)\mathrm{sCl}(f^{-1}(j\mathrm{Int}(i\mathrm{Cl}(B)))) \setminus f^{-1}(i\mathrm{Cl}(B))\} \\ &= \textstyle\bigcup_{R\in\mathcal{R}} \{(i,j)\mathrm{sCl}(f^{-1}(j\mathrm{Int}(R))) \setminus f^{-1}(R)\}, \\ &= \textstyle\bigcup_{G\in\sigma_j}\{(i,j)\mathrm{sCl}(f^{-1}(G)) \setminus f^{-1}(i\mathrm{Cl}(G))\}, \end{aligned}$$

where $\mathcal{F}$ *is the family of* σ_i*-closed sets of* Y *and* $\mathcal{R}$ *is the family of* (i,j) *regular closed sets of* Y.

REMARK 6.13. If we set as follows:

(1) $m_X = (i,j)\mathrm{PO}(X)$, $m^1_Y = \sigma_1$ and $m^2_Y = \sigma_2$,
(2) $m_X = (i,j)\mathrm{SO}(X)$, $m^1_Y = (i,j)\mathrm{SO}(Y)$ and $m^2_Y = (j,i)\mathrm{SO}(Y)$,
(3) $m_X = \tau_i$, $m^1_Y = (i,j)\mathrm{SO}(Y)$ and $m^2_Y = (j,i)\mathrm{SO}(Y)$,

then by Theorems 4.8 and 4.11 we obtain the similar corollaries with Corollary 6.12 concerning (1) (i,j)-weakly precontinuity, (2) (i,j)-quasi irresoluteness, and (3) (i,j)-almost s-continuity, respectively.

7. Some properties of weak $M(i,j)$-continuity

DEFINITION 7.1. A bi-m-space (X, m^1_X, m^2_X) is said to be M_{ij}*-Urysohn* if for each distinct points x, y of X there exist an m^i_X-open set U and an m^j_X-open set V such that $x \in U$ and $y \in V$ and $\mathrm{m^i_X Cl}(U) \cap \mathrm{m^i_X Cl}(V) = \emptyset$ for $i \neq j, i, j = 1, 2$.

Remark 7.2. If (X, τ_1, τ_2) is a bitopological space, then we obtain the definition of a pairwise Urysohn space [7].

Definition 7.3. A function $f : (X, m_X) \to (Y, m_Y)$ is said to have a *strongly M-closed graph* [49] if for each $(x, y) \in (X \times Y) - \mathrm{G}(f)$, there exist $U \in m_X$ containing x and $V \in m_Y$ containing y such that $[U \times \mathrm{m_Y Cl}(V)] \cap \mathrm{G}(f) = \emptyset$.

Lemma 7.4. (Popa and Noiri [49]) A function $f : (X, m_X) \to (Y, m_Y)$ has a strongly M-closed graph if and only if for each $(x, y) \in (X \times Y) - \mathrm{G}(f)$, there exist $U \in m_X$ containing x and $V \in m_Y$ containing y such that $f(U) \cap \mathrm{m_Y Cl}(V) = \emptyset$.

Definition 7.5. A function $f : (X, m_X) \to (Y, m_Y)$ is said to have a *strongly M-semi-closed graph* [33] (resp. *strongly M-preclosed graph* [34]) if for each $(x, y) \in (X \times Y) - \mathrm{G}(f)$, there exist an m-semi-open (resp. m-preopen) set U containing x and $V \in m_Y$ containing y such that $[U \times \mathrm{m_Y Cl}(V)] \cap \mathrm{G}(f) = \emptyset$.

Remark 7.6. If $m_X = \mathrm{mSO}(X)$ (resp. $\mathrm{mPO}(X)$) in Definition 7.3, then we obtain Definition 7.5.

Definition 7.7. A function $f : (X, m_X) \to (Y, m_Y^1, m_Y^2)$ is said to have an *M_{ij}-strongly closed graph* if $f : (X, m_X) \to (Y, m_Y^i)$ has a strongly M-closed graph.

Theorem 7.8. *If a function $f : (X, m_X) \to (Y, m_Y^1, m_Y^2)$ is weakly $M(i, j)$-continuous and Y is M_{ij}-Urysohn, then f has an M_{ij}-strongly closed graph.*

Proof. Let $(x, y) \in (X \times Y) - \mathrm{G}(f)$. Then $y \neq f(x)$. Since Y is M_{ij}-Urysohn, there exist an m_Y^i-open set U and an m_Y^j-open set V such that $f(x) \in U$ and $y \in V$, respectively, such that $\mathrm{m_Y^j Cl}(U) \cap \mathrm{m_Y^i Cl}(V) = \emptyset$. By weak $M(i, j)$-continuity of f, there exists an m_X-open set G containing x such that $f(G) \subset \mathrm{m_Y^j Cl}(U)$; hence $f(G) \cap \mathrm{m_Y^i Cl}(V) = \emptyset$. By Lemma 7.4, f has an M_{ij}-strongly closed graph. □

Remark 7.9. If $m_X = \mathrm{mSO}(X)$ (resp. $\mathrm{mPO}(X)$), then we obtain Theorem 3.9 of [33] (resp. Theorem 3.9 of [34])

Definition 7.10. An m-space (X, m_X) is said to be m-T_2 [47] if for each distinct points x, y of X there exist m_X-open sets U and V such that $x \in U$ and $y \in V$, respectively, such that $U \cap V = \emptyset$.

