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EXISTENCE OF THREE SOLUTIONS OF NON-HOMOGENEOUS BVPS FOR SINGULAR DIFFERENTIAL SYSTEMS WITH LAPLACIAN OPERATORS

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Abstract. This paper is concerned with a kind of non-homogeneous boundary value problems for singular second order differential systems with Laplacian operators. Using multiple fixed point theorems, sufficient conditions to guarantee the existence of at least three solutions of this kind of boundary value problems are established. An example is presented to illustrate the main results.

1. Introduction

In recent years, existence of solutions of multi-point boundary-value problems (BVPs for short) for second order differential equations or higher order differential equations on finite interval has been studied by different authors, see papers [4-8, 12-19, 22-28]. Different methods are used in these papers such as the Guo-Krasnoselskii fixed point theorem [6], the fixed-point theorem due to Avery and Peterson [3], the Leggett-Williams fixed point theorem [2, 3], the upper and lower solution methods with monotone iterative techniques [24, 25], the five functional fixed point theorem [1], and the Mawhin coincidence degree theory [6], see for examples Ma in [7, 20, 21] and the text book [6]. In many applications, BVPs consist of differential equations coupled with nonhomogeneous BCs, see [16, 17]. Papers [9-11] may be first group

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of papers concerned with BVPs with two parameter multi-point non-homogeneous BCs for second order ordinary differential equations. In these papers, the existence of lower and upper solutions with certain relations are supposed.

In [28, 29, 30, 31], by using the upper and lower solution method and fixed point index theory, authors studied the existence, non-existence and multiplicity of positive solutions of the following BVP of second order differential systems

$$(1.1)\qquad \begin{cases} [\phi_p(u'(t))]' + \lambda h_1(t) f(u(t), v(t)) = 0, & t \in (0,1), \\ [\phi_p(v'(t))]' + \mu h_2(t) g(u(t), v(t)) = 0, & t \in (0,1), \\ u(0) = a \geq 0,\ v(0) = b \geq 0,\ u(1) = 0,\ v(1) = 0, \end{cases}$$

where $\phi_p(x) = |x|^{p-2}x$ with $p > 1$, λ, μ are non-negative real parameters, $h_1, h_2 \in C((0,1),(0,+\infty))$, $f, g \in C([0,\infty) \times [0,+\infty),[0,+\infty))$, $h_i(i = 1,2)$ are supposed to be singular at $t = 0$ in [31], while $h_i \in C([0,1],[0,+\infty))$ and $\phi_2(x) = x$ in [30].

Motivated by [29, 30, 31], the purpose of this paper is to investigate the following BVP for singular second order differential systems with Laplacian operators

$$(1.2)\qquad \begin{cases} (p(t)\phi(x'(t)))' + f(t, y(t), y'(t)) = 0, & a.e., t \in (0,1), \\ (q(t)\psi(y'(t)))' + g(t, x(t), x'(t)) = 0, & a.e., t \in (0,1), \\ x(0) = \sum_{i=1}^m a_i x(\xi_i) + A, \\ x(1) = \sum_{i=1}^m b_i x(\xi_i) + B, \\ y(0) = \sum_{i=1}^m c_i y(\xi_i) + C, \\ y(1) = \sum_{i=1}^m d_i y(\xi_i) + D \end{cases}$$

where

- $0 < \xi_1 < \cdots < \xi_m < 1$, $A, B, C, D \in R$, $a_i, b_i, c_i, d_i \geq 0$ for all $i = 1, \cdots, m$;
- f and g defined on $(0,1) \times R \times R$, f is a $p-$Caratheodory function and g a $q-$Caratheodory function (see the definitions in Section 2) and f, g may be singular at $t = 0, 1$;
- both p and q are continuous on $(0,1)$ and nonnegative, both p and q may be singular at $t = 0, 1$ or vanish at $t = 0, 1$, i.e., $\lim_{t\to 0} p(t) = \lim_{t\to 1} p(t) = 0$, $\lim_{t\to 0} q(t) = \lim_{t\to 1} q(t) = 0$;
- $\phi(x) = |x|^{m-2}x$ with $m > 1$ and $\psi(x) = |x|^{n-2}x$ with $n > 1$ are Laplacian operators and their inverse operators are denoted by ϕ^{-1} and ψ^{-1} respectively.

A pair of functions $x, y : [0,1] \to R$ is called a solution of BVP(1.2) if $x, y \in C^0[0,1]$, $x', y' \in C^0(0,1)$, $[p\phi(x')]', [q\Phi(y')]' \in L^1(0,1)$ and all equations in (1.2) are satisfied, where $C^0(I)$ is the set of all functions continuous on the interval I and $L^1(0,1)$ the set of all functions on $\sigma:(0,1)\to R$ such that $\int_0^1|\sigma(s)|ds<+\infty$.

Sufficient conditions for the existence of at least three solutions of BVP(1.2) are established by using the five functional fixed point theorem [2]. The Green′s functions are not used in the proofs of the main results. Different from [28-31], we don't use the assumptions of the existence of lower and upper solutions and f,g may be singular at $t=0,1$.

The remainder of this paper is organized as follows: the preliminary results are given in Section 2, the main results and their proofs are presented in Section 3, and an example is given in Section 4.

2. Preliminary results

To the reader′s convenience, some background definitions in Banach spaces and an important fixed point theorem are presented.

As usual, let X be a real Banach space. The nonempty convex closed subset P of X is called a cone in X if $ax\in P$ and $x+y\in P$ for all $x,y\in P$ and $a\geq 0$, and $x\in X$ and $-x\in X$ imply $x=0$. A map $\psi:P\to[0,+\infty)$ is a nonnegative continuous concave (or convex) functional map provided ψ is nonnegative, continuous and satisfies

$$\psi(tx+(1-t)y)\geq\ (\text{ or }\ \leq\ )\ t\psi(x)+(1-t)\psi(y)\text{ for all }x,y\in P,t\in[0,1].$$

An operator $T:X\to X$ is completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Let $c_1,c_2,c_3,c_4,c_5>0$ be positive constants, α_1,α_2 be two nonnegative continuous concave functionals on the cone P, β_1,β_2,β_3 be three nonnegative continuous convex functionals on the cone P. Define the convex sets as follows:

$$P_{c_5}=\{x\in P:||x||<c_5\},$$

$$P(\beta_1,\alpha_1;c_2,c_5)=\{x\in P:\alpha_1(x)\geq c_2,\ \beta_1(x)\leq c_5\},$$

$$P(\beta_1,\beta_3,\alpha_1;c_2,c_4,c_5)=\{x\in P:\alpha_1(x)\geq c_2,\ \beta_3(x)\leq c_4,\ \beta_1(x)\leq c_5\},$$

$$Q(\beta_1,\beta_2;,c_1,c_5)=\{x\in P:\beta_2(x)\leq c_1,\ \beta_1(x)\leq c_5\},$$

$$Q(\beta_1,\beta_2,\alpha_2;c_3,c_1,c_5)=\{x\in P:\alpha_2(x)\geq c_3,\ \beta_2(x)\leq c_1,\ \beta_1(x)\leq c_5\}.$$

Lemma 2.1. ([2]) *Let X be a real Banach space, P be a cone in X, α_1, α_2 be two nonnegative continuous concave functionals on the cone P, β_1, β_2, β be three nonnegative continuous convex functionals on the cone P. If*

(A1) *$T: X \to X$ is a completely continuous operator;*
(A2) *there exist a constant $M > 0$ such that*

$$\alpha_1(x) \le \beta_2(x),\ ||x|| \le M\beta_1(x) \text{ for all } x \in P;$$

(A3) *there exist positive numbers c_1, c_2, c_3, c_4, c_5 with $c_1 < c_2$ such that*
 (i) $T\overline{P_{c_5}} \subset \overline{P_{c_5}}$;
 (ii) $\{y \in P(\beta_1, \beta_3, \alpha_1; c_2, c_4, c_5)|\alpha_1(x) > c_2\} \ne \emptyset$ *and*

$$\alpha_1(Tx) > c_2 \text{ for every } x \in P(\beta_1, \beta_3, \alpha_1; c_2, c_4, c_5);$$

 (iii) $\{y \in Q(\beta_1, \beta_2, \alpha_2; c_3, c_1, c_5)|\beta_2(x) < c_1\} \ne \emptyset$ *and*

$$\beta_2(Tx) < c_1 \text{ for every } x \in Q(\beta_1, \beta_2, \alpha_2; c_3, c_1, c_5);$$

 (iv) $\alpha_1(Ty) > c_2$ *for* $y \in P(\beta_1, \alpha_1; c_2, c_5)$ *with* $\beta_3(Ty) > c_4$;
 (v) $\beta_2(Tx) < c_1$ *for each* $x \in Q(\beta_1, \beta_2; c_1, c_5)$ *with* $\alpha_2(Tx) < c_3$.

Then T has at least three fixed points y_1, y_2 and y_3 such that

$$\beta_2(y_1) < c_1,\ \alpha_1(y_2) > c_2,\ \beta_2(y_3) > c_1,\ \alpha_1(y_3) < c_2.$$

A function $F : (0, 1) \times R^2 \to R$ is called a $p-$Caratheodory function if

(i) $t \to F\left(t, x, \frac{1}{\phi^{-1}(p(t))}y\right)$ is measurable on $(0, 1)$ for each $x, y \in R$;
(ii) $(x, y) \to F\left(t, x, \phi^{-1}(p(t))y\right)$ is continuous for almost all $t \in (0, 1)$;
(iii) for each $r > 0$ there exists a function $\phi_r \in L^1(0, 1)$ such that

$$\left|F\left(t, x, \phi^{-1}(p(t))y\right)\right| \le \phi_r(t),\ t \in (0, 1), |x|, |y| \le r.$$

A function $G : (0, 1) \times R^2 \to R$ is called a $q-$Caratheodory function if

(i) $t \to F\left(t, x, \frac{1}{\psi^{-1}(q(t))}y\right)$ is measurable on $(0, 1)$ for each $x, y \in R$;
(ii) $(x, y) \to F\left(t, x, \frac{1}{\psi^{-1}(q(t))}y\right)$ is continuous for almost all $t \in (0, 1)$;
(iii) for each $r > 0$ there exists a function $\phi_r \in L^1(0, 1)$ such that

$$\left|F\left(t, x, \frac{1}{\psi^{-1}(q(t))}y\right)\right| \le \phi_r(t),\ t \in (0, 1), |x|, |y| \le r.$$

Suppose that

(B1) $f : (0, 1) \times [h_2, +\infty) \times R \to [0, +\infty)$, where $h_2 = \frac{C}{1-\sum_{i=1}^m c_i}$, is a $p-$Caratheodory function with $f(t, 0, 0) \not\equiv 0$ on each sub-interval of $(0, 1)$;

$g:(0,1)\times[h_1,+\infty)\times R\to[0,+\infty)$, where $h_1=\frac{A}{1-\sum_{i=1}^m a_i}$, is a $q-$Caratheodory function with $g(t,0,0)\not\equiv 0$ on each sub-interval of $(0,1)$;

(B2) $p,q:(0,1)\to[0,+\infty)$ are continuous on $(0,1)$, both p and q may be singular at $t=0,1$ or vanish at $t=0,1$, i.e., $\lim_{t\to 0}p(t)=\lim_{t\to 1}p(t)=0$, $\lim_{t\to 0}q(t)=\lim_{t\to 1}q(t)=0$, $\int_0^1\phi^{-1}\left(\frac{1}{p(s)}\right)ds<+\infty$ and $\int_0^1\psi^{-1}\left(\frac{1}{q(s)}\right)ds<+\infty$;

(B3) $a_i,b_i,c_i,d_i\geq 0$ for all $i=1,\cdots,m$;

