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FINITE TYPE OF THE PEDAL OF REVOLUTION SURFACES IN E^3

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Abstract. Chen and Ishikawa studied the surfaces of revolution of the polynomial and the rational kind of finite type in Euclidean 3-space E^3 [10]. Here, the pedal of revolution surfaces of the polynomial and the rational kind are discussed. Also, as a special case of general revolution surfaces, the sphere and catenoid are studied for the kind of finite type.

1. Introduction

The study of submanifolds of finite type began in the late 1970's through the author's attempts to find the best possible estimate of the total mean curvature of a compact submanifold of a Euclidean space and to find a notion of "degree" for submanifolds of a Euclidean space.

The first results on this subject have been collected in [4] and [5]. Since that time, the subject has had a rapid development.

The class of submanifolds of finite type is large, it consists of nice submanifolds of Euclidean spaces. For example, all minimal submanifolds of a Euclidean space and all minimal submanifolds of hyperspheres are of 1-type and vice versa. Also, all parallel submanifolds of a Euclidean space and all compact homogeneous Riemannian manifolds equivariantly immersed in a Euclidean space are of finite type.

On one hand, the study of finite type submanifolds provides a natural way to combine the spectral theory with the geometry of submanifolds and smooth maps; in particular, with the Gauss map. On the other hand, the tools of geometry of submanifolds can then be applied to the study of spectral geometry via the study of finite type submanifolds. The notion of finite type immersion is naturally extended in particular to the Gauss map $\mathbf{G}$ on a surface M in Euclidean space [11], such that finite type Gauss map is an especially useful tool in the study of submanifolds [1] and [25]. Baikoussis et al. [2] have shown that the Gauss map of the cyclides of dupin is of infinite type. Also, Chen

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et al. [9] introduced the notion of pointwise 1-type Gauss map of the first and second kinds and study surfaces of revolution with such Gauss map. Soliman et al. [22] have studied ruled surfaces which are generated by a linear combination of Frenet frame along a base curve and found the conditions which determine Frenet surfaces of 1-type or pointwise 1-type Gauss map of the first kind depending on the kind of base curve in E^3. Also Baikoussis et al. [3] studied finite type of spiral surfaces, which are a generalization of the revolution surfaces in E^3.

The notion of the pedal of a given surface M in E^3 with respect to a chosen origin is well known in literature [13, 14, 15] and [19, 20, 21].

Georgiou et al [13] have studied the differential geometry of the pedal surface of M and they investigated the application in geometric optics. Kuruoğlu [19] has studied the pedal surface with respect to a point in the interior of a closed, convex and smooth surface in E^3 and given some new characteristic properties of the pedal surface of M. The pedal surface has been generalized by Kuruoğlu and Sarioğlugil, [20]. In addition, ruled surfaces were investigated first by G. Monge who established the partial differential equation satisfied by all ruled surfaces. Thus, ruled surfaces were formed by a one-parameter set of lines and investigated by Hlavaty [16] and Hoschek [17]. Furthermore, In [18] authors look for the answer of the question of what the pedal of the developable is ruled surface M in E^3 and they show that the pedal of M is a curve. Soliman et al [23] studied the variation problem of examples of pedal surfaces and pedal hypersurfaces in E^{n+1}.

In this work our main aim is to obtain a classification of the pedal of revolution surfaces of polynomial and rational kind in Euclidean 3-space E^3 of finite type, and to research if the character finite type is inherited or not. We found that the pedal of revolution surfaces of polynomial kind are infinite type. Also, we research in the revolution surfaces of rational kind which have degree of denominator larger than the degree of the numerator and got the pedal surfaces of them of infinite type. Finally, we prove that a pedal of surfaces does not preserve the property of finite type for these surfaces by given some examples.

2. Preliminaries

Here, and in the sequel, we assume that the index i, $j \geq 1$ unless otherwise stated.

Let a surface $M : \mathbf{X} = \mathbf{X}(u, v)$ in an Euclidean 3-space E^3. The map $\mathbf{G} : M \to S^2(1) \subset E^3$ which sends each point of M to the unit normal vector to M at the point is called the Gauss map of a surface M; where $S^2(1)$ denotes the unit sphere of E^3. And the Gauss map is given by

$$\mathbf{G} = \frac{\mathbf{X}_u \times \mathbf{X}_v}{\mid \mathbf{X}_u \times \mathbf{X}_v \mid}, \tag{1}$$

where $\mathbf{X}_u$ and $\mathbf{X}_v$ are the first partial derivatives with respect to the parameters u, v of $\mathbf{X}$. For the matrix (g_{ij}) of the Riemannian metric on M we denote by

(g^{ij}) the inverse matrix and g is the determinant of the matrix (g_{ij}). The Laplacian Δ associated with the induced metric g on M is given by

$$\Delta = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(\sqrt{g}\ g^{ij} \frac{\partial}{\partial x_j}). \tag{2}$$

The mean curvature H of the surface is defined by

$$H = \frac{1}{2} \sum_{i,j=1}^{2} g^{ij} L_{ij}, \tag{3}$$

where L_{ij} are the coefficients of the second fundamental form.

An isometric immersion $\mathbf{X} : M \to E^3$ of a submanifold M in E^3 is said to be of finite type if $\mathbf{X}$ identified with the position vector field of M in E^3 can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is,

$$\mathbf{X} = X_0 + \sum_{i=1}^{j} X_i, \tag{4}$$

where X_0 is a constant map and X_1, X_2, $\ldots$, X_j non-constant maps such that

$$\Delta\, X_i = \lambda_i\, X_i, \quad \lambda_i \in R, \quad 1 \leq i \leq j. \tag{5}$$

If $\lambda_1, \lambda_2, \ldots, \lambda_j$ are different eigen values, then M is said to be of j-type. If in particular, one of λ_i is zero then M is said to be of null j-type. If all coordinate function of E^3, restricted to M, are of finite type, then M is said to be finite type. Otherwise, M is said to be infinite type. Similarly, a smooth map ϕ an 2-dimensional Riemannian manifold M of E^3 is said to be finite type if ϕ is a finite sum of E^3-valued eigenfunctions of Δ [4] and [5].