Theorem 7.11. *If a function* $f : (X, m_X) \to (Y, m_Y^1, m_Y^2)$ *is a weakly* $M(i,j)$*-continuous injection, where* $m_Y^i \subset m_Y^j$*, and* f *has an* M_{ij}*-strongly closed graph, then* (X, m_X) *is* m-T_2.

Proof. Let $x_1 \neq x_2$. Then $f(x_1) \neq f(x_2)$ and $(x_1, f(x_2)) \notin G(f)$. Since $G(f)$ is M_{ij}-strongly closed, there exist $U \in m_X$ containing x_1 and $V \in m_Y^i$ containing $f(x_2)$ such that $f(U) \cap \mathrm{m^i_Y Cl}(V) = \emptyset$. Since f is weakly $M(i,j)$-continuous and $f(x_2) \in V \in m_Y^i$, there exists an m_X-open set W containing x_2 such that $f(W) \subset \mathrm{m^j_Y Cl}(V)$. Since $m_Y^i \subset m_Y^j$, it follows that $\mathrm{m^j_Y Cl}(V) \subset \mathrm{m^i_Y Cl}(V)$. Hence $f(U) \cap f(W) = \emptyset$. This implies that $U \cap W = \emptyset$ and hence (X, m_X) is m-T_2. □

Remark 7.12. If $f : (X, \mathrm{mSO}(X)) \to (Y, m_Y, \mathrm{mSO}(Y))$ (resp. $f : (X, \mathrm{mPO}(X)) \to (Y, m_Y, \mathrm{mPO}(Y))$, then we obtain Theorem 3.10 of [33] (resp. Theorem 3.10 of [34]).

Theorem 7.13. *If* $f : (X, m_X) \to (Y, m_Y^1, m_Y^2)$ *is a function such that*

(1) (Y, m_Y^1, m_Y^2) *is* M_{ij}*-Urysohn,*

(2) $f(x_i) \neq f(x_j)$,

(3) f *is weakly* $M(i,j)$*-continuous at* x_i *and weakly* $M(j,i)$*-continuous at* x_j *for distinct points* $x_i, x_j \in X$,

then (X, m_X) *is* m-T_2.

Proof. Let x_1 and x_2 be distinct points and $y_i = f(x_i)$ for $i = 1, 2$. Then $y_1 \neq y_2$. Since (Y, m_Y^1, m_Y^2) is M_{ij}-Urysohn, there exist an m_Y^i-open set V_i and an m_Y^j-open set V_j such that $y_i \in V_i$, $y_j \in V_j$ and $\mathrm{m^j_X Cl}(V_i) \cap \mathrm{m^i_X Cl}(V_j) = \emptyset$. Since f is weakly $M(i,j)$-continuous at x_i and $f(x_i) \in V_i \in m_Y^i$, there exists $U_i \in m_X$ containing x_i such that $f(U_i) \subset \mathrm{m^j_Y Cl}(V_i)$. Since f is weakly $M(j,i)$-continuous at x_j and $f(x_j) \in V_j \in m_Y^j$, there exists $U_j \in m_X$ containing x_j such that $f(U_j) \subset \mathrm{m^i_Y Cl}(V_j)$. Hence $f(U_i) \cap f(U_j) = \emptyset$ which implies $U_i \cap U_j = \emptyset$. Therefore, (X, m_X) is m-T_2. □

Definition 7.14. An m-space (X, m_X) is said to be *m-connected* [47], [48] if it is not expressed as the union of two disjoint nonempty m-open sets of X.

Definition 7.15. A bi-m-space (X, m_X^1, m_X^2) is said to be *pairwise m-connected* if it cannot be expressed as the union of two nonempty disjoint sets $U \in m_X^1$ and $V \in m_X^2$.