(B4i) $\sum_{i=1}^m a_i<1$, $\sum_{i=1}^m b_i<1$ and $\frac{A}{1-\sum_{i=1}^m a_i}\leq\frac{B}{1-\sum_{i=1}^m b_i}$;

(B4ii) $\sum_{i=1}^m c_i<1$, $\sum_{i=1}^m d_i<1$ and $\frac{C}{1-\sum_{i=1}^m c_i}\leq\frac{D}{1-\sum_{i=1}^m d_i}$;

(B4i)$'$ $\sum_{i=1}^m a_i<1$, $\sum_{i=1}^m b_i<1$ and $\frac{A}{1-\sum_{i=1}^m a_i}\leq\frac{B}{1-\sum_{i=1}^m b_i}$;

(B4ii)$'$ $\sum_{i=1}^m c_i<1$, $\sum_{i=1}^m d_i<1$ and $\frac{C}{1-\sum_{i=1}^m c_i}\geq\frac{D}{1-\sum_{i=1}^m d_i}$;

(B4i)$''$ $\sum_{i=1}^m a_i<1$, $\sum_{i=1}^m b_i<1$ and $\frac{A}{1-\sum_{i=1}^m a_i}\geq\frac{B}{1-\sum_{i=1}^m b_i}$;

(B4ii)$''$ $\sum_{i=1}^m c_i<1$, $\sum_{i=1}^m d_i<1$ and $\frac{C}{1-\sum_{i=1}^m c_i}\leq\frac{D}{1-\sum_{i=1}^m d_i}$;

(B4i)$'''$ $\sum_{i=1}^m a_i<1$, $\sum_{i=1}^m b_i<1$ and $\frac{A}{1-\sum_{i=1}^m a_i}\geq\frac{B}{1-\sum_{i=1}^m b_i}$;

(B4ii)$'''$ $\sum_{i=1}^m c_i<1$, $\sum_{i=1}^m d_i<1$ and $\frac{C}{1-\sum_{i=1}^m c_i}\geq\frac{D}{1-\sum_{i=1}^m d_i}$;

(B5) $\sigma,\varrho:(0,1)\to[0,+\infty)$ are continuous functions and $\sigma,\varrho\in L^1(0,1)$ with $\varrho(t),\sigma(t)\not\equiv 0$ on each subinterval of $(0,1)$.

Choose

$$X=\left\{\begin{array}{l}x\in C^0[0,1]\\ \phi^{-1}(p)x'\in C^0(0,1)\end{array}:\begin{array}{c}\text{the limits }\lim_{t\to 0}\phi^{-1}(p(t))x'(t),\\ \\ \text{and }\lim_{t\to 1}\phi^{-1}(p(t))x'(t)\text{ exist}\end{array}\right\}.$$

Define its norm by

$$||x||=||x||_X=\max\left\{\max_{t\in[0,1]}|x(t)|,\ \sup_{t\in(0,1)}\phi^{-1}(p(t))|x'(t)|\right\}.$$

It is easy to see that X is a real Banach space.

Choose

$$Y=\left\{\begin{array}{l}y\in C^0[0,1]\\ \psi^{-1}(q)y'\in C^0(0,1)\end{array}:\begin{array}{c}\text{the limits }\lim_{t\to 0}\psi^{-1}(q(t))y'(t),\\ \\ \text{and }\lim_{t\to 1}\psi^{-1}(q(t))y'(t)\text{ exist}\end{array}\right\}.$$

Define its norm by

$$||y|| = ||y||_Y = \max\left\{\max_{t\in[0,1]}|y(t)|,\ \sup_{t\in(0,1)}\psi^{-1}(q(t))|y'(t)|\right\}.$$

It is easy to see that Y is a real Banach space.

Then $E = X \times Y$ with the norm $||(x,y)|| = \max\{||x||,\ ||y||\}$ for $(x,y)\in E$ is a Banach space.

Consider the following BVP

$$\begin{cases} [p(t)\phi(u'(t))]' + \sigma(t) = 0, \ \ t\in(0,1), \\ u(0) = \sum_{i=1}^m a_i u(\xi_i), \\ u(1) = \sum_{i=1}^m b_i u(\xi_i) + B - \frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A. \end{cases} \tag{2.1}$$

Lemma 2.2. *Suppose that (B2), (B3), (B4i) and (B5) hold. Then*

(i) *$u\in X$ is a solution of BVP(2.1) implies that u is concave with respect to τ, where τ is defined by*

$$\tau = \tau(t) = \frac{\int_0^t \phi^{-1}\left(\frac{1}{p(s)}\right)ds}{\int_0^1 \phi^{-1}\left(\frac{1}{p(s)}\right)ds};$$

(ii) *$u\in X$ is a solution of BVP(2.1) implies that u is positive on $(0,1)$;*

(iii) *Let $k\in(0,1/2)$. $u\in X$ is a solution of BVP(2.1) implies that u satisfies that*

$$\min_{t\in[k,1-k]}u(t) \ge \mu \max_{t\in[0,1]}u(t),$$

where μ is defined by

$$\mu = \min\left\{\frac{\int_0^{1-k}\phi^{-1}\left(\frac{1}{p(s)}\right)ds}{2\int_0^1\phi^{-1}\left(\frac{1}{p(s)}\right)ds}, \frac{\int_0^{k}\phi^{-1}\left(\frac{1}{p(s)}\right)ds}{2\int_0^1\phi^{-1}\left(\frac{1}{p(s)}\right)ds}\right\};$$

(iv) *$u\in X$ is a solution of BVP(2.1) if and only if there exist unique numbers A_σ, B_σ such that*

$$u(t) = B_\sigma + \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(A_\sigma - \int_0^s \sigma(w)dw\right)ds, \ \ t\in[0,1], \tag{2.2}$$

where

$$A_\sigma \in \left[\phi(M_0), \phi(M_0) + \int_0^1 \sigma(w)dw\right]$$

such that

$$(2.3)\quad \begin{aligned} &\tfrac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\tfrac{1}{p(s)}\right)\phi^{-1}\left(A_\sigma-\int_0^s\sigma(w)dw\right)ds\\ &+\left(1-\sum_{i=1}^m b_i\right)\int_0^1\phi^{-1}\left(\tfrac{1}{p(s)}\right)\phi^{-1}\left(A_\sigma-\int_0^s\sigma(w)dw\right)ds\\ &+\sum_{i=1}^m b_i\int_{\xi_i}^1\phi^{-1}\left(\tfrac{1}{p(s)}\right)\phi^{-1}\left(A_\sigma-\int_0^s\sigma(w)dw\right)ds\\ &=B-\tfrac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A \end{aligned}$$

and B_σ satisfies

$$(2.4)\qquad B_\sigma=\frac{\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\tfrac{1}{p(s)}\right)\phi^{-1}\left(A_\sigma-\int_0^s\sigma(w)dw\right)ds}{1-\sum_{i=1}^m a_i}$$

with

$$M=\frac{1-\sum\limits_{i=1}^m b_i}{1-\sum\limits_{i=1}^m a_i}\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\tfrac{1}{p(s)}\right)ds+\left(1-\sum_{i=1}^m b_i\right)\int_0^1\phi^{-1}\left(\tfrac{1}{p(s)}\right)ds$$

$$+\sum_{i=1}^m b_i\int_{\xi_i}^1\phi^{-1}\left(\tfrac{1}{p(s)}\right)ds,$$

$$M_0=\frac{\left(1-\sum_{i=1}^m a_i\right)B-\left(1-\sum_{i=1}^m b_i\right)A}{\left(1-\sum_{i=1}^m a_i\right)M}.$$

Proof. Firstly we prove that u is concave with respect to τ. It is easy to see that $\tau\in C^0([0,1],[0,1])$ and

$$(2.5)\qquad \phi\left(\frac{du}{d\tau}\right)=p(t)\phi\left(\frac{du}{dt}\right)\phi\left(\int_0^1\phi^{-1}\left(\frac{1}{p(s)}\right)ds\right).$$

Since $[p(t)\phi(u'(t))]'=-\sigma(t)\le 0$, then we get $p(t)\phi(u'(t))$ is decreasing (non-increasing) on $(0,1)$. Then (2.5) implies that $\phi\left(\frac{du}{d\tau}\right)$ decreases as t increases. By definition of τ, we know that τ increases if and only if t increases. Hence $\phi\left(\frac{du}{d\tau}\right)$ decreases as τ increases. So u is concave with respect to τ on $[0,1]$.

Secondly, we prove that u is positive on $(0,1)$. In fact, since $[p(t)\phi(u'(t))]'\le 0$, we see that $p(t)\phi(u'(t))$ is non-increasing on $(0,1)$. Then $\phi^{-1}(p(t))u'(t)$ is non-increasing on $(0,1)$. We prove that $u'(0)\ge 0$.

In fact, if $u'(0) < 0$, then $u'(t) < 0$ for all $t \in [0,1]$. So

$$u(0) = \sum_{i=1}^{m} a_i u(\xi_i) \leq \sum_{i=1}^{m} a_i u(0).$$

It follows that $u(0) \leq 0$. On the other hand, we have from (B4i) that

$$u(1) = \sum_{i=1}^{m} b_i u(\xi_i) + B - \frac{1 - \sum_{i=1}^{m} b_i}{1 - \sum_{i=1}^{m} a_i} A \geq \sum_{i=1}^{m} b_i u(1).$$

We get $u(1) \geq 0$. Hence $u(t) \equiv 0$ on $[0,1]$. Then $\sigma(t) \equiv 0$ on $[0,1]$, a contradiction. So $u'(0) \geq 0$.

Case 1. $u'(1) \geq 0$.

It is easy to see that u is increasing on $[0,1]$. From

$$u(0) = \sum_{i=1}^{m} a_i u(\xi_i) \geq \sum_{i=1}^{m} a_i u(0),$$

we get $u(0) \geq 0$. One gets that $u(t) > 0$ for all $t \in (0,1]$. Otherwise, there exists $t_0 \in (0,1]$ such that $u(t_0) = 0$, then $u(t) \equiv 0$ on $(0,t_0]$. Hence $\sigma(t) \equiv 0$ on $(0,t_0]$, a contradiction.

Case 2. $u'(1) < 0$.

It is easy to see that there $t_0 \in [0,1)$ such that $u'(t_0) = 0$. So

$$\phi(u'(t)) = \frac{1}{p(t)} \begin{cases} -\int_{\xi}^{t} \sigma(w)dw, & t \geq \xi, \\ \int_{t}^{\xi} \sigma(w)dw, & t \leq \xi. \end{cases}$$

Then

$$u(t) = \begin{cases} u(1) + \int_{t}^{1} \phi^{-1}\left(\frac{1}{p(s)} \int_{\xi}^{s} \sigma(w)dw\right) ds, & t \geq \xi, \\ u(0) + \int_{0}^{t} \phi^{-1}\left(\frac{1}{p(s)} \int_{s}^{\xi} \sigma(w)dw\right) ds, & t \leq \xi. \end{cases}$$

It follows that $u(t) \geq \min\{u(0), u(1)\}$ for all $t \in [0,1]$. Hence

$$u(0) = \sum_{i=1}^{m} a_i u(\xi_i) \geq \sum_{i=1}^{m} a_i \min\{u(0), u(1)\}$$

and

$$u(1) = \sum_{i=1}^{m} b_i u(\xi_i) + B - \frac{1 - \sum_{i=1}^{m} b_i}{1 - \sum_{i=1}^{m} a_i} A \geq \sum_{i=1}^{m} b_i \min\{u(0), u(1)\}.$$

We get that

$$\min\{u(0), u(1)\} \geq \min\left\{\sum_{i=1}^{m} a_i, \sum_{i=1}^{m} b_i\right\} \min\{u(0), u(1)\}.$$

Thus $\min\{u(0), u(1)\} \geq 0$. It is easy to show that $u(t) > 0$ on $(0, 1)$.

