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NORMAL EIGENVALUES IN EVOLUTIONARY PROCESS

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Abstract. Firstly, we establish spectral mapping theorems for normal eigenvalues (due to Browder) of a C_0-semigroup and its generator. Secondly, we discuss relationships between normal eigenvalues of the compact monodromy operator and the generator of the evolution semigroup on $P_\tau(X)$ associated with the τ-periodic evolutionary process on a Banach space X, where $P_\tau(X)$ stands for the space of all τ-periodic continuous functions mapping $\mathbb{R}$ to X.

1. Introduction and preliminaries

1.1. Introduction

Let X be a Banach space. We denote by $P_\tau(X)$ the set of all τ-periodic continuous X-valued functions on $\mathbb{R} := (-\infty, \infty)$. For a given τ-periodic evolutionary process $\{U(t,s)\}_{t\geq s}$ on X the monodromy operator $V(0)$ is given by $V(0) = U(0, -\tau)$. Denote by L the (infinitesmal) generator of the C_0-semigroup $\{T^h\}_{h\geq 0}$ (see (11)) on $P_\tau(X)$ associated with $\{U(t,s)\}_{t\geq s}$. It is important to study the spectral properties of the generator L. Roughly speaking, $(Lu)(t) = -\frac{du}{dt} + A(t)u(t), u \in D(L) \subset P_\tau(X)$ if $\{U(t,s)\}_{t\geq s}$ arise from a τ-periodic evolution equation of the form $\frac{du}{dt} = A(t)u$.

In particular, let $A(t) = A + \beta(t)I$, where A is the generator of a C_0-semigroup $T(t)$, I is the identity operator, and $\beta(t)$ is a τ-periodic, continuous scalar-valued function. Then the evolutionary process and the monodromy operator become

$$U(t,s) = e^{\int_s^t \beta(r)dr}T(t-s) \quad\text{and}\quad V(0) = e^{\int_0^\tau \beta(r)dr}T(\tau).$$

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In this case, the evolution equation may serve as a model for the following PDE

$$(1) \qquad \frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + \beta(t)u(t,x), \quad 0 \le x \le \pi, \ t \ge 0$$

$$(2) \qquad u(t,0) = u(t,\pi) = 0, \ t \ge 0.$$

We would like to decide the set of all normal eigenvalues (see [1] for definition) of the generator L arising from the equation (1) with the boundary condition (2). In general, the spectra of the generator L is deeply concerned with spectra of the monodromy operator $V(0)$, for example, cf. [5, 8].

The purpose of this paper is to give the spectral mapping theorems for normal eigenvalues in a C_0-semigroups and relationships between spectra of $V(0)$ and L. We denote by $\sigma_n(H)$ the set of all normal eigenvalues for a linear operator $H : X \to X$ and by $\sigma_p(H)$ the point spectrum of H.

First, in Section 2 we will contribute new results to the theory of spectral properties of a C_0-semigroup $T(t)$ and its generator A. In particular, we give relationships between the ascents of $\mu I - T(t)$ and $\lambda I - A$, and show that the order of pole of $(\lambda I - A)^{-1}$ for some λ coincides with the order of pole of $(\mu I - T(t))^{-1}$, provided that $\mu = e^{\lambda t}, t > 0$ (Theorem 2.6 and Theorem 2.10). These are new results which is not found in the literatures [2, 4, 12, 15], etc.. As an application, we prove that the inclusion $\sigma_n(T(t)) \setminus \{0\} \subset e^{t\sigma_n(A)}, t > 0$ holds (Theorem 2.11).

Second, in Section 3 we give a relationship between $\sigma_n(V(0))$ and $\sigma_n(L)$ and some additional results on other spectra. More recently, it was proved that if $1 \in \sigma_n(V(0))$, then $0 \in \sigma_n(L)$ in [5], which is important to obtain criteria of the existence of τ-periodic solutions for τ-periodic systems with nonlinear perturbation. On the other hand, in the sequential paper [6] we proved that the equality

$$\dim N((\alpha I - L)^m) = \dim N((e^{\tau\alpha} I - V(0))^m)$$

holds by using a representation of elements in the null space $N((\alpha I - L)^m)$. Summing up those results in Section 3, we shall prove that $e^{\alpha\tau} \in \sigma_n(V(0))$ if and only if $\alpha \in \sigma_n(L)$ (Theorem 3.5), provided that the monodromy operator $V(0)$ is compact. As additional results, we give spectral properties on the resolvent set, the continuous spectrum and the residual spectrum in connection with $V(0)$ and L.