THEOREM 7.16. *Let a function $f : (X, m_X) \to (Y, m_Y^1, m_Y^2)$ be a function, where m_X, m_Y^1 and m_Y^2 have property $\mathcal{B}$. If f is a weakly $M(i,j)$-continuous and weakly $M(j,i)$-continuous surjection and (X, m_X) is m-connected, then (Y, m_Y^1, m_Y^2) is pairwise m-connected.*

Proof. Suppose that (Y, m_Y^1, m_Y^2) is not pairwise m-connected. Then, there exist a σ_Y^i-open set U and a σ_Y^j-open set V such that $U \neq \emptyset, V \neq \emptyset, U \cap V = \emptyset$ and $U \cup V = Y$. Since f is surjective, $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty. Moreover $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and $f^{-1}(U) \cup f^{-1}(V) = X$. Since f is weakly $M(i,j)$-continuous and weakly $M(j,i)$-continuous, by Theorem 4.14 we have $f^{-1}(U) \subset \mathrm{m_X Int}(f^{-1}(\mathrm{m_Y^j Cl}(U)))$ and $f^{-1}(V) \subset \mathrm{m_X Int}(f^{-1}(\mathrm{m_Y^i Cl}(V)))$. Since $V = X \setminus U \in m_Y^j$ and $U = X \setminus V \in m_Y^i$, by Lemma 3.9 $U = \mathrm{m_Y^j Cl}(U)$ and $V = \mathrm{m_Y^i Cl}(V)$. Therefore, we have $f^{-1}(U) \subset \mathrm{m_X Int}(f^{-1}(U))$ and $f^{-1}(V) \subset \mathrm{m_X Int}(f^{-1}(V))$. Hence by Lemma 3.5 $f^{-1}(U) = \mathrm{m_X Int}(f^{-1}(U))$ and $f^{-1}(V) = \mathrm{m_X Int}(f^{-1}(V))$. By Lemma 3.9, $f^{-1}(U)$ and $f^{-1}(V)$ are m_X-open sets in (X, m_X). This shows that (X, m_X) is not m-connected. □

DEFINITION 7.17. A subset K of an m-space (X, m_X) is said to be *m-compact* if [47], [48] if every cover of K by m_X-open sets has a finite subcover.

DEFINITION 7.18. A subset K of a bi-m-space (Y, m_Y^1, m_Y^2) is said to be *M_{ij}-quasi H-closed relative to Y* if for each cover $\{U_\alpha : \alpha \in \Delta\}$ of K by m_Y^i-open sets of Y, there exists a finite subset Δ_0 of Δ such that $K \subset \cup\{\mathrm{m_Y^j Cl}(U_\alpha) : \alpha \in \Delta_0\}$.

THEOREM 7.19. *If $f : (X, m_X) \to (Y, m_Y^1, m_Y^2)$ is weakly $M(i,j)$-continuous and K is an m-compact set in (X, m_X), then $f(K)$ is M_{ij}-quasi H-closed relative to Y.*

Proof. Let K be an m-compact set of X and $\{V_\alpha : \alpha \in \Delta\}$ any cover of $f(K)$ by m_Y^i-open sets of Y. For each $x \in K$, there exists $\alpha(x) \in \Delta$ such that $f(x) \in V_{\alpha(x)}$. Since f is weakly $M(i,j)$-continuous, there exists $U_x \in m_X$ containing x such that $f(U_x) \subset \mathrm{m_Y^j Cl}(V_{\alpha(x)})$. The family $\{U_x : x \in K\}$ is a cover of K by m_X-open sets. Since K is m-compact, there exist a finite number of points, say $x_1, x_2, ..., x_n$ in K such that $K \subset \cup\{U_{x_k} : x_k \in K, k = 1, 2, ..., n\}$. Therefore, we obtain $f(K) \subset \cup\{f(U_{x_k}) : x_k \in K, k = 1, 2, ..., n\} \subset \cup\{\mathrm{m_Y^j Cl}(V_{\alpha(x_k)}) : x_k \in K, k = 1, 2, ..., n\}$. This shows that $f(K)$ is M_{ij}-quasi H-closed relative to Y. □

REMARK 7.20. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces.

(1) If we set $m_X = (i,j)\mathrm{PO}(X)$, $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$, then by Theorem 7.19 we obtain the results established in Theorem 6.3 of [42].
(2) If we set $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$, then by Theorem 7.19 we obtain the results established in Theorem 5.3 of [43].
(3) If we set $m_X = (i,j)\mathrm{SO}(X)$, $m_Y^1 = (i,j)\mathrm{SO}(Y)$ and $m_Y^2 = (j,i)\mathrm{SO}(Y)$, then by Theorem 7.19 we obtain the results established in Proposition 17 of [21].
(4) If we set $m_X = \tau_i$, $m_Y^1 = (i,j)\mathrm{SO}(Y)$ and $m_Y^2 = (j,i)\mathrm{SO}(Y)$, then by Theorem 7.19 we obtain the results established in Theorem 7 of [22].

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2979-1 Shiokita-cho, Hinagu
Yatsushiro-shi, Kumamoto-ken, 869-5142 JAPAN
E-mail: `t.noiri@nifty.com`

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Department of Mathematics, Faculty of Sciences
"Vasile Alecsandri" University of Bacău
Bacău 6001115, ROMANIA
E-mail: `vpopa@ub.ro`