Let M be a connected (not necessary compact) surface in E^3. Then the position vector $\mathbf{X}$ and the mean curvature vector H of M in E^3 satisfy [4]

$$\Delta\, \mathbf{X} = -2\, \mathbf{H}, \tag{6}$$

where $\mathbf{H} = H\mathbf{G}$. This formula yields the following well-known result: A surface M in E^3 is minimal if and only if all coordinate functions of E^3, restricted to M, are harmonic functions, that is,

$$\Delta\, \mathbf{X} = 0. \tag{7}$$

We recall theorem of T. Takahashi [24] and [7] which states that a submanifold M in a Euclidean space is of 1-type, i.e., the position vector field of the submanifold in the Euclidean space satisfies the differential equation

$$\Delta\, \mathbf{X} = \lambda\, \mathbf{X}, \tag{8}$$

for some real number λ, if and only if either the submanifold is a minimal submanifold of the Euclidean space ($\lambda = 0$) or it is a minimal submanifold of a hypersphere of the Euclidean space centered at the origin ($\lambda \neq 0$).

We mention the following known result for later use.

Proposition 2.1 ([4, 6, 8] and [10]). *Let M be a j-type surfaces whose spectral decomposition is given by Eq.* (4)*. If we put*

$$P(T) = \prod_{i=1}^{j} (T - \lambda_i), \tag{9}$$

then

$$P(\Delta)(\mathbf{X} - \mathbf{X}_0) = 0. \tag{10}$$

We can rewrite the previous equation as follows

$$\Delta^{j+1}\,\mathbf{X} + d_1\,\Delta^j\,\mathbf{X} + \cdots + d_j\,\Delta\,\mathbf{X} = 0, \tag{11}$$

where $d_1, d_2, \ldots, d_j$ are constants.

And the monic polynomial P is called the minimal polynomial which plays a very important role to find out whether or nor a surface is of finite type.

We say M be a surface of revolution if it is generated by a plane curve $\boldsymbol{\alpha}(u)$ when it is rotated around a straight line in the same plane. Let the plane be $x\,z$ and the line be z-axis. Then, the parametrization of the plane curve takes the following form [12]:

$$\boldsymbol{\alpha}(u) = \{f(u)\,,\; h(u)\}. \tag{12}$$

Hence the parametrization of the surface of revolution is given by

$$\mathbf{X}(u, v) = \{f(u)\,\cos v\,,\; f(u)\,\sin v\,,\; h(u)\}. \tag{13}$$

Chen and Ishikawa introduced in [10] the notion of surfaces of revolution of polynomial and rational kinds: A surface of revolution which is given by Eq. (13) is said to be of the polynomial kind if $f(u)$ and $h(u)$ are polynomial functions in u; and it is said to be of rational kind if $h(u)$ is a rational function.

Let M be a smooth, convex surface in E^3 and O be a point not on M. If $\mathbf{X}$ is the position vector of a point P on M with respect to O as origin and $\mathbf{G}$ is the inner unit normal vector of the surface at $P \in M$, then the support function F of M is defined by

$$F = - < \mathbf{X}\,,\, \mathbf{G} >, \tag{14}$$

where $<\,,\,>$ is an inner product in E^3. Geometrically, F is the distance from the origin 0 to the tangent plane of M at the point of M described by $\mathbf{X}$.

Suppose now that there exists a point 0 that lies on no tangent plane of M; we call such a point an admissible origin for M. If we choose an admissible 0 as origin, the corresponding support function clearly never vanishes. Thus, by connectivity, either $F > 0$ or $F < 0$.

The surface $\overline{M}$ with the position vector

$$\overline{\mathbf{X}} = -F\,\mathbf{G}, \tag{15}$$

of an arbitrary point $\overline{P}$ on the tangent plane $TM(P)$ of M with respect to O as origin is called the pedal surface of M with respect to O. Geometrically, we can construct the pedal surface $\overline{M}$ as follows: We draw tangent plane $TM(P)$ and we get the normal to that plane from O. The normal contacts the plane $TM(P)$ at a point $\overline{P}$. The locus of all points $\overline{P}$ for all $P \in M$ will give the pedal surface.

3. Pedal of the revolution surfaces of polynomial kind

The goal of this section is to study pedal of revolution surfaces M of the polynomial kind in the Euclidean 3-space E^3 and to look the property of finite type of them.

Let M be a surfaces of revolution of the polynomial kind. Hence the position vector of M is given as Eq. (13).

Then, the unite normal vector filed of M is defined by

$$\mathbf{G} = \frac{1}{\sqrt{\gamma}}\{-\cos v\, h'(u)\,,\ -\sin v\, h'(u)\,,\ f'(u)\}, \tag{16}$$

where, $\gamma = f'^2 + h'^2 \neq 0$.

Using Eqs. (13)–(16), we find that the pedal surface of revolution of the polynomial kind can be written in the form

$$\overline{\mathbf{X}}(u,v) = \frac{1}{\gamma}\{\lambda\, h'\ \cos v\,,\ \lambda\, h'\ \sin v\,,\ -\lambda\, f'\}, \tag{17}$$

where, $\lambda = f\,h' - h\,f' \neq 0$. Therefore we get:

Corollary 3.1. *The Pedal of revolution surface of polynomial kind is a revolution surface.*

The unite normal vector filed of $\overline{M}$ is given by

$$\overline{\mathbf{G}} = \frac{1}{\gamma\,\beta}\{\omega\ \cos v\,,\ \omega\ \sin v\,,\ \mu\}, \tag{18}$$

where

(19)

$$\omega = 2\,f'h\,h' + f\,(f'^2 - h'^2),\ \mu = 2\,f\,f'h' + h\,(h'^2 - f'^2),\ \beta = \sqrt{f^2 + h^2} \neq 0.$$

Therefore, using Mathematica program, we can be written the Laplacian $\overline{\Delta}$ of $\overline{M}$ as

$$\overline{\Delta} = \frac{\gamma^2}{\beta^4\epsilon^3\lambda^2h'^2}\Big(h'\lambda\big(f^3h'^2\psi + fhh'^2\theta\big)\frac{\partial}{\partial u} - \epsilon\beta^2\lambda^2h'^2\frac{\partial^2}{\partial u^2} - \beta^4\epsilon^3\frac{\partial^2}{\partial v^2}\Big), \tag{20}$$

where

$$\begin{aligned}
&\epsilon = h'f'' - f'h'' \neq 0,\\
&\phi = 2\,f'h''^2 + h'^2f^{(3)} - h'(2\,f''h'' + f'^{(3)}),\\
&\psi = h\,h'^2f''^2 + f'h'^2(h'f''^{(3)}) - f'^2\big(h''\,(h'^2 + h\,h'') - h\,h'^{(3)}\big),\\
&\rho = \ -f'^2h'f'' + f'^3h'' + h'\ \big(f''(h'^2 - 2\,h\,h'') + h\,h'^{(3)}\big)\\
&\qquad - f'\big(h''(h'^2 - 2\,h\,h'') + h\,h'^{(3)}\big),
\end{aligned} \tag{21}$$

$$\theta = h\,h'^2 f''^2 - f'h'^2(h'\,f''^{(3)}) + f'^2\big(h''(h'^2 - h\,h'') + h\,h'^{(3)}\big).$$