Thirdly, we prove (iii). Let the inverse function of $\tau = \tau(t)$ be $t = t(\tau)$. It follows from above discussion that $\sup_{t\in[0,1]} u(t) = u(t_0)$. One sees

$$\min_{t\in[k,1-k]} u(t) = \min\left\{u(k),\ u(1-k)\right\}.$$

If $\min\{u(k),\ u(1-k)\} = u(k) = u(t(\tau(k)))$, we have $t_0 \geq k$. Then for $t \in [k, 1-k]$, one has

$$u(t) \geq u(t(\tau(k))) = u\left(\ t\left(\tfrac{1-\tau(k)+\tau(t_0)}{1+\tau(t_0)}\tfrac{\tau(k)}{1-\tau(k)+\tau(t_0)} + \tfrac{\tau(k)}{1+\tau(t_0)}\tau(t_0)\right)\right).$$

Noting that $1 > \tau(k)$ and $u(t)$ is concave with respect to τ, then, for $t \in [k, 1-k]$,

$$\begin{aligned}
u(t) &\geq \tfrac{1-\tau(k)+\tau(t_0)}{1+\tau(t_0)} u\left(t\left(\tfrac{\tau(k)}{1-\tau(k)+\tau(t_0)}\right)\right) + \tfrac{\tau(k)}{1+\tau(t_0)} u\left(\ t\left(\tau(t_0)\right)\right)\\
&\geq \textstyle\int_0^k \phi^{-1}\left(\tfrac{1}{p(s)}\right) ds \tfrac{1}{2\int_0^1 \phi^{-1}\left(\frac{1}{p(s)}\right) ds} u(t_0) = \mu \sup_{t\in[0,1]} u(t).
\end{aligned}$$

Similarly, if $\min\{y(k),\ y(1-k)\} = (y(t(\tau(1-k)))$, we have $t_0 \leq 1-k$. For $t \in [k, 1-k]$, from $0 < k < \frac{1}{2}$, one has

$$\begin{aligned}
&u(t) \geq u(t(\tau(1-k)))\\
&= u\left(t\left(\tfrac{1+\tau(t_0)-\tau(1-k)}{1+\tau(t_0)}\tfrac{\tau(1-k)}{1+\tau(t_0)-\tau(1-k)} + \tfrac{\tau(1-k)}{1+\tau(t_0)}\tau(t_0)\right)\right)\\
&\geq \tfrac{1+\tau(t_0)-\tau(1-k)}{1+\tau(t_0)} u\left(t\left(\tfrac{\tau(1-k)}{1+\tau(t_0)-\tau(1-k)}\right)\right) + \tfrac{\tau(1-k)}{1+\tau(\tau_0)} u\left(t\left(\tau(t_0)\right)\right)\\
&\geq \textstyle\int_0^{1-k} \phi^{-1}\left(\tfrac{1}{p(s)}\right) ds \tfrac{1}{2\int_0^1 \phi^{-1}\left(\frac{1}{p(s)}\right) ds} u(t_0) > \mu \sup_{t\in[0,1]} u(t).
\end{aligned}$$

Hence (iii) holds.

Finally, we prove (iv). If $u \in X$ is a solution of (2.1), we get from $\sigma \in L^1(0, 1)$ that

$$p(t)\phi(u'(t)) = \lim_{t\to 0} p(t)\phi(u'(t)) - \int_0^t \sigma(w)dw,\ \ t \in [0, 1].$$

Then there exists an unique number $B_\sigma = u(0)$ such that

$$u(t) = u(0) + \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(\lim_{t\to 0} p(t)\phi(u'(t)) - \int_0^s \sigma(w)dw\right) ds.$$

The BCs in (2.1) imply that

$$u(0)=u(0)\sum_{i=1}^m a_i$$

$$+\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(\lim_{t\to 0}p(t)\phi(u'(t))-\int_0^s\sigma(w)dw\right)ds$$

and

$$\begin{aligned}&u(0)+\int_0^1\phi^{-1}\left(\tfrac{1}{p(s)}\right)\phi^{-1}\left(\lim_{t\to 0}p(t)\phi(u'(t))-\int_0^s\sigma(w)dw\right)ds\\&=u(0)\sum_{i=1}^m b_i+B-\tfrac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A\end{aligned}$$

$$+\sum_{i=1}^m b_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(\lim_{t\to 0}p(t)\phi(u'(t))-\int_0^s\sigma(w)dw\right)ds.$$

It follows that

$$\frac{1-\sum\limits_{i=1}^m b_i}{1-\sum\limits_{i=1}^m a_i}\left(\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(\lim_{t\to 0}p(t)\phi(u'(t))-\int_0^s\sigma(w)dw\right)ds\right)$$

$$=-\int_0^1\phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(\lim_{t\to 0}p(t)\phi(u'(t))-\int_0^s\sigma(w)dw\right)ds$$

$$+\sum_{i=1}^m b_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(\lim_{t\to 0}p(t)\phi(u'(t))-\int_0^s\sigma(w)dw\right)ds$$

$$+B-\frac{1-\sum\limits_{i=1}^m b_i}{1-\sum\limits_{i=1}^m a_i}A.$$

Let

$$G(c)=\frac{1-\sum\limits_{i=1}^m b_i}{1-\sum\limits_{i=1}^m a_i}\left(\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(c-\int_0^s\sigma(u)du\right)ds\right)$$

$$+\int_0^1\phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(c-\int_0^s\sigma(u)du\right)ds$$

$$-\sum_{i=1}^m b_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(c-\int_0^s\sigma(u)du\right)ds-\left(B-\frac{1-\sum\limits_{i=1}^m b_i}{1-\sum\limits_{i=1}^m a_i}A\right)$$

$$= \frac{1-\sum\limits_{i=1}^{m} b_i}{1-\sum\limits_{i=1}^{m} a_i} \left(\sum_{i=1}^{m} a_i \int_0^{\xi_i} \phi^{-1} \left(\frac{1}{p(s)} \right) \phi^{-1} \left(c - \int_0^s \sigma(u) du \right) ds \right)$$

$$+ \left(1 - \sum_{i=1}^{m} b_i \right) \int_0^1 \phi^{-1} \left(\frac{1}{p(s)} \right) \phi^{-1} \left(c - \int_0^s \sigma(u) du \right) ds$$

$$+ \sum_{i=1}^{m} b_i \int_{\xi_i}^1 \phi^{-1} \left(\frac{1}{p(s)} \right) \phi^{-1} \left(c - \int_0^s \sigma(u) du \right) ds - \left(B - \frac{1-\sum\limits_{i=1}^{m} b_i}{1-\sum\limits_{i=1}^{m} a_i} A \right).$$

It is easy to see that $G(c)$ is increasing on $(-\infty, +\infty)$. Using $(B5)$, one gets easily that

$$G(\phi(M_0)) \quad \leq \quad 0,$$

and

$$G\left(\phi(M_0) + \int_0^1 \sigma(w) dw \right) \geq 0.$$

Hence we get that there exists a unique constant

$$A_\sigma \in \left[\phi(M_0), \phi(M_0) + \int_0^1 \sigma(w) dw \right]$$

such that A_σ satisfies (2.3) and $A_\sigma = \lim_{t \to 0} p(t) \phi(u'(t))$. It is easy to see that B_σ satisfies (2.4). These imply that u satisfies (2.2) and A_σ, B_σ satisfy (2.3) and (2.4).

On the other hand, if u satisfies (2.2) with A_σ, B_σ satisfying (2.3) and (2.4), it is easy to show that $u \in X$ and u is a solution of BVP(2.1).

The proof is complete. □

Similarly we consider the following BVP

$$\begin{cases} [q(t)\psi(v(t))]' + \varrho(t) = 0, \quad t \in (0,1), \\ v(0) = \sum\limits_{i=1}^{m} a_i v \xi_i), \\ v(1) = \sum\limits_{i=1}^{m} b_i v(\xi_i) + D - \dfrac{1-\sum\limits_{i=1}^{m} d_i}{1-\sum\limits_{i=1}^{m} c_i} C. \end{cases} \tag{2.6}$$

We get the following lemma:

Lemma 2.3. *Suppose that (B2), (B3), (B4ii) and (B5) hold. Then*

(i) *$v \in X$ is a solution of BVP(2.6) implies that v is concave with respect to ς, where ς is defined by*

$$\varsigma = \varsigma(t) = \frac{\int_0^t \psi^{-1}\left(\frac{1}{q(s)}\right) ds}{\int_0^1 \psi^{-1}\left(\frac{1}{q(s)}\right) ds};$$

(ii) *$v \in X$ is a solution of BVP(2.6) implies that v is positive on $(0,1)$;*

(iii) *Let $k \in (0, 1/2)$. $v \in X$ is a solution of BVP(2.6) implies that v satisfies that*

$$\min_{t\in[k,1-k]} v(t) \geq \nu \max_{t\in[0,1]} v(t),$$

where ν is defined by

$$\nu = \min\left\{\frac{\int_0^{1-k} \psi^{-1}\left(\frac{1}{q(s)}\right) ds}{2\int_0^1 \psi^{-1}\left(\frac{1}{q(s)}\right) ds}, \frac{\int_0^{k} \psi^{-1}\left(\frac{1}{q(s)}\right) ds}{2\int_0^1 \psi^{-1}\left(\frac{1}{q(s)}\right) ds}\right\};$$

(iv) *$v \in Y$ is a solution of BVP(2.6) if and only if there exist unique numbers A_ϱ, B_ϱ such that*

$$v(t) = B_\varrho + \int_0^t \psi^{-1}\left(\frac{1}{q(s)}\right)\psi^{-1}\left(A_\varrho - \int_0^s \varrho(w)dw\right) ds, \;\; t \in [0,1],$$

where

$$A_\varrho \in \left[\psi\left(N_0\right), \psi\left(N_0\right) + \int_0^1 \varrho(w)dw\right]$$

such that

$$\frac{1-\sum\limits_{i=1}^m d_i}{1-\sum\limits_{i=1}^m c_i}\sum_{i=1}^m c_i \int_0^{\xi_i} \psi^{-1}\left(\frac{1}{q(s)}\right)\psi^{-1}\left(A_\varrho - \int_0^s \varrho(w)dw\right) ds$$

$$+\left(1-\sum_{i=1}^m d_i\right)\int_0^1 \psi^{-1}\left(\frac{1}{q(s)}\right)\psi^{-1}\left(A_\varrho - \int_0^s \varrho(w)dw\right) ds$$

$$+\sum_{i=1}^m d_i \int_{\xi_i}^1 \psi^{-1}\left(\frac{1}{q(s)}\right)\psi^{-1}\left(A_\varrho - \int_0^s \varrho(w)dw\right) ds = D - \frac{1-\sum\limits_{i=1}^m d_i}{1-\sum\limits_{i=1}^m c_i}C.$$

and B_ϱ satisfies

$$B_\varrho = \frac{\sum\limits_{i=1}^m c_i \int_0^{\xi_i} \psi^{-1}\left(\frac{1}{q(s)}\right)\psi^{-1}\left(A_\varrho - \int_0^s \varrho(w)dw\right) ds}{1-\sum\limits_{i=1}^m c_i}$$

with

$$N=\frac{1-\sum\limits_{i=1}^{m}d_i}{1-\sum\limits_{i=1}^{m}c_i}\sum_{i=1}^{m}c_i\int_0^{\xi_i}\psi^{-1}\left(\frac{1}{q(s)}\right)ds+\left(1-\sum_{i=1}^{m}d_i\right)\int_0^1\psi^{-1}\left(\frac{1}{q(s)}\right)ds$$

$$+\sum_{i=1}^{m}d_i\int_{\xi_i}^1\psi^{-1}\left(\frac{1}{q(s)}\right)ds,\quad N_0=\frac{\left(1-\sum\limits_{i=1}^{m}c_i\right)D-\left(1-\sum\limits_{i=1}^{m}d_i\right)C}{\left(1-\sum\limits_{i=1}^{m}c_i\right)N}.$$