The results up to this point are illustrated in the equation (1) with $\beta(t) = \alpha(t) - \gamma$ in Section 4.

1.2. Preliminaries

Let T be a closed linear operator with dense domain $D(T) \subset X$. Set $N(T) = \{x \in D(T) \mid Tx = 0\}$ and $R(T) = \{Tx \in X \mid x \in D(T)\}$. The complex number ζ is called a normal eigenvalue of the operator T if the following conditions are satisfied:

(i) $R(\zeta I - T)$ is closed;

(ii) $\cup_{m\in\mathbb{N}} N((\zeta I - T)^m)$ is of finite dimension, where $\mathbb{N} = \{1, 2, 3, \ldots\}$; and
(iii) The point ζ is an isolated point of the spectrum of T.

Let $\rho(T)$ denote the resolvent set of T, $\sigma(T)$ the spectrum of T, $\sigma_p(T)$ the point spectrum of T and $\sigma_n(T)$ the set of all normal eigenvalues of T. Note that if T is a compact operator, then

$$\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\} = \sigma_n(T). \tag{3}$$

If the smallest nonnegative integer m such that

$$N(T^m) = N(T^{m+1}), \quad T^0 = I$$

exists, it is called the ascent of the operator T and denoted by $\eta(T)$. If $m = \eta(T)$ is a positive integer, then

$$N(T^{m-1}) \subsetneqq N(T^m) = N(T^{m+1})$$

holds. If no such integer exists, we say that $\eta(T) = \infty$. Note that $\eta(T) = 0$ if and only if T^{-1} exists. The generalized eigenspace of T with respect to $\zeta_0 \in \sigma_p(T)$, denoted by $N_{\zeta_0}(T)$, is the smallest closed subspace of X containing $\cup_{k=1}^{\infty} N((\zeta_0 I - T)^k)$. If $\zeta_0 I - T$ has the ascent m, then $N_{\zeta_0}(T) = N((\zeta_0 I - T)^m)$. If the smallest nonnegative integer m such that

$$R(T^m) = R(T^{m+1})$$

exists, it is called the descent of the operator T and denoted by $\delta(T)$. We say that $\delta(T) = \infty$ if for each n, $R(T^{n+1}) \subsetneqq R(T^n)$. Note that $\delta(T) = 0$ if and only if $R(T) = X$.

If λ_0 is an isolated singular point of the resovent $R(\lambda, T) = (\lambda I - T)^{-1}$, and if the Laurent expansion of $R(\lambda, T)$ in powers of $\lambda - \lambda_0$ is

$$R(\lambda, T) = \sum_{n \geq -k} (\lambda - \lambda_0)^n P_n$$

with $P_{-k} \neq 0$, we shall say that λ_0 is a pole of $R(\lambda, T)$ of order k. The following two results show relationships between the order of a pole λ_0 of $R(\lambda, T)$ and the ascent and descent of $\lambda_0 I - T$, under the assumption $\rho(T) \neq \emptyset$.

Lemma 1.1 ([13, Theorem 10.1, Sec. 10, Chap. 5]). *If λ_0 is a pole of $R(\lambda, T)$ of order p, then $\lambda_0 \in \sigma_p(T)$ and the ascent and descent of $\lambda_0 I - T$ are both equal to p.*

Lemma 1.2 ([13, Theorem 10.2, Sec. 10, Chap. 5]). *Suppose that $\lambda_0 \in \sigma(T)$ and $\lambda_0 I - T$ has finite ascent and descent. Then λ_0 is a pole of $R(\lambda, T)$*

The fundamental result on the normal eigenvalues of T is found in [1, 15] as follows.

Lemma 1.3 ([1, Lemma 17]). *Let T be a closed linear operator densely defined in the Banach space X with* $\dim N_{\lambda_0}(T) < \infty$ *for the complex number λ_0. Then $\lambda_0 \in \sigma_n(T)$ if and only if the resolvent $R(\lambda, T)$ is analytic in the neighborhood of λ_0 and has a pole at λ_0.*

Clearly, it follows from Lemma 1.3 and Lemma 1.1 that if $\lambda_0 \in \sigma_n(T)$, then λ_0 is a pole of $R(\lambda, T)$ of some order m, and hence, $\lambda_0 I - T$ has the ascent m.