Let $\overline{X}_1$, $\overline{X}_2$, and $\overline{X}_3$ be the three components functions of the $\overline{\mathbf{X}}$. Here, we take

$$\overline{X}_3 = -\frac{f'\,\lambda}{\gamma}. \tag{22}$$

Then, by direct computation, it leads immediately to

$$\overline{\Delta}\ \overline{X}_3 = \frac{R_1(u)}{Q_1(u)}, \tag{23}$$

where

$$\begin{aligned} R_1 &= \mu\,\big(h'^2\,\gamma - \beta^2\,\eta\,f''^2\,f'\,\eta\,h''\big), \quad Q_1 = -\beta^4\,h'\,\lambda\,\gamma\,\epsilon, \\ \eta &= 4\,h\,f'h' + f\,(f'^2 - 3\,h'^2). \end{aligned} \tag{24}$$

We give the following lemma to help us in our work.

Lemma 3.1. *Let $R_1(u)$ and $Q_1(u)$ be polynomial functions in u and $\overline{M}$ is the pedal surface of a revolution M of polynomial kind in E^3 which is parametrized by Eq.* (13). *Then*

$$\overline{\Delta}\,(\frac{R_1(u)}{Q_1(u)}) = \frac{R_2(u)}{Q_2(u)}$$

for some polynomial functions $R_2(u)$ and $Q_2(u)$ with,

$$\begin{aligned} &\deg R_2(u) - \deg Q_2(u) \\ &\leq\ \deg R_1(u) - \deg Q_1(u) - 2\,(q + r) + 2\ \max\{r\,,\ q-1\} + s, \end{aligned} \tag{25}$$

where s is a positive integer.

Proof. By using Eq. (20) and straight-forward computations, we get

$$\overline{\Delta}\,(\frac{R_1(u)}{Q_1(u)}) = \frac{R_2(u)}{Q_2(u)}, \tag{26}$$

where

$$\begin{aligned} R_2(u) = -\gamma^2\,\Big(&-\beta^2\,h'\lambda\epsilon\big(R_1(2\,Q_1'^{\,2} - Q_1Q_1'') + Q_1(Q_1R_1'' - 2\,Q_1'R_1')\big) \\ &+ Q_1(Q_1\,R_1' - R_1Q_1')(f^3\,h'\phi + f^2\psi + f\,hh'\rho + h^2\theta)\Big), \end{aligned}$$

$$Q_2(u) = -\beta^4 Q_1^3 h'\lambda\epsilon^3. \tag{27}$$

For convenient, put $\deg h = q$ and $\deg f = r$ where $\deg h$ and $\deg f$ are degrees of $h(u)$ and $f(u)$, respectively. Then, it is easy to see that

$$\begin{aligned} &\deg\beta^2 = 2\max\{q\,,\,r\}, \quad \deg\gamma = 2\ \max\{q\,,\,r\} - 2, \quad \deg\lambda \leq q + r - 1, \\ &\deg\phi = 2\,q + r - 5, \quad \deg\psi \leq 3\,q + 2\,r - 6, \quad \deg\epsilon \leq q + r - 3, \\ &\deg\rho \leq 2\ \max\{q\,,\,r\} + q + r - 5, \quad \deg\mu \leq 2\ \max\{q\,,\,r\} + q - 2, \\ &\deg\eta \leq 2\ \max\{q\,,\,r\} + r - 2, \quad \deg\theta \leq 3\,q + 2\,r - 6, \\ &\deg R_1 \leq 6\ \max\{q\,,\,r\} + 2q + 2r - 7, \quad \deg Q_1 \leq 6\max\{q\,,\,r\} + 3\,q + 2\,r - 7, \end{aligned}$$

$\deg R_2 \leq 6\ \max\{q\,,\, r\} + \deg R_1 + 2\ \deg Q_1 + 3\,q + 2\,r - 11,$
$\deg Q_2 \leq 4\ \max\{q\,,\, r\} + 3\ \deg Q_1 + 5\,q + 4\,r - 11.$

Let $s = 4\ \max\{q\,,\, r\} + 3\deg Q_1 + 5\,q + 4\,r - 11 - \deg Q_2 \geq 0$. Hence,

$$\begin{aligned}\deg(\frac{R_2}{Q_2}) &= \deg R_2 - \deg Q_2 \\ &\leq \deg R_1 - \deg Q_1 - 2\,(q + r) + 2\ \max\{q\,,\, r\} + s.\end{aligned}$$

For $\deg R_2 - \deg Q_2 \leq \deg R_1 - \deg Q_1$, it must be $2\max\{q\,,\, r\} - 2(q+r) + s \leq 0$. Therefore, we have the following probabilities.

(i) If $r \geq q \Rightarrow s \leq 2\,q$.
(ii) If $q > r \Rightarrow s \leq 2\,r$. □

Moreover, by Eqs. (23) and (26), we may conclude inductively that

$$\overline{\Delta}^j\, \overline{X}_3 = \frac{R_j}{Q_j} \tag{28}$$

for some polynomial functions R_i and Q_i. Therefore, by Lemma 3.1 we have

$$\deg R_{j+1} - \deg Q_{j+1} \leq \deg R_j - \deg Q_j \leq\ \cdots\ \leq \deg R_1 - \deg Q_1. \tag{29}$$

Suppose $\overline{M}$ is of finite type, say of $j-$type. Let

$$P(T) = T^j + d_1\, T^{j-1} +\ \cdots\ + d_{j-1}\, T + d_j, \tag{30}$$

be minimal polynomial of $\overline{M}$ given in Proposition 2.1. Then P has k distinct real roots. Consequently, from Eq. (28) and Proposition 2.1 we have

$$\frac{R_{j+1}}{Q_{j+1}} + d_1\, \frac{R_j}{Q_j} +\ \cdots\ + d_j\, \frac{R_1}{Q_1} = 0. \tag{31}$$