Let $h_1=\frac{A}{1-\sum\limits_{i=1}^{m}a_i}$, $h_2=\frac{C}{1-\sum\limits_{i=1}^{m}c_i}$ and $x(t)=u(t)+h_1$ and $y(t)=v(t)+h_2$. Then BVP(1.2) is transformed into the following BVP

$$\text{(2.7)}\qquad \begin{cases}(p(t)\phi(u'(t)))'+f\left(t,v(t)+h_2,v'(t)\right)=0,\ \ t\in(0,1),\\ (q(t)\psi(v'(t)))'+g\left(t,u(t)+h_1,u'(t)\right)=0,\ \ t\in(0,1),\\ u(0)=\sum\limits_{i=1}^{m}a_iu(\xi_i),\ v(0)=\sum\limits_{i=1}^{m}c_iv(\xi_i),\\ u(1)=\sum\limits_{i=1}^{m}b_iu(\xi_i)+B-\dfrac{1-\sum\limits_{i=1}^{m}b_i}{1-\sum\limits_{i=1}^{m}a_i}A,\\ v(1)=\sum\limits_{i=1}^{m}d_iv(\xi_i)+D-\dfrac{1-\sum\limits_{i=1}^{m}d_i}{1-\sum\limits_{i=1}^{m}c_i}C.\end{cases}$$

Let $k\in(0,1/2)$. Define

$$P=\left\{(u,v)\in E:\ \begin{array}{l}u(t),v(t)\geq 0,\ t\in[0,1],\\ \min\limits_{t\in[k,1-k]}u(t)\geq\mu\max_{t\in[0,1]}u(t),\\ \min\limits_{t\in[k,1-k]}v(t)\geq\nu\max_{t\in[0,1]}v(t),\\ u(0)=\sum\limits_{i=1}^{m}a_iu(\xi_i),\\ v(0)=\sum\limits_{i=1}^{m}c_iv(\xi_i),\end{array}\right\}.$$

It is easy to see that P is a cone in the Banach space E.

Define the functionals on $P\to R$ by

$$\beta_1(u,v)=\max\left\{\sup_{t\in(0,1)}\phi^{-1}(p(t))|u'(t)|,\ \sup_{t\in(0,1)}\psi^{-1}(q(t))|v'(t)|\right\},\ (u,v)\in P,$$

$$\beta_2(u,v)=\max\left\{\max_{t\in[0,1]}|u(t)|,\ \max_{t\in[0,1]}|v(t)|\right\},\ (u,v)\in P,$$

$$\begin{aligned}
\beta_3(u,v) &= \max\left\{\max_{t\in[0,1]}|u(t)|,\ \max_{t\in[0,1]}|v(t)|\right\},\ (u,v)\in P,\\
\alpha_1(u,v) &= \min\left\{\min_{t\in[k,1-k]}|u(t)|,\ \min_{t\in[k,1-k]}|v(t)|\right\},\ (u,v)\in P,\\
\alpha_2(u,v) &= \min\left\{\min_{t\in[k,1-k]}|u(t)|,\ \min_{t\in[k,1-k]}|v(t)|\right\},\ (u,v)\in P.
\end{aligned}$$

Define the nonlinear operator $T : P \to X$ by $T(u,v) = (T_1v, T_2u)$ with

$$(T_1v)(t) = B_v + \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(A_v - \int_0^s f(w,v(w)+h_2,v'(w))dw\right)ds,$$

$$(T_2u)(t) = B_u + \int_0^t \psi^{-1}\left(\frac{1}{q(s)}\right)\psi^{-1}\left(A_u - \int_0^s g(w,u(w)+h_1,u'(w))dw\right)ds,$$

where A_v, B_v, A_u, B_u satisfy

$$\frac{1-\sum\limits_{i=1}^m b_i}{1-\sum\limits_{i=1}^m a_i}\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(A_v - \int_0^s f(w,v(w)+h_2,v'(w))dw\right)ds$$

$$+\left(1-\sum_{i=1}^m b_i\right)\int_0^1 \phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(A_v - \int_0^s f(w,v(w)+h_2,v'(w))dw\right)ds$$

$$+\sum_{i=1}^m b_i\int_{\xi_i}^1 \phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(A_v - \int_0^s f(w,v(w)+h_2,v'(w))dw\right)ds$$

$$= B - \frac{1-\sum\limits_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A,$$

$$B_v = \frac{\sum\limits_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(A_v-\int_0^s f(w,v(w)+h_2,v'(w))dw\right)ds}{1-\sum_{i=1}^m a_i},$$

and

$$\frac{1-\sum\limits_{i=1}^m d_i}{1-\sum\limits_{i=1}^m c_i}\sum_{i=1}^m c_i\int_0^{\xi_i}\psi^{-1}\left(\frac{1}{q(s)}\right)\psi^{-1}\left(A_u - \int_0^s g(w,u(w)+h_1,u'(w))dw\right)ds$$

$$+\left(1-\sum_{i=1}^m d_i\right)\int_0^1 \psi^{-1}\left(\frac{1}{q(s)}\right)\psi^{-1}\left(A_u - \int_0^s g(w,u(w)+h_1,u'(w))dw\right)ds$$

$$+\sum_{i=1}^{m} d_i \int_{\xi_i}^{1} \psi^{-1}\left(\frac{1}{q(s)}\right)\psi^{-1}\left(A_u - \int_0^s g(w,u(w)+h_1,u'(w))dw\right) ds$$

$$= D - \frac{1-\sum\limits_{i=1}^{m} d_i}{1-\sum_{i=1}^{m} c_i} C,$$

$$B_u = \frac{\sum_{i=1}^{m} c_i \int_0^{\xi_i} \psi^{-1}\left(\frac{1}{q(s)}\right)\psi^{-1}\left(A_u - \int_0^s g(w,u(w)+h_1,u'(w))dw\right)ds}{1-\sum_{i=1}^{m} c_i}$$

with

$$A_v \in \left[\phi\left(M_0\right), \phi\left(M_0\right) + \int_0^1 f(w,v(w)+h_2,v'(w))dw\right],$$

$$A_u \in \left[\psi\left(N_0\right), \psi\left(N_0\right) + \int_0^1 g(w,u(w)+h_1,u'(w))dw\right]$$

with M, M_0, N, N_0 be define by

$$M = \frac{1-\sum\limits_{i=1}^{m} b_i}{1-\sum\limits_{i=1}^{m} a_i} \sum_{i=1}^{m} a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1}{p(s)}\right) ds + \left(1 - \sum_{i=1}^{m} b_i\right) \int_0^1 \phi^{-1}\left(\frac{1}{p(s)}\right) ds$$

$$+\sum_{i=1}^{m} b_i \int_{\xi_i}^{1} \phi^{-1}\left(\frac{1}{p(s)}\right) ds, M_0 = \frac{\left(1-\sum_{i=1}^{m} a_i\right)B - \left(1-\sum_{i=1}^{m} b_i\right)A}{\left(1-\sum_{i=1}^{m} a_i\right)M},$$

$$N = \frac{1-\sum\limits_{i=1}^{m} d_i}{1-\sum\limits_{i=1}^{m} c_i} \sum_{i=1}^{m} c_i \int_0^{\xi_i} \psi^{-1}\left(\frac{1}{q(s)}\right) ds + \left(1 - \sum_{i=1}^{m} d_i\right) \int_0^1 \psi^{-1}\left(\frac{1}{q(s)}\right) ds$$

$$+\sum_{i=1}^{m} d_i \int_{\xi_i}^{1} \psi^{-1}\left(\frac{1}{q(s)}\right) ds, N_0 = \frac{\left(1-\sum\limits_{i=1}^{m} c_i\right)D - \left(1-\sum\limits_{i=1}^{m} d_i\right)C}{\left(1-\sum_{i=1}^{m} c_i\right)N}.$$

LEMMA 2.4. *Suppose that (B1), (B2), (B3), (B4i) and (B4ii) hold. Then*

(i) $T_1 v, T_2 u$ *satisfy the following equalities:*

$$\begin{cases} (p(t)\phi((T_1 v)'))' + f\left(t, v(t) + h_2, v'(t)\right) = 0, \quad t \in (0,1), \\ (q(t)\psi((T_2 u)'))' + g\left(t, u(t) + h_1, u'(t)\right) = 0, \quad t \in (0,1), \\ (T_1 u)(0) = \sum\limits_{i=1}^{m} a_i (T_1 u)(\xi_i), \end{cases}$$

$$\begin{cases} (T_1u)(1) = \sum\limits_{i=1}^{m} b_i(T_1u)(\xi_i) + B - \frac{1-\sum\limits_{i=1}^{m} b_i}{1-\sum\limits_{i=1}^{m} a_i}A; \\ (T_2v)(0) = \sum\limits_{i=1}^{m} c_i(T_2v)(\xi_i), \\ (T_2v)(1) = \sum\limits_{i=1}^{m} d_i(T_2v)(\xi_i) + D - \frac{1-\sum\limits_{i=1}^{m} d_i}{1-\sum\limits_{i=1}^{m} c_i}C; \end{cases}$$

(ii) *$T(u,v) \in P$ for each $(u,v) \in P$;*

(iii) *(x,y) is a solution of BVP(1.2) if and only if $x = u+h_1$, $y = v+h_2$ and (u,v) is a solution of the operator equation $(u,v) = T(u,v)$ in P;*

(iv) *T is completely continuous.*

Proof. The proofs of (i), (ii) and (iii) are simple, To prove (iv), it suffices to prove that T is continuous on P and for each bounded subset $D \subset P$, $T(D) = \{T(u,v) : (u,v) \in D\}$ is relative compact. We divide the proof into two steps:

Step 1. for $(u,v) \in P$, since both f and g are continuous, we can prove that T is continuous at y.

Step 2. We note that $\Omega \in E$ is relatively compact if it is bounded equi-continuous on $[0,1]$. We need to prove that $T(D)$ is relative compact.

The proofs are similar to those of the proofs of lemmas in [16, 23, 26, 27] and are omitted. □

3. Main results

In this section, we give the main results under the assumptions (B1), (B2), (B3), (B4i), (B4ii) and their proofs. Denote

$$\mathbf{B}(m,n) = \int_0^1 w^{m-1}(1-w)^{n-1}dw \text{ for } m,n > 0,$$

$$\mu = \min\left\{ \frac{\int_0^{1-k} \phi^{-1}\left(\frac{1}{p(s)}\right)ds}{2\int_0^1 \phi^{-1}\left(\frac{1}{p(s)}\right)ds}, \frac{\int_0^{k} \phi^{-1}\left(\frac{1}{p(s)}\right)ds}{2\int_0^1 \phi^{-1}\left(\frac{1}{p(s)}\right)ds} \right\},$$

$$\nu = \min\left\{ \frac{\int_0^{1-k} \psi^{-1}\left(\frac{1}{q(s)}\right)ds}{2\int_0^1 \psi^{-1}\left(\frac{1}{q(s)}\right)ds}, \frac{\int_0^{k} \psi^{-1}\left(\frac{1}{q(s)}\right)ds}{2\int_0^1 \psi^{-1}\left(\frac{1}{q(s)}\right)ds} \right\},$$

$$\prod_1 = \int_0^1 \phi^{-1}\left(\frac{1}{p(s)}\right) ds + \frac{1}{1-\sum\limits_{i=1}^{m} a_i}\left(\sum_{i=1}^{m} \frac{a_i\xi_i}{\phi^{-1}(p(\xi_i))}\right),$$

$$\prod_2 = \int_0^1 \psi^{-1}\left(\frac{1}{q(s)}\right) ds + \frac{1}{1-\sum\limits_{i=1}^{m} c_i}\left(\sum_{i=1}^{m} \frac{c_i\xi_i}{\psi^{-1}(q(\xi_i))}\right),$$