2. Normal eigenvalues in C_0-semigroups

Let $T(t)$ be a C_0-semigroup on X with the generator A in this section.

First, we will state some fundamental facts on spectral properties in C_0-semigroups. We define an operator $B_\lambda(t), \lambda \in \mathbb{C}, t > 0$ as

$$B_\lambda(t)x = \int_0^t e^{\lambda(t-s)}T(s)x\, ds, \quad x \in X.$$

Then $B_\lambda(t)$ is a bounded linear operator on X with the following properties:

$$(\lambda I - A)B_\lambda(t)x = (e^{t\lambda}I - T(t))x, \quad x \in X,$$

$$B_\lambda(t)(\lambda I - A)x = (e^{t\lambda}I - T(t))x, \quad x \in D(A),$$

cf. [12, Lemma 2.2, Chap. 2]. These relations work effectively in the proof of the statement 1) in the following lemma; the statement 2) is proved through the technique of Fourier series.

Lemma 2.1 ([2, Theorems 3.7, Chap. IV, pp. 277–278], [15, Proposition 4.13])**.** *The following statements hold true*:

1) $\rho(T(t)) \setminus \{0\} \subset e^{t\rho(A)}$ *for* $t \geq 0$; *more precisely, if* $e^{\lambda t} \in \rho((T(t))$, *then* $\lambda \in \rho(A)$, *which implies* $e^{t\sigma(A)} \subset \sigma(T(t))$ *for* $t \geq 0$.

2)
$$\sigma_p(T(t)) \setminus \{0\} = e^{t\sigma_p(A)} \quad \textit{for} \quad t \geq 0.$$
More precisely, if $\lambda \in \sigma_p(A)$, *then* $e^{\lambda t} \in \sigma_p(T(t))$, *and conversely, if* $e^{\lambda t} \in \sigma_p(T(t))$, *then there exists* $k \in \mathbb{Z}$ *such that* $\lambda + \frac{2k\pi}{t} i \in \sigma_p(A)$, $i = \sqrt{-1}$, *where* $\mathbb{Z}$ *stands for the set of all integers.*

For $\mu \in \sigma_p(T(t)) \setminus \{0\}$, $t > 0$ we denote by $\Lambda_t(\mu)$ the set of all $\lambda \in \sigma_p(A)$ such that $\mu = e^{\lambda t}$. Then $\Lambda_t(\mu) \neq \emptyset$. The following result shows relationships between the eigenspaces corresponding to $\mu \in \sigma_p(T(t)) \setminus \{0\}$ for each $t > 0$ and the eigenspaces corresponding to $\lambda \in \Lambda_t(\mu)$.

Lemma 2.2 ([11, Lemma 2.1])**.** *If* $(A - \lambda I)^m x = 0$, *then*

$$T(t)x = e^{\lambda t} \sum_{k=0}^{m-1} \frac{t^k}{k!}(A - \lambda I)^k x.$$

Lemma 2.3. *The following statements hold true*:

1) *Let* $\mu = e^{\lambda t}, t \geq 0$. *Then*

$$N((\lambda I - A)^n) \subset N((\mu I - T(t))^n), \quad n = 1, 2, \ldots,$$

and

$$N_\lambda(A) \subset N_\mu(T(t)).$$

2) *Let* $\mu \in \sigma_p(T(t)) \setminus \{0\}, t > 0$. *Then* $N((\mu I - T(t))^n)$ *is the minimal closed subspace containing the linear independent subspaces* $N((\lambda I - A)^n)$ *for all* $\lambda \in \Lambda_t(\mu)$, *that is*

$$N((\mu I - T(t))^n) = \overline{\bigoplus_{\lambda \in \Lambda_t(\mu)} N((\lambda I - A)^n)}, \quad n = 1, 2, \ldots, \tag{4}$$

where $\overline{D}$ *stands for the closure of the set* D.