Put $D = Q_{j+1}\, Q_j\ \cdots\ Q_1$. Therefore, we get

$$D\,\frac{R_{j+1}}{Q_{j+1}} + d_1\, D\,\frac{R_j}{Q_j} +\ \cdots\ + d_j\, D\,\frac{R_1}{Q_1} = 0. \tag{32}$$

From Eq. (29) we find

$$\deg D\,\frac{R_{j+1}}{Q_{j+1}} \leq \deg D\,\frac{R_j}{Q_j} \leq\ \cdots\ \leq \deg D\,\frac{R_1}{Q_1} \tag{33}$$

which is impossible. Based on the above results, we conclude the following

(i) If $q = 0$, then $\mathbf{X} = \{f(u)\ \cos v\,,\ f(u)\ \sin v\,,\ c\}$; where c is constant. Therefore M is a plane and $\overline{M}$ is a degenerate pedal surface to point $(0,\, 0,\, c)$.
(ii) If $r = 0$, then $\mathbf{X} = \{c\,\cos v\,,\ c\,\sin v\,,\ h(u)\}$; where c is constant. Therefore
$$\overline{\mathbf{X}} = \{c\cos v\,,\ c\sin v\,,\ 0\}.$$
That is, $\overline{M}$ degenerate to a circular curve of 1-type. It is clear that for $q = 1$, then M is circular cylinder.

(iii) If $r = q = 1$, then $\mathbf{X} = \{(a\, s + b)\, \cos v\,,\, (a\, s + b)\, \sin v\,,\, c\, s + d\}$ where a, b, c, and d are constants. Therefore M is a cone (infinite type [12]). Then

$$\overline{\mathbf{X}} = \{\frac{c\,(bc - ad)\, \cos v}{a^2 + c^2}\,,\, \frac{c\,(bc - ad)\, \sin v}{a^2 + c^2}\,,\, \frac{a\,(ad - bc)}{a^2 + c^2}\},$$

one can see $\overline{M}$ degenerate to a circular curve of 1-type where $bc - ad \neq 0$. However, if $bc - ad = 0$, then $\overline{M}$ degenerates to a point $\{0,\, 0,\, 0\}$.

(iv) If $r, q \geq 1$ otherwise $r = q = 1$ then, from (33) and Lemma 3.1 we conclude that it is impossible. To illustrate this we give the following example.

Example 3.1. Let $f(u) = 2u - 3$ and $h(u) = 2u^4 + 7u^3 - 4u - 6$. Then

$$r = 1,\ q = 4,\ r_1 = 27,\ q_1 = 31,\ r_{i+1} = 2q_i + r_i + 27,\ q_{i+1} = 3q_i + 29,\ \forall\, i.$$

Substituting the above values in Eq. (32) and taking some cases we get

- If $j = 1 \;\Rightarrow\; D\,\frac{R_2}{Q_2} + d_1 D\,\frac{R_1}{Q_1} = 0$, the resulting equation from degree 149.
- If $j = 2 \;\Rightarrow\; D\,\frac{R_3}{Q_3} + d_1\, D\,\frac{R_2}{Q_2} + d_2\, D\,\frac{R_1}{Q_1} = 0$, and this equation has degree 540 where $D\,\frac{R_3}{Q_3}$ has the largest degree, and these cases lead to contradiction with (33). See Figure 1.

(A) Surface M (B) The pedal surface $\overline{M}$

$u\ \in [0.001, 1],\ v\ \in [-3\pi, 3\pi]$

FIGURE 1. The pedal of revolution surfaces of polynomial kind

(v) If $\overline{M}$ is a plane, then $\overline{X}_3 = c =$ constant. Hence $c = -\frac{\lambda\, f'}{\gamma} = \frac{f'\,(h\, f' - f\, h')}{f'^2 + h'^2}$, and this implies

(1) If $c = 0 \;\Rightarrow\; f'\,(h\, f' - f\, h') = 0$. Then
- Either $f' = 0 \;\Rightarrow\; f =$ constant $\Rightarrow\; r = 0$, this contrasts with Case (ii).
- Or $(h\, f' - f\, h') = 0 \;\Rightarrow\; \frac{h'}{h} = \frac{f'}{f} \;\Rightarrow\; h = cf \;\Rightarrow\; \overline{\mathbf{X}} = (0,\, 0,\, 0)$, this contradiction.

(2) If $c \neq 0 \;\Rightarrow\; f'(h\,f' - f\,h') = c\,(f'^2 + h'^2) \;\Rightarrow\; h'\,(c\,h' + f\,f') = f'^2\,(h-c) \;\Rightarrow\; 2\deg f \;\leq\; \max\{2\deg f\,,\ \deg h\}$. Then, either $\deg f \geq \frac{\deg h}{2}$ or $\deg h > \deg f$, and this is impossible as the following example shows.

Example 3.2. Let $f(u) = u^3 - 2u - 3$ and $h(u) = 2u^4 + 7u^3 - 4u - 6$, where $r = 3,\ q = 4$. From Figure 2 one can see $\overline{M}$ is not plane.

(A) Surface M (B) The pedal surface $\overline{M}$

$u \in [0.001, 1],\ v \in [-3\pi, 3\pi]$

FIGURE 2. The pedal of revolution surfaces of polynomial kind

(vi) If $\overline{M}$ is a circular cylinder, then from Eq. (17) we find $\frac{\lambda\,h'}{\gamma} =$ constant. Consequently, by derivative we get

$$\frac{(\lambda'\,h' + \lambda\,h'')\gamma - \lambda\,h'\,\gamma'}{\gamma^2} = 0 \quad\Rightarrow\quad \frac{\lambda'}{\lambda} + \frac{h''}{h'} = \frac{\gamma'}{\gamma} \quad\Rightarrow\quad \ln h' = \ln\frac{c\,\gamma}{\lambda}$$

$$\Rightarrow \quad h'^{\frac{c\,\gamma}{\lambda}},$$

This means that h is an exponential function and this is a contradiction where $\frac{c\,\gamma}{\lambda} \neq constant$. Because if $\frac{c\,\gamma}{\lambda} = constant = c_1$, then $c\,(f'^2 + h'^2) = c_1(f\,h' - h\,f')$. Therefore,

$$c\,f'^2 + c\,h'^2 - c_1\,f\,h' + c_1\,h\,f' = 0.$$

Consequently, one can have the following cases

- If $r = q \;\Rightarrow\; \max\{2(r-1)\,, 2(q-1)\,, r+q-1\} = 2r-1 = 0 \;\Rightarrow\; r = \frac{1}{2}$ and this is impossible.
- If $r > q \;\Rightarrow\; \max\{2(r-1)\,,\ 2(q-1)\,,\ r+q-1\} \;=\; 2r-2 \;=\; 0 \;\Rightarrow\; r = 1,\ q = 0$ and this gives us Case (i) and this contradiction.
- If $q > r \;\Rightarrow\; \max\{2(r-1)\,,\ 2(q-1)\,,\ r+q-1\} \;=\; 2q-2 \;=\; 0 \;\Rightarrow\; q = 1,\ r = 0$ and its back us to Case (ii) and this contradiction.