$$M = \frac{1-\sum\limits_{i=1}^{m} b_i}{1-\sum\limits_{i=1}^{m} a_i}\sum_{i=1}^{m} a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1}{p(s)}\right) ds + \left(1-\sum_{i=1}^{m} b_i\right)\int_0^1 \phi^{-1}\left(\frac{1}{p(s)}\right) ds$$

$$+\sum_{i=1}^{m} b_i \int_{\xi_i}^1 \phi^{-1}\left(\frac{1}{p(s)}\right) ds,\ M_0 = \frac{\left(1-\sum\limits_{i=1}^{m} a_i\right)B-\left(1-\sum\limits_{i=1}^{m} b_i\right)A}{\left(1-\sum\limits_{i=1}^{m} a_i\right)M},$$

$$N = \frac{1-\sum\limits_{i=1}^{m} d_i}{1-\sum\limits_{i=1}^{m} c_i}\sum_{i=1}^{m} c_i \int_0^{\xi_i} \psi^{-1}\left(\frac{1}{q(s)}\right) ds + \left(1-\sum_{i=1}^{m} d_i\right)\int_0^1 \phi^{-1}\left(\frac{1}{p(s)}\right) ds$$

$$+\sum_{i=1}^{m} d_i \int_{\xi_i}^1 \psi^{-1}\left(\frac{1}{q(s)}\right) ds,\ N_0 = \frac{\left(1-\sum\limits_{i=1}^{m} c_i\right)D-\left(1-\sum\limits_{i=1}^{m} d_i\right)C}{\left(1-\sum\limits_{i=1}^{m} c_i\right)N},$$

$$W_1 = \mu \min\left\{\int_k^{\frac{1}{2}} \phi^{-1}\left(\frac{\int_s^{\frac{1}{2}} w^\alpha(1-w)^\beta dw}{p(s)}\right) ds,\ \int_{\frac{1}{2}}^{1-k} \phi^{-1}\left(\frac{\int_{\frac{1}{2}}^{s} w^\alpha(1-w)^\beta dw}{p(s)}\right) ds\right\},$$

$$W_2 = \nu \min\left\{\int_k^{\frac{1}{2}} \psi^{-1}\left(\frac{\int_s^{\frac{1}{2}} w^\gamma(1-w)^\delta dw}{q(s)}\right) ds,\ \int_{\frac{1}{2}}^{1-k} \psi^{-1}\left(\frac{\int_{\frac{1}{2}}^{s} w^\gamma(1-w)^\delta dw}{q(s)}\right) ds\right\},$$

$$L_1 = \frac{\sum\limits_{i=1}^{m} a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1}{p(s)}\right) ds}{1-\sum\limits_{i=1}^{m} a_i} + \int_0^1 \phi^{-1}\left(\frac{1}{p(s)}\right) ds,$$

$$L_2 = \frac{\sum\limits_{i=1}^{m} c_i \int_0^{\xi_i} \psi^{-1}\left(\frac{1}{q(s)}\right) ds}{1-\sum\limits_{i=1}^{m} c_i} + \int_0^1 \psi^{-1}\left(\frac{1}{q(s)}\right) ds.$$

THEOREM 3.1. *Suppose that (B1), (B2), (B3), (B4i) and (B4ii) hold, e_1, e_2, c are positive numbers, $\alpha, \beta, \gamma, \delta > -1$. Let Q, W and E be given by*

$$Q = \min\left\{ \frac{\phi(c)-\phi(M_0)}{2\mathbf{B}(\alpha+1,\beta+1)}, \frac{\psi(c)-\phi(N_0)}{2\mathbf{B}(\gamma+1,\delta+1)}, \quad \frac{\phi\left(\frac{c}{\prod_1}\right)-\phi(M_0)}{2\mathbf{B}(\alpha+1,\beta+1)}, \frac{\psi\left(\frac{c}{\prod_2}\right)-\phi(N_0)}{2\mathbf{B}(\gamma+1,\delta+1)} \right\};$$

$$E = \min\left\{ \frac{\phi\left(\frac{e_1}{L_1}\right)-\phi(M_0)}{\mathbf{B}(\alpha+1,\beta+1)}, \frac{\psi\left(\frac{e_1}{L_2}\right)-\psi(N_0)}{\mathbf{B}(\gamma+1,\delta+1)} \right\}$$

and

$$W = \max\left\{ \phi\left(\frac{e_2}{W_1}\right), \ \psi\left(\frac{e_2}{W_2}\right), \ \phi\left(\frac{e_2}{\mu\int_k^{1-k} \phi^{-1}\left(\frac{\int_s^{1-k} w^\alpha(1-w)^\beta dw}{p(s)}\right)ds} \right), \right.$$

$$\left. \psi\left(\frac{e_2}{\nu\int_k^{1-k} \psi^{-1}\left(\frac{\int_s^{1-k} w^\gamma(1-w)^\delta dw}{q(s)}\right)ds} \right) \right\}.$$

If $Q \geq W$ and

$$c > \max\left\{M_0, N_0, \prod\nolimits_1 M_0, \prod\nolimits_2 N_0\right\}, \ e_1 > \max\{L_1 M_0, L_2 N_0\},$$

$$c > \frac{e_2}{\min\{\mu,\nu\}} > e_2 > e_1 > 0$$

and

(B6)

$$f(t,w,z) \leq Qt^\alpha(1-t)^\beta \text{ for all } t \in (0,1), w \in [h_2, c+h_2], z \in [-c,c],$$

$$g(t,w,z) \leq Qt^\gamma(1-t)^\delta \text{ for all } t \in (0,1), w \in [h_1, c+h_1], z \in [-c,c];$$

(B7)

$$f(t,w,z) \geq Wt^\alpha(1-t)^\beta \text{ for all } t \in [k,1-k], z \in [-c,c],$$

$$w \in \left[e_2+h_2, \frac{e_2}{\min\{\mu,\nu\}}+h_2\right],$$

$$g(t,w,z) \geq Wt^\gamma(1-t)^\delta \text{ for all } t \in [k,1-k], z \in [-c,c],$$

$$w \in \left[e_2+h_1, \frac{e_2}{\min\{\mu,\nu\}}+h_1\right];$$

(B8)

$$f(t,w,z) \le Et^{\alpha}(1-t)^{\beta} \text{ for all } t \in (0,1), w \in [h_2, e_1+h_2], z \in [-c,c],$$

$$g(t,w,z) \le Et^{\gamma}(1-t)^{\delta} \text{ for all } t \in (0,1), w \in [h_1, e_1+h_1], z \in [-c,c];$$

then BVP(1.2) has at least three pair of solutions $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ *such that*

$$\begin{aligned}
&\max_{t\in[0,1]} x_1(t) < e_1+h_1, \quad \max_{t\in[0,1]} y_1(t) < e_1+h_2,\\
&\min_{t\in[k,1-k]} x_2(t) > e_2+h_1, \quad \min_{t\in[k,1-k]} y_2(t) > e_2+h_2,\\
&\text{either } \max_{t\in[0,1]} x_3(t) > e_1+h_1 \text{ or } \max_{t\in[0,1]} y_3(t) > e_1+h_2,\\
&\text{either } \min_{t\in[k,1-k]} x_3(t) < e_2+h_1 \text{ or } \min_{t\in[k,1-k]} y_3(t) < e_2+h_2.
\end{aligned} \tag{3.1}$$

Proof. To apply Lemma 2.1, we prove that all hypotheses in Lemma 2.1 are satisfied.

By the definitions, it is easy to see that α_1, α_2 are nonnegative continuous concave functional on the cone P, $\beta_1, \beta_2, \beta_3$ nonnegative continuous convex functional on the cone P, and $\alpha_1(u,v) \le \beta_2(u,v)$ for all $(u,v) \in P$. Lemma 2.3 implies that $(x,y) = (x(t),y(t))$ is a pair of solutions of BVP(1.2) if and only if $x(t) = u(t)+h_1$, $y(t) = v(t)+h_2$ and $(u,v) = (u(t),v(t))$ is a pair of positive solutions of the operator equation $(u,v) = T(u,v)$ in P and T is completely continuous.

We claim that

$$\left.\begin{aligned}
&\max_{t\in[0,1]} |u(t)| \le \textstyle\prod_1 \sup_{t\in(0,1)} \phi^{-1}(p(t))|u'(t)|,\\
&\max_{t\in[0,1]} |v(t)| \le \textstyle\prod_2 \sup_{t\in(0,1)} \psi^{-1}(q(t))|v'(t)|
\end{aligned}\right\} \text{ for } (u,v) \in P. \tag{3.2}$$

In fact, for $(u,v) \in P$, we have that

$$u(0) - \sum_{i=1}^{m} a_i u(\xi_i) = 0, \; v(0) - \sum_{i=1}^{m} c_i v(\xi_i) = 0,$$

we get that

$$|u(0)| = \left|\frac{u(0)-u(0)\sum_{i=1}^m a_i}{1-\sum_{i=1}^m a_i}\right| = \left|\frac{\sum_{i=1}^m a_i u(\xi_i)-u(0)\sum_{i=1}^m a_i}{1-\sum_{i=1}^m a_i}\right|$$

$$= \frac{1}{1-\sum_{i=1}^m a_i}\left|\sum_{i=1}^m a_i\xi_i u'(\eta_i)\right| \text{ where } \eta_i \in [0,\xi_i]$$

$$\le \frac{1}{1-\sum_{i=1}^m a_i}\left(\sum_{i=1}^m \frac{a_i\xi_i}{\phi^{-1}(p(\xi_i))}\right)\sup_{t\in(0,1)}\phi^{-1}(p(t))|u'(t)|.$$

It follows that

$$|u(t)| \le |u(t)-u(0)| + |u(0)|$$

$$\le \int_0^t |u'(w)|dw + \frac{1}{1-\sum_{i=1}^m a_i}\left(\sum_{i=1}^m \frac{a_i\xi_i}{\phi^{-1}(p(\xi_i))}\right)\sup_{t\in(0,1)}\phi^{-1}(p(t))|u'(t)|$$

$$\le \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}(p(s))|u'(w)|dw$$

$$+\frac{1}{1-\sum_{i=1}^m a_i}\left(\sum_{i=1}^m \frac{a_i\xi_i}{\phi^{-1}(p(\xi_i))}\right)\sup_{t\in(0,1)}\phi^{-1}(p(t))|u'(t)|$$

$$\le \left(\int_0^1 \phi^{-1}\left(\frac{1}{p(s)}\right)ds + \frac{1}{1-\sum_{i=1}^m a_i}\left(\sum_{i=1}^m \frac{a_i\xi_i}{\phi^{-1}(p(\xi_i))}\right)\right)\sup_{t\in(0,1)}\phi^{-1}(p(t))|u'(t)|.$$

Then

$$\max_{t\in[0,1]}|u(t)| \le \prod_1 \sup_{t\in(0,1)}\phi^{-1}(p(t))|u'(t)|.$$

Similarly we get that

$$\max_{t\in[0,1]}|v(t)| \le \prod_2 \sup_{t\in(0,1)}\psi^{-1}(q(t))|v'(t)|.$$

Hence (3.2) holds. It follows that

$$||(u,v)|| = \max\left\{\max_{t\in[0,1]}|u(t)|, \sup_{t\in(0,1)}\phi^{-1}(p(t))|u'(t)|,\right.$$

(3.3)

$$\left.\max_{t\in[0,1]}|v(t)|, \sup_{t\in(0,1)}\psi^{-1}(q(t))|v'(t)|\right\} \le \max\{\prod_1,\prod_2\}\beta_1(u,v)$$

for all $(u,v)\in P$. From the above discussion, we see that $(A1)$ and $(A2)$ of Lemma 2.1 are satisfied.