Note that the assertion 1) in Lemma 2.3 is easily proved by using Lemma 2.2. The assertion 2) for $n = 1$ is proved in the book in [15, Proposition 4.13]. For the general $n \geq 1$ the assertion 2) is found in [3, Lemma 6.1, Chap. 7, p. 213] without proof.

Next, we give relationships between the ascent of $e^{\lambda t} I - T(t), t > 0$ and the ascent of $\lambda I - A$. For this purpose the following result is needed.

Lemma 2.4. *For* $t > 0$,

$$\begin{aligned} &N((A - \lambda I)^m) \setminus N((A - \lambda I)^{m-1}) \\ &\subset N((T(t) - e^{\lambda t} I)^m) \setminus N((T(t) - e^{\lambda t} I)^{m-1}), \quad m = 1, 2, \ldots. \end{aligned} \tag{5}$$

Proof. Let $x \in N((A - \lambda I)^m)$. Then $x \in D((A - \lambda I)^n) = D(A^n)$ for $n = 0, 1, 2, \ldots$, and for $j = 0, 1, 2, \ldots$,

$$\begin{aligned} &(T(t) - e^{\lambda t} I)^j x \\ &= e^{j\lambda t} \left(t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \cdots + \frac{t^{m-1}}{(m-1)!}(A - \lambda I)^{m-1} \right)^j x \\ &= e^{j\lambda t} \left(t^j (A - \lambda I)^j x + j \frac{t^{j+1}}{2!}(A - \lambda I)^{j+1} x + \cdots \right. \\ &\qquad\qquad \left. + \frac{t^{j(m-1)}}{((m-1)!)^j}(A - \lambda I)^{j(m-1)} x \right). \end{aligned}$$

Here we have used Lemma 2.2. Hence, if $x \in N(A - \lambda I)^m)$, then $(T(t) - e^{\lambda t} I)^m x = 0$ for $t \geq 0$; if $x \in N((A - \lambda I)^m) \setminus N((A - \lambda I)^{m-1}) \neq \emptyset$, then

$$(T(t) - e^{\lambda t} I)^{m-1} x = e^{(m-1)\lambda t} t^{m-1} (A - \lambda I)^{m-1} x \neq 0$$

for $t > 0$. The proof is complete. □

Corollary 2.5. $\eta(A - \lambda I) \leq \eta(T(t) - e^{\lambda t} I)$.

Theorem 2.6. *Let* $\mu \in \sigma_p(T(t)) \setminus \{0\}, t > 0$. *If* $\mu I - T(t)$ *has the finite ascent* m *(and hence,* $1 \leq m$*), then the maximal ascent of* $\lambda I - A$ *for all* $\lambda \in \Lambda_t(\mu)$ *is* m, *and vice versa. Then*

$$N_\mu(T(t)) = \overline{\bigoplus_{\lambda \in \Lambda_t(\mu)} N_\lambda(A)}. \tag{6}$$

Proof. Assume that $\mu I - T(t)$ has the finite ascent m. Then $\eta(\lambda I - A) \le m$ for all $\lambda \in \Lambda_t(\mu)$ by Corollary 2.5. It suffices to prove that there exists a $\lambda_0 \in \Lambda_t(\mu)$ such that $\eta(\lambda_0 I - A) = m$. Assume that $\eta(\lambda I - A) \le m - 1$ for all $\lambda \in \Lambda_t(\mu)$. Then we have

$$\begin{aligned} N((\mu I - T(t))^m) &= \overline{\bigoplus_{\lambda \in \Lambda_t(\mu)} N((\lambda I - A)^m)} \\ &= \overline{\bigoplus_{\lambda \in \Lambda_t(\mu)} N((\lambda I - A)^{m-1})} \\ &= N((\mu I - T(t))^{m-1}). \end{aligned}$$

This is a contradiction since $N((\mu I - T(t))^{m-1}) \subsetneqq N((\mu I - T(t))^m)$.

Conversely, we assume that the maximal ascent of $\lambda I - A$ for all $\lambda \in \Lambda_t(\mu)$ is m. Then, for $n \ge m$, we have

$$\begin{aligned} N((\mu I - T(t))^n) &= \overline{\bigoplus_{\lambda \in \Lambda_t(\mu)} N((\lambda I - A)^n)} \\ &= \overline{\bigoplus_{\lambda \in \Lambda_t(\mu)} N(\lambda I - A)^m)} \\ &= N((\mu I - T(t))^m). \end{aligned}$$

Hence $\eta(\mu I - T(t)) \le m$. Since there exists $\lambda_0 \in \Lambda_t(\mu)$ such that $\eta(\lambda_0 I - A) = m$, it follows that $m \le \eta(e^{\lambda_0 t} I - T(t)) = \eta(\mu I - T(t))$. Therefore $\eta(\mu I - T(t)) = m$. □

The next result immediately follows from Theorem 2.6.