According to the result in [10] and our result, we can reformulate the following theorem.

Theorem 3.1. *The pedal surface of revolution $\overline{M}$ is infinite type if and only if its original surface M is infinite type except M is cone.*

4. Pedal of the revolution surfaces of rational kind

In this section, we shall study pedal $\overline{M}$ of revolution surface M of rational kind in the Euclidean 3-space E^3.

Assume M be a revolution surface of rational kind. Then, without loss of generality, we may assume M is parameterized by

$$\mathbf{X}(u,v)=\big\{u\,\cos v\,,\;u\,\sin v\,,\;\frac{A(u)}{B(u)}\big\},\quad B(u)\neq 0, \tag{34}$$

where, $A(u)$ and $B(u)$ are polynomials. Therefore, the unite normal vector field on M is given by

$$\mathbf{G}=\frac{1}{\sqrt{Q(u)}}\{-Z(u)\,\cos v\,,\;-Z(u)\,\sin v\,,\;B^2(u)\}, \tag{35}$$

where

$$Z(u)=B(u)\,A'(u)-A(u)\,B'^4(u)+Z^2(u)\neq 0,\quad Z(u)\neq 0. \tag{36}$$

Hence, we get the pedal of the revolution surface of rational kind writing as follows

$$\overline{\mathbf{X}}(u,v)=\frac{1}{Q(u)}\{R(u)\,\cos v\,,\;R(u)\,\sin v\,,\;\gamma(u)\}, \tag{37}$$

where

$$R(u)=Z(u)\big(u\,Z(u)-A(u)\,B(u)\big),\;\gamma(u)=B^2(u)\,\big(A(u)\,B(u)-u\,Z(u)\big). \tag{38}$$

From above we obtain:

Corollary 4.1. *The Pedal of revolution surface of rational kind is a revolution surface.*

The unite normal vector filed on $\overline{M}$ is

$$\overline{\mathbf{G}}=\frac{1}{\sqrt{\eta(u)}}\{\epsilon(u)\,\cos v\,,\;\epsilon(u)\,\sin v\,,\;-\delta(u)\}, \tag{39}$$

where

$$\begin{aligned}&\epsilon=\gamma(u)\,Q'(u)-Q(u)\,\gamma'(u)\neq 0,\quad \delta=R(u)\,Q'(u)-Q(u)\,R'(u)\neq 0,\\ &\eta=\epsilon^2(u)+\delta^2(u)\neq 0.\end{aligned} \tag{40}$$

Then, by direct computations, we can find the Laplacian $\overline{\Delta}$ of $\overline{M}$ is given by

$$\overline{\Delta}=\frac{Q^2}{2\,\eta^2\,R^2}\Big(Q\,\big(2R\,\eta\,(Q\,R)'+Q\,\eta'^2\big)\,\frac{\partial}{\partial u}-2\,Q^2\,\eta\,R^2\,\frac{\partial^2}{\partial u^2}-2\,\eta^2\,\frac{\partial^2}{\partial v^2}\Big). \tag{41}$$

Let $\overline{X}_1$, $\overline{X}_2$, and $\overline{X}_3$ be the three components functions of $\overline{\mathbf{X}}$. Then, we take

$$\overline{X}_1=\frac{R}{Q}\,\cos v. \tag{42}$$

Easy computations, we can get

$$\overline{\Delta}\ \overline{X}_1 = \frac{R_1}{Q_1}\ \cos v, \tag{43}$$

where

$$R_1 = Q\,\Big(-\eta^2 - \delta\,(Q\,\eta\,R\,)' - R\,\eta\big(R\,(Q\,Q')' + Q\,(Q\,R')'\,\big) + \frac{3}{2}\,Q\,R\,\delta\,\eta' \\ + 3\,Q'R\,\eta(Q'\,R - Q\,R')\Big),$$

(44) $Q_1 = R\,\eta^2$.

Consequently

$$\overline{\Delta}^i\ \overline{X}_1 = \frac{R_i}{Q_i}\ \cos v, \tag{45}$$

where

$$\begin{aligned} R_{i+1} = {} & Q^{\,3}\,Q_i\,R\,(Q\,\eta\,R_i\,RQ'_i)'^4\,Q_i{}'^2\,R^{\,2}\,R_i\,\eta \\ & + Q_i^{\,2}\,Q^{\,2}\,\big(R_i\,\eta^2 - Q\,R\,(R'_i\,Q\,\eta\,R\,)'\,\big) \\ & + Q^{\,4}\,Q_i\,Q'_i\,R^{\,2}\,R'_i\,\eta - \frac{3}{2}Q^{\,4}\,Q_i\,R^{\,2}\,\eta'(R_i\,Q'_i - R'_i\,Q_i), \end{aligned}$$

$$Q_{i+1} = Q_i^3\,R^{\,2}\,\eta^2, \quad \forall\, i. \tag{46}$$

Assume $\overline{M}$ is of finite type, say j-type. From Eqs. (11) and (45) we obtain

$$\frac{R_{j+1}}{Q_{j+1}} + d_1\,\frac{R_j}{Q_j} + \ \cdots\ + d_j\,\frac{R_1}{Q_1} = 0. \tag{47}$$

Let $D = Q_{j+1}\,Q_j\ \cdots\ Q_1$. Hence

$$D\,\frac{R_{j+1}}{Q_{j+1}} + d_1\,D\,\frac{R_j}{Q_j} + \ \cdots\ + d_j\,D\,\frac{R_1}{Q_1} = 0. \tag{48}$$

For simplicity, we put $a = \deg A(u)$, $b = \deg B(u)$, $r = \deg R(u)$, $q = \deg Q(u)$, $r_1 = \deg R_1(u)$, $q_1 = \deg Q_1(u)$, $r_i = \deg R_i(u)$, and $q_i = \deg Q_i(u)$.