Now we prove that $(A3)$ holds. Corresponding to Lemma 2.1, choose

$$c_1 = e_1, \;\; c_2 = e_2, \;\; c_3 = \min\{\mu,\nu\}e_1, \;\; c_4 = \frac{e_2}{\min\{\mu,\nu\}}, \;\; c_5 = c.$$

One sees that $c_1 < c_2$ since $e_1 < e_2$. The remainder is divided into five steps.

Step 1. Prove that $T\overline{P_{c_5}} \subset \overline{P_{c_5}}$;

For $(u,v) \in \overline{P_{c_5}}$, we have $||(u,v)|| \le c$. Then

$0 \le u(t), v(t) \le c, \;\; -c \le \phi^{-1}(p(t))u'(t), \psi^{-1}(q(t))v'(t) \le c$ for all $t \in (0,1)$.

So (B6) implies for $t \in (0,1)$ that

$$f\left\{\begin{array}{l} f\left(t, v(t)+h_2, \frac{1}{\phi^{-1}(p(t))}\phi^{-1}(p(t))v'(t)\right) \ge Qt^{\alpha}(1-t)^{\beta}, \\ g\left(t, u(t)+h_1, \frac{1}{\psi^{-1}(q(t))}\psi^{-1}(q(t))u'(t)\right) \ge Qt^{\gamma}(1-t)^{\delta}. \end{array}\right.$$

Lemma 2.2 implies

$$A_v \in \left[\phi\left(M_0\right), \phi\left(M_0\right) + \int_0^1 f(w, v(w)+h_2, v'(w))dw\right].$$

The definition of T_1 imlies that

$$\phi^{-1}(p(t))(T_1v)'(t) = \phi^{-1}\left(A_v - \int_0^t f(w, v(w)+h_2, v'(w))dw\right).$$

Then

$$\begin{aligned} \phi^{-1}(p(t))|(T_1v)'(t)| &\le \phi^{-1}\left(A_v + \textstyle\int_0^1 f(w, v(w)+h_2, v'(w))dw\right) \\ &\le \phi^{-1}\left(\phi(M_0) + 2\textstyle\int_0^1 f(w, v(w)+h_2, v'(w))dw\right) \\ &\le \phi^{-1}\left(\phi(M_0) + 2Q\textstyle\int_0^1 s^{\alpha}(1-s)^{\beta}ds\right) \\ &= \phi^{-1}\left(\phi(M_0) + 2Q\mathbf{B}(\alpha+1, \beta+1)\right) \\ &\le c. \end{aligned}$$

On the other hand, we have from $T(u,v) = (T_1v, T_2u) \in P$ and (3.2) that

$$\begin{aligned} |(T_1v)(t)| &\le \textstyle\prod_1 \sup_{t\in(0,1)} \phi^{-1}(p(t))|(T_1v)'(t)| \\ &\le \textstyle\prod_1 \phi^{-1}\left(\phi(M_0) + 2Q\mathbf{B}(\alpha+1, \beta+1)\right) \le c. \end{aligned}$$

It follows that

$$||T_1v|| = \max\left\{\max_{t\in[0,1]}|(T_1v)(t)|, \sup_{t\in(0,1)}\phi^{-1}(p(t))|(T_1v)'(t)|\right\} \le c.$$

Similarly we get that

$$||T_2u|| = \max\left\{\max_{t\in[0,1]}|(T_2u)(t)|, \sup_{t\in(0,1)}\psi^{-1}(q(t))|(T_2u)'(t)|\right\} \le c.$$

Then

$$||T(u,v)|| = \max\{||T_1v||, ||T_2u||\} \le c.$$

So $T(\overline{P_{c_5}}) \subseteq \overline{P_{c_5}}$. This completes the proof of (A3)(i) of Lemma 2.1.

Step 2. Prove that $\{(u,v) \in P(\beta_1,\beta_3,\alpha_1;c_2,c_4,c_5)|\alpha_1(u,v) > c_2\} \neq \emptyset$ and

$$\alpha_1(T(u,v)) > c_2 \text{ for every } (u,v) \in P(\beta_1,\beta_3,\alpha_1;c_2,c_4,c_5);$$

It is easy to show that $\{(u,v) \in P(\beta_1,\beta_3,\alpha_1;c_2,c_4,c_5)|\alpha(u,v) > c_2\} \neq \emptyset$.

For $(u,v) \in P(\beta_1,\beta_3,\alpha_1;c_2,c_4,c_5)$, one has that

$$\alpha_1(u,v) = \min\left\{\min_{t\in[k,1-k]}u(t), \min_{t\in[k,1-k]}v(t)\right\} \ge e_2,$$

$$\beta_3(u,v) = \max\left\{\max_{t\in[0,1]}u(t), \max_{t\in[0,1]}v(t)\right\} \le \frac{e_2}{\min\{\mu,\nu\}},$$

$$\beta_1(u,v) = \max\left\{\sup_{t\in(0,1)}\phi^{-1}(p(t))|u'(t)|, \sup_{t\in(0,1)}\psi^{-1}(q(t))|v'(t)|\right\} \le c.$$

Then

$$e_2 \le u(t), v(t) \le \frac{e_2}{\min\{\mu,\nu\}},\ t \in [k,1-k],$$

$$\phi^{-1}(p(t))|u'(t)|, \psi^{-1}(q(t))|v'(t)| \le c, t \in [0,1].$$

Thus $(B7)$ implies for $t \in [k,1-k]$ that

$$f\left(t, v(t)+h_2, \frac{1}{\phi^{-1}(p(t))}\phi^{-1}(p(t))v'(t)\right) \ge Wt^{\alpha}(1-t)^{\beta},$$

$$g\left(t, u(t)+h_1, \frac{1}{\psi^{-1}(q(t))}\psi^{-1}(q(t))u'(t)\right) \ge Wt^{\gamma}(1-t)^{\delta}.$$

Since $T(u,v) = (T_1v, T_2u) \in P$, we get

$$\alpha_1(T(u,v)) = \min\left\{\min_{t\in[k,1-k]}(T_1v)(t), \min_{t\in[k,1-k]}(T_2u)(t)\right\}$$

$$\ge \min\left\{\mu\max_{t\in[0,1]}(T_1v)(t), \nu\max_{t\in[0,1]}(T_2u)(t)\right\}.$$

We can prove that $(T_1v)'(0) \geq 0$. In fact, if $(T_1v)'(0) < 0$ then $(T_1v)'(t) < 0$ for all $t \in [0,1]$ by the fact $[p(t)\phi((T_1v)'(t))]' \leq 0$ and $p(t)\phi((T_1v)'(t))$ is decreasing with $p(0)\phi((T_1v)'(0)) < 0$. So, using Lemma 2.4,

$$(T_1v)(0) = \sum_{i=1}^{m} c_i(T_1v)(\xi_i) \leq \sum_{i=1}^{m} c_i(T_1v)(0).$$

Then $(T_1v)(0) \leq 0$. Similarly we get that

$$(T_1v)(1) = \sum_{i=1}^{m} d_i(T_1v)(\xi_i) + D - \frac{1-\sum_{i=1}^m d_i}{1-\sum_{i=1}^m c_i}C \geq \sum_{i=1}^{m} d_i(T_1v)(1).$$

It follows that $(T_1v)(1) \geq 0$. Together with $(T_1v)(0) \leq 0$, we get $(T_1v)(t) \equiv 0$ on $[0,1]$. This contradicts to $(T_1v)'(t) < 0$ for all $t \in [0,1]$. Hence $(T_1v)'(0) \geq 0$.

Case 1. there exists $\xi \in [0,1]$ such that $(T_1v)'(\xi) = 0$. Then the definition of T_1 implies that

$$\phi((T_1v)'(t)) = \frac{1}{p(t)} \begin{cases} -\int_\xi^t f(w,v(w)+h_2,v'(w))dw, & t \geq \xi, \\ \int_t^\xi f(w,v(w)+h_2,v'(w))dw, & t \leq \xi. \end{cases}$$

Then

$$(T_1v)(t) = \begin{cases} (T_1v)(1) + \int_t^1 \phi^{-1}\left(\frac{1}{p(s)}\int_\xi^s f(w,v(w)+h_2,v'(w))dw\right)ds, & t \geq \xi, \\ (T_1v)(0) + \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\int_s^\xi f(w,v(w)+h_2,v'(w))dw\right)ds, & t \leq \xi. \end{cases}$$

Hence

$$\max_{t\in[0,1]}(T_1v)(t) = (T_1v)(1) + \int_\xi^1 \phi^{-1}\left(\frac{1}{p(s)}\int_\xi^s f(w,v(w)+h_2,v'(w))dw\right)ds$$

$$= ((T_1v)(0) + \int_0^\xi \phi^{-1}\left(\frac{1}{p(s)}\int_s^\xi f(w,v(w)+h_2,v'(w))dw\right)ds.$$

Then $(T_1v)(0) \geq 0$ and $(T_1v)(1) \geq 0$ imply that

$$\min_{t\in[k,1-k]}(T_1v)(t) \geq \mu \max_{t\in[0,1]}(T_1v)(t)$$

$$\geq \mu \max\left\{\int_\xi^1 \phi^{-1}\left(\frac{1}{p(s)}\int_\xi^s f(w,v(w)+h_2,v'(w))dw\right)ds,\right.$$

$$\left.\int_0^\xi \phi^{-1}\left(\frac{1}{p(s)}\int_s^\xi f(w,v(w)+h_2,v'(w))dw\right)ds\right\}.$$

It is easy to see that

$$\int_\xi^1 \phi^{-1}\left(\frac{1}{p(s)}\int_\xi^s f(w,v(w)+h_2,v'(w))dw\right)ds$$

$$\geq \int_{\frac{1}{2}}^{1-k} \phi^{-1}\left(\frac{1}{p(s)}\int_{\frac{1}{2}}^s f(w,v(w)+h_2,v'(w))dw\right)ds$$

if $\xi \leq \frac{1}{2}$ and

$$\int_0^\xi \phi^{-1}\left(\frac{1}{p(s)}\int_s^\xi f(w,v(w)+h_2,v'(w))dw\right)ds \geq$$

$$\int_k^{\frac{1}{2}} \phi^{-1}\left(\frac{1}{p(s)}\int_s^{\frac{1}{2}} f(w,v(w)+h_2,v'(w))dw\right)ds$$

if $\xi \geq \frac{1}{2}$. We get that

$$\min_{t\in[k,1-k]}(T_1v)(t) \geq \mu \min\left\{\int_{\frac{1}{2}}^{1-k} \phi^{-1}\left(\frac{1}{p(s)}\int_{\frac{1}{2}}^s f(w,v(w)+h_2,v'(w))dw\right)ds,\right.$$

$$\left.\int_k^{\frac{1}{2}} \phi^{-1}\left(\frac{1}{p(s)}\int_s^{\frac{1}{2}} f(w,v(w)+h_2,v'(w))dw\right)ds\right\}$$

$$\geq \mu \min\left\{\int_k^{\frac{1}{2}} \phi^{-1}\left(W\frac{\int_s^{\frac{1}{2}} w^\alpha(1-w)^\beta dw}{p(s)}\right)ds,\ \int_{\frac{1}{2}}^{1-k} \phi^{-1}\left(W\frac{\int_{\frac{1}{2}}^s w^\alpha(1-w)^\beta dw}{p(s)}\right)ds\right\}$$

$$\geq e_2.$$

Case 2. $(T_1v)'(t) > 0$ on $[0,1]$. From Lemma 2.4(i), we get

$$p(t)\phi((T_1v)'(t)) = p(1)\phi((T_1v)'(1)) + \int_t^1 f(w,v(w)+h_2,v'(w))dw.$$

So

$$(T_1v)(t) = (T_1v)(0) + \int_0^t \phi^{-1}\left(\frac{p(1)\phi((T_1v)'(1))+\int_s^1 f(w,v(w)+h_2,v'(w))dw}{p(s)}\right)ds.$$

Hence

$$\min_{t\in[k,1-k]}(T_1v)(t) \geq \mu \max_{t\in[0,1]}(T_1v)(t)$$

$$\geq \mu \int_0^1 \phi^{-1}\left(\frac{p(1)\phi((T_1v)'(1))+\int_s^1 f(w,v(w)+h_2,v'(w))dw}{p(s)}\right) ds$$

$$\geq \mu \int_k^{1-k} \phi^{-1}\left(\frac{\int_s^{1-k} f(w,v(w)+h_2,v'(w))dw}{p(s)}\right) ds$$

$$\geq \mu \int_k^{1-k} \phi^{-1}\left(\frac{\int_s^{1-k} Ww^{\alpha}(1-w)^{\beta}dw}{p(s)}\right) ds \geq e_2.$$

Similarly we get that $\min_{t\in[k,1-k]}(T_2u)(t) \geq e_2$. It follows that $\alpha_1(T(u,v)) > c_2$ for every $(u,v) \in P(\beta_1,\beta_3,\alpha_1;c_2,c_4,c_5)$. This completes the proof of (A3)(ii) of Lemma 2.1.