Corollary 2.7. *Let $\mu \in \sigma_p(T(t)) \setminus \{0\}, t > 0$. Then the ascent of $\lambda I - A$ is 1 for every $\lambda \in \Lambda_t(\mu)$ if and only if the ascent of $\mu I - T(t)$ is 1.*

Lemma 2.8. *Let $\mu \in \sigma_p(T(t)) \setminus \{0\}, t > 0$. Then $\dim N_\mu(T(t)) < \infty$ if and only if $\Lambda_t(\mu)$ is finite and $\dim N_\lambda(A) < \infty$ for all $\lambda \in \Lambda_t(\mu)$. If one of the above equivalent conditions is satisfied, then the ascent of $\mu I - T(t)$ coincides with the maximal ascent of $\lambda I - A$ for all $\lambda \in \Lambda_t(\mu)$ and*

$$N_\mu(T(t)) = \bigoplus_{\lambda \in \Lambda_t(\mu)} N_\lambda(A). \tag{7}$$

Proof. Assume that $\dim N_\mu(T(t)) < \infty$. Then there is an ascent m of $\mu I - T(t)$ for which $N_\mu(T(t)) = N((\mu I - T(t))^m)$. By the assertion 1) in Lemma 2.3 we have $N_\lambda(A) \subset N_\mu(T(t))$ for all $\lambda \in \Lambda_t(\mu)$. Since $\dim N_\mu(T(t)) < \infty$, we have $1 \le \dim N_\lambda(A) < \infty$; and hence, $\Lambda_t(\mu)$ is finite.

Conversely, assume that $\Lambda_t(\mu)$ is a finite set and $\dim N_\lambda(A) < \infty$ for $\lambda \in \Lambda_t(\mu)$. Set $m = \max\{\eta(\lambda I - A) \mid \lambda \in \Lambda_t(\mu)\}$. Then the assertion 2) in Lemma

2.3 implies

$$(8) \qquad N((\mu I - T(t))^m) = \bigoplus_{\lambda \in \Lambda_t(\mu)} N((\lambda I - A)^m) = \bigoplus_{\lambda \in \Lambda_t(\mu)} N_\lambda(A).$$

By using the same argument as in the proof of Theorem 2.6 we have (7) and hence, $\dim N_\mu(T(t)) < \infty$. □

Finally, we consider the orders of poles for $R(\mu, T(t))$ and $R(\lambda, A)$ provided that $\mu = e^{t\lambda}, t > 0$. The following result was shown independently by using the same idea in [9, Theorem 4.2] and [2, Theorem 3.6, Chap. IV, pp. 276–277].

Lemma 2.9. *Suppose that $\mu_0 \neq 0, t > 0$ and μ_0 is a pole of $R(\mu, T(t))$ of order k. If $\lambda_0 \in \Lambda_t(\mu_0)$, then λ_0 is a pole of $R(\lambda, A)$ with the order $\leq k :$ as a result, if $k = 1$, then λ_0 is a pole of $R(\lambda, A)$ of order* 1.

Lemma 2.9 is improved as follows.

Theorem 2.10. *Suppose that $\mu_0 \neq 0, t > 0$ and μ_0 is a pole of $R(\mu, T(t))$ of order k. Then there exists a $\lambda_m \in \Lambda_t(\mu_0)$ such that λ_m is a pole of $R(\lambda, A)$ of order k.*

Proof. From the assumption together with Lemma 1.1 we see that $\mu_0 I - T(t)$ has the ascent k. Hence it follows from Theorem 2.6 that there exists a $\lambda_m \in \Lambda_t(\mu_0)$ satisfying $\eta(\lambda_m I - A) = k$. Since λ_m is a pole of $R(\lambda, A)$ by Lemma 1.3, the order of the pole λ_m is k by Lemma 1.1. The reminder is obvious. □

Let

$$\sigma_e(A) = \sigma(A) \setminus \sigma_n(A), \ \sigma_e(T(t)) = \sigma(T(t)) \setminus \sigma_n(T(t)).$$

Then $e^{t\sigma_e(A)} \subset \sigma_e(T(t))$ for $t > 0$; see [15, Proposition 4.13]. From this inclusion a spectral mapping theorem for normal eigenvalues is not derived, generally. Using Lemma 2.8 and Lemma 2.9, we will give a spectral mapping theorem for normal eigenvalues.