We will study the following cases.

Case (i). if $a > b$

Lemma 4.1. *If $a > b$, then, for any $i \geq 1$, we have*

$$r_i - q_i \leq 1 - 2i + s, \tag{49}$$

where s is a positive integer.

Proof. From Eqs. (36), (38), and (44) we get

$$r \leq 2(a+b) - 1, \quad q = 2(a+b) - 2, \quad r_1 \leq 18(a+b) - 18, \quad q_1 \leq 18(a+b) - 17.$$

Let $s = 18(a+b) - 17 - q_1 \geq 0$. Then, $r_1 - q_1 \leq -1 + s$. Assume (49) holds for some $i \geq 1$. Therefore, by Eqs. (46) we get

$$r_{i+1} \leq 2q_i + r_i + 20(a+b) - 20, \quad q_{i+1} = 3q_i + 20(a+b) - 18 - s \geq 0. \tag{50}$$

This implies $r_{i+1} - q_{i+1} \le r_i - q_i - 2 + s \le 1 - 2(i+1) + s$. $\square$

From the above lemma, we have a large number of possibilities, so we give an illustrative example.

Example 4.1. Let $A(u) = u^3 + 2$ and $B(u) = u^{\,2} + u \neq 0$. Then

$$r = 8,\ q = 8,\ \deg \eta = 28,\ r_1 = 66,\ q_1 = 64,\ r_{i+1} = 2q_i + r_i + 74,$$
$$q_{i+1} = 3q_i + 72, \quad \forall\, i.$$

Inserting the above values in Eq. (48) will imply some cases.

- If $j = 1 \ \Rightarrow\ D\,\frac{R_2}{Q_2} + d_1\, D\,\frac{R_1}{Q_1} = 0$, we find this equation from degree 332.
- If $j = 2 \ \Rightarrow\ D\,\frac{R_3}{Q_3} + d_1\, D\,\frac{R_2}{Q_2} + d_2\, D\,\frac{R_1}{Q_1} = 0$, also this equation has degree 1198.

We note if the value of j is increased, then the degree of Eq. (48) is greatly increased. Therefore, this case is impossible. According to the results in [10] so M is infinite type and according to our results $\overline{M}$ is infinite type and Figure 3 show that.

(A) Surface M (B) The pedal surface $\overline{M}$

$u \in]0,3],\ v \in\ [0.5, \pi]$

FIGURE 3. The pedal of revolution surfaces of rational kind of Case (i)

Case (ii). if $b > a$

Lemma 4.2. *If $b > a$, then, for any $i \ge 1$, we have*

$$r_i - q_i = (2i-1)(q-r). \tag{51}$$

Proof. From Eqs. (36), (38), and (44) we get

$$r = 2(a+b)-1,\quad q = 4b,\quad r_1 = 7q+2r-2,\quad q_1 = 6q+3r-2,\ \deg \eta = 3q+r-1.$$

Thus, Eq. (51) holds for $i = 1$. Assume Eq. (51) holds for some $i \ge 1$. Therefore, from above and Eqs. (46) we get

$$r_{i+1} = 2q_i + r_i + 8q + 2r - 2, \quad q_{i+1} = 3q_i + 4r + 6q - 2. \tag{52}$$

This implies $r_{i+1} - q_{i+1} = (2i+1)(q-r) > 0$. □

Using above Lemma and Eq. (48) we can conclude that Case (ii) is impossible.

Here, we will provide an example to support our result.

Example 4.2. Let $A(u) = u^2 - u + 2$ and $B(u) = 3u^5 - 7u^4 + u^2 - 2u - 3$ where $u \neq 2.35667$. Then

$$r = 13,\ q = 20,\ \deg\eta = 72,\ r_1 = 164,\ q_1 = 157,\ r_{i+1} = 2q_i + r_i + 184,$$
$$q_{i+1} = 3q_i + 170, \quad \forall\, i.$$

Substituting the above values in Eq. (48) will obtain the surface of infinite type where the origin surface is infinite type as shown in Figure 4.

(A) Surface M (B) The pedal surface $\overline{M}$

$u \in [0.1, 2],\ v \in [\pi, 2\pi]$

FIGURE 4. The pedal of revolution surfaces of rational kind of Case (ii)

Case (iii). if $a = b$

Let $m = \deg(A+B)$. We divide this case into three subcases:

Case (iii-a). *if $a = b = m$.* Without loss of generality, we may assume the leading coefficient of $B(u)$ is 1.

Lemma 4.3. *Assume $a = b = m$. Then we get*

$$r_i - q_i \leq 2(2i-1) + s, \quad \forall\, i, \tag{53}$$

where s is a positive integer.

Proof. Since $a = b = m$, we have

$$r \leq 4m - 2,\quad q = 4m,\quad \deg\eta \leq 16m - 4,\quad q_1 \leq 36m - 10,\quad r_1 \leq 36m - 8.$$

Let $s = 36m - 10 - q_1 \geq 0$. Thus, (53) holds for $i = 1$. Assume (53) holds for some $i \geq 1$. Therefore, from above and Eqs. (46) we get

$$r_{i+1} \leq 2q_i + r_i + 40m - 8,\qquad q_{i+1} = 3q_i + 40m - 12 - s. \tag{54}$$

This implies $r_{i+1} - q_{i+1} \leq 2(2i+1) + s$. □

Also, we have a lot of possibilities, such as Case (i).

The following example gives illustrate the previous case.

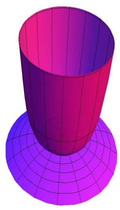
Example 4.3. Let $A(u) = 3u^3 + u - 1$ and $B(u) = u^3 + u^2 - 7u + 5 \neq 0$. Then

$$r = 10,\ q = 12,\ \deg \eta = 44,\ r_1 = 100,\ q_1 = 98,$$
$$r_{i+1} = 2q_i + r_i + 112,\ q_{i+1} = 3q_i + 108, \quad \forall\, i.$$

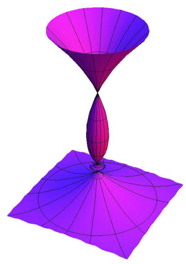
Making these values in Eq. (48) and taking some properties we get

- If $j = 1 \ \Rightarrow\ D\,\frac{R_2}{Q_2} + d_1\, D\,\frac{R_1}{Q_1} = 0$, the resulting equation from degree 506.
- If $j = 2 \ \Rightarrow\ D\,\frac{R_3}{Q_3} + d_1\, D\,\frac{R_2}{Q_2} + d_2\, D\,\frac{R_1}{Q_1} = 0$, too, this equation has degree 1824.