Step 3. Prove that $\{(u,v) \in Q(\beta_1,\beta_2,\alpha_2;c_3,c_1,c_5)|\beta_2(u,v) < c_1\} \neq \emptyset$ and

$$\beta_2(T(u,v)) < c_1 \text{ for every } (u,v) \in Q(\beta_1,\beta_2,\alpha_2;c_3,c_1,c_5);$$

It is easy to show that $\{(u,v) \in P(\beta_1,\beta_2,\alpha_2;c_3,c_1,c_5) : \alpha_2(u,v) < c_1\} \neq \emptyset$.

For $(u,v) \in Q(\beta_1,\beta_2,\alpha_2;c_3,c_1,c_5)$, one has that

$$\alpha_2(u,v) = \min\left\{\min_{t\in[k,1-k]} u(t), \min_{t\in[k,1-k]} v(t)\right\} \geq c_3,$$

$$\beta_2(u,v) = \max\left\{\max_{t\in[0,1]} u(t), \max_{t\in[0,1]} v(t)\right\} \leq c_1,$$

$$\beta_1(u,v) = \max\left\{\sup_{t\in(0,1)} \phi^{-1}(p(t))|u'(t)|, \sup_{t\in(0,1)} \psi^{-1}(q(t))|v'(t)|\right\} \leq c_5.$$

Then

$$0 \leq u(t),v(t) \leq e_1, \quad \phi^{-1}(p(t))|u'(t)|, \psi^{-1}(q(t))|v'(t)| \leq c, \ \ t \in [0,1].$$

Thus (B8) implies that

$$f(t,v(t)+h_2,v'(t)) \leq Et^{\gamma}(1-t)^{\delta}, \ \ t \in (0,1).$$

We note that

$$A_v \in \left[\phi\left(M_0\right), \phi\left(M_0\right) + \int_0^1 f(w,v(w)+h_2,v'(w))dw\right].$$

So

$$(T_1v)(t) = B_v + \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(A_v - \int_0^s f(w, v(w) + h_2, v'(w))dw\right)ds$$

$$= \frac{\sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(A_v - \int_0^s f(w,v(w)+h_2,v'(w))dw\right)ds}{1-\sum_{i=1}^m a_i}$$

$$+\int_0^t \phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(A_v - \int_0^s f(w, v(w) + h_2, v'(w))dw\right)ds$$

$$\leq \frac{\sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1}{p(s)}\right)ds}{1-\sum_{i=1}^m a_i}\phi^{-1}\left(\phi(M_0) + \int_0^1 f(w, v(w) + h_2, v'(w))dw\right)$$

$$+\int_0^1 \phi^{-1}\left(\frac{1}{p(s)}\right)ds\phi^{-1}\left(\phi(M_0) + \int_0^1 f(w, v(w) + h_2, v'(w))dw\right)$$

$$\leq \left[\frac{\sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1}{p(s)}\right)ds}{1-\sum_{i=1}^m a_i} + \int_0^1 \phi^{-1}\left(\frac{1}{p(s)}\right)ds\right]$$
$$\times\phi^{-1}\left(\phi(M_0) + \int_0^1 Ew^\alpha(1-w)^\beta dw\right)$$

$$\leq \left[\frac{\sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1}{p(s)}\right)ds}{1-\sum_{i=1}^m a_i} + \int_0^1 \phi^{-1}\left(\frac{1}{p(s)}\right)ds\right]$$
$$\times\phi^{-1}\left(\phi(M_0) + E\mathbf{B}(\alpha+1, \beta+1)\right) \leq e_1 = c_1.$$

Hence $\max_{t\in[0,1]}(T_1v)(t) \leq c_1$. Similarly, we have $\max_{t\in[0,1]}(T_2u)(t) \leq c_1$. It follows that $\beta_2(T(u,v)) < c_1$. This completes the proof of (A3)(iii) of Lemma 2.1.

Step 4. Prove that $\alpha_1(T(u,v)) > c_2$ for $(u,v) \in P(\beta_1, \alpha_1; c_2, c_5)$ with $\beta_3(T(u,v)) > c_4$;

For $(u,v) \in P(\beta_1, \alpha_1; c_2, c_5)$ with $\beta_3(T(u,v)) > c_4$, we have that

$$\alpha_1(u,v) = \min\left\{\min_{t\in[k,1-k]} u(t), \min_{t\in[k,1-k]} v(t)\right\} \geq c_2 = e_2$$

$$\beta_1(u,v) = \max\left\{\sup_{t\in(0,1)} \phi^{-1}(p(t))|y'(t)|, \sup_{t\in(0,1)} \psi^{-1}(q(t))|v'(t)|\right\} \leq c_5$$

and

$$\beta_3(T(u,v)) = \max\left\{\max_{t\in[0,1]}(T_1v)(t), \max_{t\in[0,1]}(T_2u)(t)\right\} > \frac{e_2}{\min\{\mu,\nu\}} = c_4.$$

Then

$$\alpha_1(T(u,v)) = \min\left\{\min_{t\in[k,1-k]}(T_1v)(t), \min_{t\in[k,1-k]}(T_2u)(t)\right\}$$

$$\geq \min\left\{\mu \max_{t\in[k,1-k]}(T_1v)(t), \nu \max_{t\in[k,1-k]}(T_2u)(t)\right\}$$

$$\geq \min\{\mu,\nu\}\beta_2(T(u,v)) > \min\{\mu,\nu\}\tfrac{e_2}{\min\{\mu,\nu\}} = e_2 = c_2.$$

This completes the proof of (A3)(iv) of Lemma 2.1.

Step 5. Prove that $\beta_2(T(u,v)) < c_1$ for each $(u,v) \in Q(\beta_1,\beta_2;c_1,c_5)$ with $\alpha_2(T(u,v)) < c_3$;

For $(u,v) \in Q(\beta_1,\beta_2;c_1,c_5)$ with $\alpha_2(T(u,v)) < c_3$, we have that

$$\beta_2(u,v) = \min\left\{\min_{t\in[k,1-k]} u(t), \min_{t\in[k,1-k]} v(t)\right\} \leq e_1$$

and

$$\beta_1(u,v) = \max\left\{\sup_{t\in(0,1)} \phi^{-1}(p(t))|u'(t)|, \sup_{t\in(0,1)} \psi^{-1}(q(t))|v'(t)|\right\} \leq c_5$$

and

$$\alpha_2(T(u,v)) = \min\left\{\min_{t\in[k,1-k]}(T_1v)(t)\ \min_{t\in[k,1-k]}(T_2u)(t)\right\} < c_3 = \min\{\mu,\nu\}e_1.$$

Since $T(u,v) \in P$, we get

$$\beta_2(T(u,v)) = \max\left\{\max_{t\in[0,1]}(T_1v)(t), \max_{t\in[0,1]}(T_2u)(t)\right\}$$

$$\leq \tfrac{1}{\min\{\mu,\nu\}} \min\left\{\min_{t\in[k,1-k]}(T_1v)(t), \min_{t\in[k,1-k]}(T_2u)(t)\right\}$$

$$< \tfrac{1}{\min\{\mu,\nu\}} \min\{\mu,\nu\}e_1 = c_1.$$

This completes the proof of (A3)(v) of Lemma 2.1.

Then Lemma 2.1 implies that T has at least three fixed points (u_1, v_1), (u_2, v_2) and (u_3, v_3) such that

$$\beta_2(u_1, v_1) = \max\left\{\max_{t\in[0,1]} u_1(t), \max_{t\in[0,1]} v_1(t)\right\} < e_1,$$

$$\alpha_1(u_2, v_2) = \min\left\{\min_{t\in[k,1-k]} u_2(t), \min_{t\in[k,1-k]} v_2(t)\right\} > e_2,$$

$$\beta_2(u_3, v_3) = \max\left\{\max_{t\in[0,1]} u_3(t), \max_{t\in[0,1]} v_3(t)\right\} > e_1,$$

$$\alpha_1(u_3, v_3) = \min\left\{\min_{t\in[k,1-k]} u_3(t), \min_{t\in[k,1-k]} v_3(t)\right\} < e_2.$$

Hence BVP(1.2) has three decreasing positive solutions $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) such that

$$\max\left\{\max_{t\in[0,1]}[x_1(t) - h_1], \max_{t\in[0,1]}[y_1(t) - h_2]\right\} < e_1,$$

$$\min\left\{\min_{t\in[k,1-k]}[x_2(t) - h_1], \min_{t\in[k,1-k]}[y_2(t) - h_2]\right\} > e_2,$$

$$\max\left\{\max_{t\in[0,1]}[x_3(t) - h_1], \max_{t\in[0,1]}[y_3(t) - h_2]\right\} > e_1,$$

$$\min\left\{\min_{t\in[k,1-k]}[x_3(t) - h_1], \min_{t\in[k,1-k]}[y_3(t) - h_2]\right\} < e_2.$$

Hence BVP(1.2) has at least three pair of solutions $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) satisfy (3.1). The proof of Theorem 3.1 is completed. □

Remark 3.2. Similarly we can establish existence results for BVP(1.2) under the assumptions (B1), (B2), (B3), (B4i)$'$, (B4ii)$'$, assumptions (B1), (B2), (B3), (B4i)$''$, (B4ii)$''$, the assumptions (B1), (B2), (B3), (B4i)$'''$, (B4ii)$'''$, respectively. We omit the details.

4. An example

Now, we present an example to illustrate the main result: Theorem 3.1.