Theorem 2.11. *If $\mu_0 \in \sigma_n(T(t)) \setminus \{0\}$, then $\Lambda_t(\mu_0) \subset \sigma_n(A)$. In particular,*

$$\sigma_n(T(t)) \setminus \{0\} \subset e^{t\sigma_n(A)}, \quad t > 0.$$

Proof. Let $\mu_0 \in \sigma_n(T(t)) \setminus \{0\}$. Then $N_{\mu_0}(T(t))$ is of finite dimension and μ_0 is a pole of $R(\mu, T(t))$ by Lemma 1.3. Thus it follows from Lemma 2.8 that $\Lambda_t(\mu_0)$ is a non-empty finite set and $\dim N_\lambda(A) < \infty$ for all $\lambda \in \Lambda_t(\mu_0)$. Then any point $\lambda_0 \in \Lambda_t(\mu_0)$ is a pole of $R(\lambda, A)$ by Lemma 2.9, so that $\lambda_0 \in \sigma_n(A)$ by Lemma 1.3 again. □

Proposition 2.12. *Let $t > 0$ be fixed. If*

$$(9) \qquad \sigma_p(T(t)) \setminus \{0\} = \sigma_n(T(t)) \setminus \{0\},$$

then

$$\sigma_p(A) = \sigma_n(A)$$

and

$$\sigma_n(T(t)) \setminus \{0\} = e^{t\sigma_n(A)}. \tag{10}$$

Proof. For the assertion $\sigma_p(A) = \sigma_n(A)$, it suffices to show $\sigma_p(A) \subset \sigma_n(A)$. Let $\lambda \in \sigma_p(A)$ and $\mu = e^{\lambda t}$. Then $\mu \in \sigma_p(T(t)) \setminus \{0\} = \sigma_n(T(t)) \setminus \{0\}$, and hence, $\lambda \in \sigma_n(A)$ by Theorem 2.11. Moreover, since $\sigma_n(T(t)) \setminus \{0\} \subset e^{t\sigma_n(A)}$ by Theorem 2.11 again, we have

$$\sigma_n(T(t)) \setminus \{0\} \subset e^{t\sigma_n(A)} = e^{t\sigma_p(A)} \subset \sigma_p(T(t)) \setminus \{0\}.$$

This means the identity (10). □

As a special case, the following results hold for a compact C_0-semigroup $T(t)$ and its generator A. The proofs are based on spectral properties of a compact operator, cf. [13].

Corollary 2.13. *Suppose that $T(t)$ is a compact C_0-semigroup on X. Let $\mu \in \sigma(T(t)) \setminus \{0\}, t > 0$. Then the following statements hold.*

1) $\dim N_\lambda(A) < \infty$ *for all* $\lambda \in \Lambda_t(\mu)$.

2) *The ascent of $\mu I - T(t)$ coincides with the maximal ascent of $\lambda I - A$ for all $\lambda \in \Lambda_t(\mu)$ and (7) holds. In particular, $\eta(\lambda I - A) = 1$ for every $\lambda \in \Lambda_t(\mu)$ if and only if $\eta(\mu I - T(t)) = 1$.*

3) *The ascent of $\mu I - T(t)$ is the order of μ as the pole of $R(\xi, T(t))$.*

Corollary 2.14. *Suppose that $T(t)$ is a compact C_0-semigroup on X. Then $\sigma_p(A) = \sigma_n(A)$ and*

$$\sigma_n(T(t)) = e^{t\sigma_n(A)}, t > 0.$$

3. Spectral properties in evolution semigroup

We give relationships between spectra of the monodromy operator $V(0)$ and the generator L.