And these cases lead to contradiction. See Figure 5.



(A) Surface M



(B) The pedal surface $\overline{M}$

$u \in [-1, 0.99]$, $v \in [0, 2\pi]$

FIGURE 5. The pedal of revolution surfaces of rational kind of Case (iii-a)

Case (iii-b). *if* $a = b = m + 1$. Let

$$A = -u^a + E(u) \text{ and } B = u^a + L(u), \tag{55}$$

where $E(u)$ and $L(u)$ are polynomials of degree $\leq a - 1 = m$. In this case we get the same result as in Case (iii-a).

This is an example of the above case.

Example 4.4. Let $A(u) = -u^4 + u^2 + 4$ and $B(u) = u^4 + 3u^2 - u + 1 \neq 0$. Then

$$r = 13,\ q = 16,\ \deg \eta = 58,\ r_1 = 132,\ q_1 = 129,\ r_{i+1} = 2q_i + r_i + 148,$$
$$q_{i+1} = 3q_i + 142, \quad \forall\, i,$$

and put these values in Eq. (48) we find

- If $j = 1 \;\Rightarrow\; D\,\frac{R_2}{Q_2} + d_1\,D\,\frac{R_1}{Q_1} = 0$, the resulting equation from degree 667.
- If $j = 2 \;\Rightarrow\; D\,\frac{R_3}{Q_3} + d_1\,D\,\frac{R_2}{Q_2} + d_2\,D\,\frac{R_1}{Q_1} = 0$, and this equation has degree 2402.

Also here, we got a contradiction as in the previous cases as shown in Figure 6.

(A) Surface M (B) The pedal surface $\overline{M}$

$u \in [0.5\pi, 2\pi],\ v \in [0, 2\pi]$

FIGURE 6. The pedal of revolution surfaces of rational kind of Case (iii-b)

Case (iii-c). *if* $a = b > m+1$. In this case we may put

$$A = -u^{m+1}\,W(u) + E(u),\ \text{and}\ B = u^{m+1}\,W(u) + L(u), \tag{56}$$

where $W(u)$ is a polynomial of degree $a-m-1$ and $E(u)$, $L(u)$ are polynomials of degree $\le m$ such that $\deg(E+L) = m$. Therefore, we give the following lemma.

Lemma 4.4. *Assume $a = b > m+1$. Then we get*

$$r_i - q_i \le 2(2i-1)(a-m) + 2i - 1 + s, \quad \forall\, i, \tag{57}$$

where s is a positive integer.

Proof. From Eqs. (36), (38), (40), and (44) we obtain

$$q = 4a,\ r \le 2a + 2m - 1,\ \deg \eta \le 4(4a-1),\ q_1 \le 34a + 2m - 9,\ r_1 \le 36a - 8.$$

Let $s = 34a + 2m - 9 - q_1 \ge 0$. Thus, (57) holds for $i = 1$. Assume (57) holds for some $i \ge 1$. Therefore, from above and Eqs. (46) we get

$$r_{i+1} \le 2q_i + r_i + 40a - 8, \quad q_{i+1} = 3q_i + 36a + 4m - 10 - s. \tag{58}$$

This implies $r_{i+1} - q_{i+1} \le 2(2i+1)(a-m) + 2i + 1 + s$. □

From Lemma we can not determine the type of these surfaces where there are many possibilities. Therefore we give the following example to illustrate this Lemma.

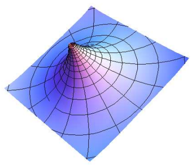
Example 4.5. Let $A(u) = -u^4 - 3u^2 + u + 1$ and $B(u) = u^4 + 3u^2 - 3u - 7 \neq 0$, where $m = 1$. Then

$$r = 12,\ q = 16,\ \deg \eta = 56,\ r_1 = 128,\ q_1 = 124,$$
$$r_{i+1} = 2q_i + r_i + 144,\ q_{i+1} = 3q_i + 136, \quad \forall\, i.$$

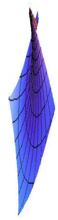
Substituting these values in Eq. (48) and we give some cases.

- If $j = 1 \Rightarrow D\,\frac{R_2}{Q_2} + d_1\, D\,\frac{R_1}{Q_1} = 0$. The result equation from degree 644.
- If $j = 2 \Rightarrow D\,\frac{R_3}{Q_3} + d_1\, D\,\frac{R_2}{Q_2} + d_2\, D\,\frac{R_1}{Q_1} = 0$. This equation has degree 2312.

Note whenever j increases the degree of Eq. (48) is increasing very significantly and this gives a contradiction. Figure 7 shows that.



(A) Surface M



(B) The pedal surface $\overline{M}$

$u \in [2, 4], v \in [\pi, 2\pi]$

FIGURE 7. The pedal of revolution surfaces of rational kind of Case (iii-c)

According the above result in Case (ii), we can deduce the following.

Theorem 4.1. *Let M be a revolution surface of rational kind, for which degree of denominator larger than the degree of the numerator, has the pedal surfaces $\overline{M}$ of infinite type.*

5. Special revolution surfaces

In the final section, we give two examples sphere and catenoid in E^3 to show the property of finite type is not a characterize inherited by the pedal transformation .

5.1. The pedal of sphere

First, we take the surface of sphere S_1^2 which has the position vector as follows

$$\mathbf{X}(u, v) = \{r\, \sin u\, \cos v\,,\ r\, \sin u\, \sin v\,,\ r\, \cos u\}, \tag{59}$$

where the unit normal vector filed of S_1^2 is given by

$$\mathbf{G} = \{\sin u\ \cos v\,,\ \sin v\ \sin u\,,\ \cos u\}.$$

As we know, by a direct computation, we find

$$\Delta = \frac{-1}{r^{2}}\Big(\frac{\partial^2}{\partial u^{2}} + \cot u\,\frac{\partial}{\partial u} + \csc^2 u\,\frac{\partial^2}{\partial v^{2}}\Big),$$

and this leads to

$$\Delta\mathbf{X} = \frac{2}{r^{2}}\,\mathbf{X}.$$

That is, $\mathbf{X}$ is 1-type where $\lambda = \frac{2}{r^2}$. From Eqs. (15) and (59) one can see that the pedal of sphere is itself, i.e., $\mathbf{X} = \overline{\mathbf{X}}$.