EXAMPLE 4.1. *Consider the following BVP of the second order differential system*

$$(4.1)\qquad \begin{cases} x''+t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}F(t,y(t),y'(t))=0, & t\in(0,1),\\ y''+t^{-\frac{1}{4}}(1-t)^{-\frac{1}{4}}G(t,x(t),x'(t))=0, & t\in(0,1),\\ x(0)=\frac{1}{2}x(1/2)+2, \\ x(1)=\frac{1}{4}x(1/4)+\frac{1}{4}x(1/2)+2, \\ y(0)=\frac{1}{4}y(1/2)+1, \\ y(1)=\frac{1}{8}y(1/4)+\frac{1}{8}y(1/2)+1, \end{cases}$$

where

$$F(t,w,z)=f_0(w)+\frac{t|z|}{8\times 10^{28}},\ G(t,w,z)=g_0(w)+\frac{t|z|}{8\times 10^{28}}$$

and

$$f_0(w)=\begin{cases} 0.159u, & u\in[0,4/3],\\ 0.2121, & u\in[4/3,10/3],\\ 0.2121+\frac{\frac{252830+156768}{2}-0.2121}{1000+4/3-10/3}\left(u-\frac{10}{3}\right), & u\in\left[\frac{10}{3},\frac{3004}{3}\right],\\ \frac{252830+156768}{2}, & u\in\left[\frac{3004}{3},10^7+\frac{4}{3}\right],\\ \frac{252830+156768}{2}e^{u-10^7-4/3}, & u\geq 10^7+4/3 \end{cases}$$

and

$$g_0(w)=\begin{cases} 0.106u, & u\in[0,4],\\ 0.2121, & u\in[4,6],\\ \frac{\frac{252830+156768}{2}-0.2121}{1004-6}(u-1004), & u\in[6,1000+4/3],\\ \frac{252830+156768}{2}, & u\in[1000+4/3,10^7+4],\\ \frac{252830+156768}{2}e^{u-10^7-4}, & u\geq 10^7+4. \end{cases}$$

Corresponding to BVP(1.2), one sees that

$$\phi(x)=\psi(x)=x,\ p(t)=q(t)\equiv 1,\ \xi_1=1/4,\xi_2=1/2,$$

$$a_1=0,a_2=1/2,b_1=1/4,b_2=1/4,\ c_1=0,c_2=1/4,$$

$$d_1=1/8,d_2=1/8,\ A=2,B=8,C=1,D=1$$

and

$$f(t,w,z)=t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}F(t,w,z),\quad g(t,w,z)=t^{-\frac{1}{4}}(1-t)^{-\frac{1}{4}}G(t,w,z).$$

Use Theorem 3.1. One sees that (B1), (B2), (B3), (B4i) and (B4ii) hold, $h_1=\frac{2}{1-1/2}=4$, $h_2=\frac{1}{1-1/4}=\frac{4}{3}$, $\alpha=\beta=-\frac{1}{2}$, $\gamma=\delta=-\frac{1}{4}$.

Choose constants $k=\frac{1}{4}$, $e_1=2, e_2=10^3, c=10^7$, then

$$\mu=\min\left\{\frac{\int_0^{1-k}\phi^{-1}\left(\frac{1}{p(s)}\right)ds}{2\int_0^1\phi^{-1}\left(\frac{1}{p(s)}\right)ds},\frac{\int_0^{k}\phi^{-1}\left(\frac{1}{p(s)}\right)ds}{2\int_0^1\phi^{-1}\left(\frac{1}{p(s)}\right)ds}\right\}=\frac{1}{8},$$

$$\nu=\min\left\{\frac{\int_0^{1-k}\psi^{-1}\left(\frac{1}{q(s)}\right)ds}{2\int_0^1\psi^{-1}\left(\frac{1}{q(s)}\right)ds},\frac{\int_0^{k}\psi^{-1}\left(\frac{1}{q(s)}\right)ds}{2\int_0^1\psi^{-1}\left(\frac{1}{q(s)}\right)ds}\right\}=\frac{1}{8},$$

$$\prod\nolimits_1=\int_0^1\phi^{-1}\left(\frac{1}{p(s)}\right)ds+\frac{1}{1-\sum\limits_{i=1}^m a_i}\left(\sum_{i=1}^m\frac{a_i\xi_i}{\phi^{-1}(p(\xi_i))}\right)=\frac{3}{2},$$

$$\prod\nolimits_2=\int_0^1\psi^{-1}\left(\frac{1}{q(s)}\right)ds+\frac{1}{1-\sum\limits_{i=1}^m c_i}\left(\sum_{i=1}^m\frac{c_i\xi_i}{\psi^{-1}(q(\xi_i))}\right)=\frac{7}{6},$$

$$M=\frac{1-\sum\limits_{i=1}^m b_i}{1-\sum\limits_{i=1}^m a_i}\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}\right)ds+\left(1-\sum_{i=1}^m b_i\right)\int_0^1\phi^{-1}\left(\frac{1}{p(s)}\right)ds$$

$$+\sum_{i=1}^m b_i\int_{\xi_i}^1\phi^{-1}\left(\frac{1}{p(s)}\right)ds,\quad M_0=\frac{\left(1-\sum\limits_{i=1}^m a_i\right)B-\left(1-\sum\limits_{i=1}^m b_i\right)A}{\left(1-\sum\limits_{i=1}^m a_i\right)M}=0,$$

$$N=\frac{1-\sum\limits_{i=1}^m d_i}{1-\sum\limits_{i=1}^m c_i}\sum_{i=1}^m c_i\int_0^{\xi_i}\psi^{-1}\left(\frac{1}{q(s)}\right)ds+\left(1-\sum_{i=1}^m d_i\right)\int_0^1\phi^{-1}\left(\frac{1}{p(s)}\right)ds$$

$$+\sum_{i=1}^m d_i\int_{\xi_i}^1\psi^{-1}\left(\frac{1}{q(s)}\right)ds,\quad N_0=\frac{\left(1-\sum\limits_{i=1}^m c_i\right)D-\left(1-\sum\limits_{i=1}^m d_i\right)C}{\left(1-\sum\limits_{i=1}^m c_i\right)N}=0,$$

$$L_1=\frac{\sum\limits_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}\right)ds}{1-\sum\limits_{i=1}^m a_i}+\int_0^1\phi^{-1}\left(\frac{1}{p(s)}\right)ds=\frac{3}{2},$$

$$L_2=\frac{\sum\limits_{i=1}^m c_i\int_0^{\xi_i}\psi^{-1}\left(\frac{1}{q(s)}\right)ds}{1-\sum\limits_{i=1}^m c_i}+\int_0^1\psi^{-1}\left(\frac{1}{q(s)}\right)ds=\frac{7}{6},$$

$$W_1 = \mu \min\left\{ \int_k^{\frac{1}{2}} \phi^{-1}\left(\frac{\int_s^{\frac{1}{2}} w^\alpha (1-w)^\beta dw}{p(s)} \right) ds,\ \int_{\frac{1}{2}}^{1-k} \phi^{-1}\left(\frac{\int_{\frac{1}{2}}^{s} w^\alpha (1-w)^\beta dw}{p(s)} \right) ds \right\}$$

$$\geq \frac{1}{8} \min\left\{ \int_{\frac{1}{4}}^{\frac{1}{2}} \int_s^{\frac{1}{2}} \sqrt{2}(1-s)^{-1/2} dw ds, \int_{\frac{1}{2}}^{\frac{3}{4}} \int_{\frac{1}{2}}^{s} s^{-1/2}\sqrt{2} dw ds \right\}$$

$$\geq \frac{\sqrt{2}}{8} \min\left\{ \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{1}{2} - s\right)(1-s)^{-1/2} ds, \int_{\frac{1}{2}}^{\frac{3}{4}} \left(s - \frac{1}{2}\right) s^{-1/2} ds \right\} \geq \frac{1}{128}\sqrt{\frac{2}{3}},$$

$$W_2 = \nu \min\left\{ \int_k^{\frac{1}{2}} \psi^{-1}\left(\frac{\int_s^{\frac{1}{2}} w^\gamma (1-w)^\delta dw}{q(s)} \right) ds,\ \int_{\frac{1}{2}}^{1-k} \psi^{-1}\left(\frac{\int_{\frac{1}{2}}^{s} w^\gamma (1-w)^\delta dw}{q(s)} \right) ds \right\}$$

$$\geq \frac{1}{32}\sqrt[4]{\frac{8}{3}}.$$

Note $\mathbf{B}(1/2, 1/2) \approx 3.1416 \leq 3.142, \mathbf{B}(3/4, 3/4) \approx 1.6944 \leq 1.695$. We get

$$Q = \min\left\{ \frac{\phi(c)-\phi(M_0)}{2\mathbf{B}(\alpha+1,\beta+1)}, \frac{\psi(c)-\phi(N_0)}{2\mathbf{B}(\gamma+1,\delta+1)}, \frac{\phi\left(\frac{c}{\prod_1}\right)-\phi(M_0)}{2\mathbf{B}(\alpha+1,\beta+1)}, \frac{\psi\left(\frac{c}{\prod_2}\right)-\phi(N_0)}{2\mathbf{B}(\gamma+1,\delta+1)} \right\}$$

$$\geq \min\left\{ \frac{333333}{\mathbf{B}(1/2,1/2)}, \frac{428571}{\mathbf{B}(3/4,3/4)} \right\} \geq \min\left\{ \frac{333333}{3.142}, \frac{428571}{1.695} \right\} \geq 252830;$$

$$E = \min\left\{ \frac{\phi\left(\frac{e_1}{L_1}\right)-\phi(M_0)}{\mathbf{B}(\alpha+1,\beta+1)}, \frac{\psi\left(\frac{e_1}{L_2}\right)-\psi(N_0)}{\mathbf{B}(\gamma+1,\delta+1)} \right\} \geq \min\left\{ \frac{4}{3\mathbf{B}(1/2,1/2)}, \frac{12}{7\mathbf{B}(3/4,3/4)} \right\}$$

$$\geq 0.4243,$$

$$W = \max\left\{ \phi\left(\frac{e_2}{W_1}\right),\ \psi\left(\frac{e_2}{W_2}\right),\ \phi\left(\frac{e_2}{\mu \int_k^{1-k} \phi^{-1}\left(\frac{\int_s^{1-k} w^\alpha (1-w)^\beta dw}{p(s)} \right) ds} \right), \right.$$

$$\psi\left(\frac{e_2}{\nu\int_k^{1-k}\psi^{-1}\left(\frac{\int_s^{1-k}w^\gamma(1-w)^\delta dw}{q(s)}\right)ds}\right)\Bigg\}\leq 64000\sqrt{6}\leq 156768.$$

If $Q\geq W$ and

$$c>\max\left\{M_0,N_0,\prod_1 M_0,\prod_2 N_0\right\},\ e_1>\max\{L_1M_0,L_2N_0\},$$

$$c>\tfrac{e_2}{\min\{\mu,\nu\}}>e_2>e_1>0$$

and

(B6)

$$f(t,w,z)\leq Qt^\alpha(1-t)^\beta, t\in(0,1), w\in[4/3,10^7+4/3], z\in[-10^7,10^7],$$

$$g(t,w,z)\leq Qt^\gamma(1-t)^\delta, t\in(0,1), w\in[4,10^7+4], z\in[-10^7,10^7];$$

(B7)

$$f(t,w,z)\geq Wt^\alpha(1-t)^\beta, t\in[1/4,3/4], z\in[-10^7,10^7],$$

$$w\in\left[10^3+4/3,8000+4/3\right]$$

$$g(t,w,z)\geq Wt^\gamma(1-t)^\delta, t\in[1/4,3/4], z\in[-10^7,10^7], w\in[1004,8004];$$

(B8)

$$f(t,w,z)\leq Et^\alpha(1-t)^\beta, t\in(0,1), w\in[4/3,10/3], z\in[-10^7,10^7],$$

$$g(t,w,z)\leq Et^\gamma(1-t)^\delta, t\in(0,1), w\in[4,6], z\in[-10^7,10^7].$$

Then applying Theorem 3.1 BVP(4.1) has at least three solutions (x_1,y_1), (x_2,y_2) and (x_3,y_3) such that

$$\max_{t\in[0,1]}x_1(t)<e_1+h_1=6,\ \max_{t\in[0,1]}y_1(t)<e_1+h_2=\tfrac{10}{3},$$

$$\min_{t\in[1/4,3/4]}x_2(t)>e_2+h_1=1004,\quad \min_{t\in[1/4,3/4]}y_2(t)>e_2+h_2=10^3+\tfrac{4}{3},$$

$$\text{either }\max_{t\in[0,1]}x_3(t)>e_1+h_1=6\text{ or }\max_{t\in[0,1]}y_3(t)>e_1+h_2=\tfrac{10}{3},$$

$$\text{either }\min_{t\in[1/4,3/4]}x_3(t)<e_2+h_1=1004,$$

$$\text{or }\min_{t\in[1/4,3/4]}y_3(t)<e_2+h_2=10^3+\tfrac{4}{3}.$$

REMARK 4.2. One can not get three solutions of BVP in Examples 4.1 by using the theorems obtained in papers [28-31].

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