3.1. Relationship between normal eigenvalues of $V(0)$ and L

A family of bounded linear operators $\{U(t,s)\}_{t\geq s}, (t, s \in \mathbb{R})$ from a Banach space X to itself is called a τ-*periodic* (*strongly continuous*) evolutionary process if the following conditions are satisfied:

(1) $U(t,t) = I$ for all $t \in \mathbb{R}$,
(2) $U(t,s)U(s,r) = U(t,r)$ for all $t \geq s \geq r$,
(3) The map $(t,s) \mapsto U(t,s)x$ is continuous for every fixed $x \in X$,
(4) $U(t+\tau, s+\tau) = U(t,s)$ for all $t \geq s$,
(5) $\|U(t,s)\| \leq M_w e^{w(t-s)}$ for some $M_w > 0$ and $w \in \mathbb{R}$ independent of $t \geq s$.

For a given τ-periodic evolutionary process $\{U(t,s)\}_{t\geq s}$ the following operator

$$V(t) = U(t, t-\tau)$$

is called a monodromy operator (sometimes, a periodic map, or Poincaré map). Then, $V(t+\tau) = V(t)$ holds for every $t \in \mathbb{R}$. For a given τ-periodic evolutionary process $\{U(t,s)\}_{t\geq s}$, the family $\{T^h\}_{h\geq 0}$ defined by

$$(T^h u)(t) := U(t, t-h)u(t-h), \forall t \in \mathbb{R}, u \in P_\tau(X) \tag{11}$$

is a C_0-semigroup on $P_\tau(X)$ (cf. [8, Lemma 2]). It is called the *evolution semigroup* associated with the τ-periodic evolutionary process $\{U(t,s)\}_{t\geq s}$ (briefly, evolution semigroup). Denote by L the (infinitesimal) generator of the C_0-semigroup $\{T^h\}_{h\geq 0}$ on $P_\tau(X)$. It is well-known that L is a closed linear operator with dense domain $D(L)$ in $P_\tau(X)$. For $\alpha \in \mathbb{C}$ we set $U_\alpha(t,s) = e^{-\alpha(t-s)}U(t,s)$. Then $U_\alpha(t,s)$ is also a τ-periodic evolutionary process. The monodromy operator $V_\alpha(0)$ and the generator L_α corresponding to $U_\alpha(t,s)$ are given by $V_\alpha(0) = e^{-\alpha\tau}V(0)$ and $L_\alpha = L - \alpha I$.

To obtain the main theorem in this section, we need the following key lemma.

Lemma 3.1 ([6, Theorem 2]). *For any complex number α,*

$$\dim N((\alpha I - L)^m) = \dim N((e^{\alpha\tau} I - V(0))^m), \; m \in \mathbb{N}. \tag{12}$$

The equation (12) shows that $e^{\alpha\tau}I - V(0)$ and $\alpha I - L$ have the same ascent.

Corollary 3.2. *$e^{\alpha\tau} \in \sigma_p(V(0))$ if and only if $\alpha \in \sigma_p(L)$. More precisely, if $\alpha \in \sigma_p(L)$, then $e^{\alpha\tau} \in \sigma_p(V(0))$, and conversely, if $e^{\alpha\tau} \in \sigma_p(V(0))$, then $\alpha + \frac{2k\pi}{\tau} i \in \sigma_p(L), k \in \mathbb{Z}$.*

Proof. It is easily derived from Lemma 3.1. □

Lemma 3.3 ([5, Theorem 3]). *If $1 \in \sigma_n(V_\alpha(0))$, then $0 \in \sigma_n(L_\alpha)$.*

Theorem 3.4. *If $e^{\alpha\tau} \in \sigma_n(V(0))$, then $\alpha \in \sigma_n(L)$, and*

$$1 \leq \eta(\alpha I - L) = \delta(\alpha I - L) = \eta(e^{\alpha\tau} I - V(0)) = \delta(e^{\alpha\tau} I - V(0)) < \infty.$$

Proof. Let $e^{\alpha\tau} \in \sigma_n(V(0))$. Then $1 \in \sigma_n(V_\alpha(0))$, since $e^{\alpha\tau}I - V(0) = e^{\alpha\tau}(I - V_\alpha(0))$. Lemma 3.3 implies $0 \in \sigma_n(L_\alpha)$, and hence $\alpha \in \sigma_n(L)$. Then we have

$$\eta(e^{\alpha\tau} I - V(0)) = \delta(e^{\alpha\tau} I - V(0)) \;\text{ and }\; \eta(\alpha I - L) = \delta(\alpha I - L).$$

Lemma 3.1 means $\eta(\alpha I - L) = \eta(e^{\alpha\tau} I - V(0))$. Summing up these, we obtain the required result. □

Now we are in a position to state the main theorem in this section.