Corollary 5.1. *The pedal of sphere preserves the property of finite type.*

5.2. The pedal of catenoid

Second, we will give the catenoid surface M which the position vector of it is defined by

$$\mathbf{X}(u,v) = \{r\ \cosh u\ \cos v\,,\ r\ \cosh u\ \sin v\,,\ r\,u\},$$

and

$$\mathbf{G} = \{-\mathrm{sech} u\ \cos v\,,\ -\mathrm{sech} u\ \sin v\,,\ \tanh u\}.$$

Based on Eq. (6), we have

$$\Delta\mathbf{X} = \{0\,,\ 0\,,\ 0\}. \tag{60}$$

That is $\mathbf{X}$ is 1-type (whereas catenoid is minimal surface).

Depending on Eq. (15) we find the parametric representation of the pedal surface $\overline{M}$ of catenoid surface as in the following.

$$\overline{\mathbf{X}} = r\,(1 - u\ \tanh u)\big\{\mathrm{sech}\ u\ \cos v,\ \mathrm{sech}\ u\ \sin v,\ -\tanh u\big\}.$$

Consequently

$$\begin{aligned}\overline{\mathbf{G}} = {} & \frac{1}{\sqrt{2}\sqrt{2u^{2} + \cosh 2u + 1}}\big\{\mathrm{sech}\ u\ \cos v\ (\cosh 2u + 4u\ \tanh u - 3)\,,\\ & \mathrm{sech}\ u\ \sin v\ (\cosh 2u + 4u\ \tanh u - 3)\,,\ 2(-u + 2\ \tanh u + 2u\,\mathrm{sech}^2\ u)\big\}.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\overline{\Delta} = {} & -\frac{1}{r^2\,\epsilon^2\,\xi^2}\Big(\big(u^3\,\epsilon\,\mathrm{sech}^4\ u - (u^2+2)\ \tanh\ u + 3u\\ & + u\,(3u\ \tanh\ u - 5)\,\mathrm{sech}^2\ u\big)\frac{\partial}{\partial u} + \epsilon^2\,\xi\,\frac{\partial^2}{\partial u^{2}} + \xi^2\ \cosh^2\ u\ \frac{\partial^2}{\partial v^{2}}\Big),\end{aligned} \tag{61}$$

where $\epsilon = u\ \tanh u - 1$, and $\xi = u^{2}\,\mathrm{sech}^2\ u + 1$. Therefore, we take $\overline{X}_3$, the third component function of $\overline{\mathbf{X}}$.

$$\overline{X}_3 = r\ \tanh u\,(u\ \tanh u - 1). \tag{62}$$

Then, by some calculations, we find

$$\overline{\Delta}\ \overline{X}_3 = \frac{P_1(\tanh u\,,\ \operatorname{sech} u)}{r\,u\,\xi^2} + \frac{Q_1(\operatorname{sech} u)}{r\,u\,\epsilon\,\xi^2},$$

where, P_1 is polynomial of two variables and Q_1 is polynomial of one variable with coefficients given by some functions of u. Consequently, by using above equation and Eq. (61) we obtain

$$\overline{\Delta}^2\ \overline{X}_3 = \frac{P_2(\operatorname{sech} u)}{r^{\,3}\,u\,\epsilon^3\,\xi^5} + \frac{Q_2(\operatorname{sech} u)}{r^{\,3}\,u\,\epsilon^2\,\xi^5}.$$

Then, by induction we find that

$$\overline{\Delta}^j\ \overline{X}_3 = \frac{P_j(\operatorname{sech} u)}{r^{\,2j-1}\,u\,\epsilon^{2j-1}\,\xi^{3j-1}} + \frac{Q_j(\operatorname{sech} u)}{r^{\,2j-1}\,u\,\epsilon^{2j-2}\,\xi^{3j-1}}.$$

Assume the pedal of catenoid is of finite type, then by Eq. (11), we get

$$\frac{P_{j+1}(\operatorname{sech} u)}{r^{\,2j+1}\,u\,\epsilon^{2j+1}\,\xi^{3j+2}} + \frac{Q_{j+1}(\operatorname{sech} u)}{r^{\,2j+1}\,u\,\epsilon^{2j}\,\xi^{3j+2}} + d_1\left(\frac{P_j(\operatorname{sech} u)}{r^{\,2j-1}\,u\,\epsilon^{2j-1}\,\xi^{3j-1}} + \frac{Q_j(\operatorname{sech} u)}{r^{\,2j-1}\,u\,\epsilon^{2j-2}\,\xi^{3j-1}}\right) + \cdots + d_j\left(\frac{P_1(\tanh u\,,\ \operatorname{sech} u)}{r\,u\,\xi^2} + \frac{Q_1(\operatorname{sech} u)}{r\,u\,\epsilon\,\xi^2}\right) = 0.$$

Simplification of the previous equation, we get

$$\frac{P_{j+1}(\operatorname{sech} u)}{\epsilon} + Q(\tanh u,\ \operatorname{sech} u) = 0,$$

where $Q(\tanh u\,,\ \operatorname{sech} u)$ is a polynomial of two variables. Above equation becomes as

$$\frac{P_{j+1}(\operatorname{sech} u)}{\epsilon} = -Q(\tanh u\,,\ \operatorname{sech} u), \quad \epsilon \neq 1,$$

and, this is impossible for $j \geq 1$ because the left hand rational function. See Figure 8.

Note: $\epsilon \neq 1$ this means, $u \neq \pm 2.06533813897470472807$. And thus, we get:

(A) Catenoid (B) The pedal of catenoid

FIGURE 8. The pedal of catenoid; $u \in [-2, 2]$, $v \in [0, 2\pi]$

Corollary 5.2. *The pedal of catenoid does not preserve the property of finite type.*

Therefore, we conclude the following theorem.

Theorem 5.1. *The pedal transformation does not necessarily preserve the property of finite type for surfaces.*

6. Conclusion

The pedal of revolution surfaces of polynomial and rational kind are revolution surfaces. Also, the pedal of revolution surfaces of polynomial kind are infinite type. Furthermore, the revolution surfaces of rational kind which have degree of denominator larger than the degree of the numerator has the pedal surfaces of infinite type. Finally, we find the pedal of surfaces does not necessarily have the same finite type as in the original surface.

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