Theorem 3.5. *Suppose that $V(0)$ is a compact operator. Then $e^{\alpha\tau} \in \sigma_n(V(0))$ if and only if $\alpha \in \sigma_n(L)$.*

Proof. Let $\alpha \in \sigma_n(L)$. Then $e^{\alpha\tau} \in \sigma_p(V(0)) \setminus \{0\}$ by Corollary 3.2. Since $V(0)$ is a compact operator, the identity (3) means $e^{\alpha\tau} \in \sigma_n(V(0))$. The converse follows from Theorem 3.4. □

The following result is derived immediately from Theorem 3.5.

Corollary 3.6. *Let $b(t)$ be a τ-periodic, continuous real function such that $b(t+\tau) = b(t) + b(\tau)$. If $T(t)$ is a compact C_0-semigroup on X, then*

$$U(t,s) := e^{b(t)-b(s)}T(t-s),\ t \geq s$$

is a compact operator. As a result, so is $V(0) := U(0,-\tau)$. Moreover, putting $\sigma_n(T(\tau)) = \{e^{\lambda_m \tau} \mid m \in \mathbb{N}\}$,

$$\sigma_n(V(0)) = \{e^{b(\tau)+\lambda_m \tau} \mid m \in \mathbb{N}\};$$

and hence,

$$\sigma_n(L) = \{\frac{b(\tau)}{\tau} + \lambda_m + \frac{2\pi k i}{\tau} \mid m \in \mathbb{N}, k \in \mathbb{Z}\}.$$

3.2. Additional results

For a closed linear operator T with dense domain in X, we denote by $\sigma_c(T)$ and $\sigma_r(T)$ the continuous spectrum and the residual spectrum, respectively (cf. [12]). In this subsection we consider relationships between these spectra of $V(0)$ and L. For the resolvent sets of $V(0)$ and L the following result is well known in [5].

Lemma 3.7 ([5, Lemma 3.10])**.** *If $e^{\alpha\tau} \in \rho(V(0))$, then $\alpha \in \rho(L)$.*

First we consider its converse. To do so, we need some of lemmas. Define

$$B_\alpha g = \int_0^\tau U_\alpha(\tau, r)g(r)dr,\ B_0 g =: Bg, \quad g \in P_\tau(X).$$

Then it is a bounded linear operator form $P_\tau(X)$ to X, which has the following property.

Lemma 3.8 ([5, Lemma 7.2] and [10, Lemma 21])**.** *$B_\alpha P_\tau(X)$ is dense in X. In particular, If $\dim X < \infty$, then $B_\alpha P_\tau(X) = X$.*

The following result is a slight extension of the above lemma.

Lemma 3.9. *Let Ω be dense in $P_\tau(X)$. Then $B_\alpha \Omega$ is also dense in X.*

Proof. Since B_α is continuous, $B_\alpha(\overline{\Omega}) \subset \overline{B_\alpha(\Omega)}$. Since Ω is dense, $B_\alpha(\overline{\Omega}) = B_\alpha(P_\tau(X))$, which is dense in X by Lemma 3.8. Thus $\overline{B_\alpha(\Omega)} = P_\tau(X)$; the proof is complete. □

Lemma 3.10 ([6, Lemma 2.1])**.** *Let $g \in P_\tau(X)$. Then $u \in D(L)$ and $(\alpha I - L)u = g$ if and only if $u \in P_\tau(X)$ is given by*

$$u(t) = U_\alpha(t,0)w + \int_0^t U_\alpha(t,s)g(s)ds, \quad t \geq 0 \tag{13}$$

with

$$(I - V_\alpha(0))w = B_\alpha g. \tag{14}$$

The following result is directly obtained from Lemma 3.10.

Corollary 3.11. *$g \in R(\alpha I - L)$ if and only if $B_\alpha g \in R(e^{\alpha\tau} I - V(0))$.*

Now we discuss the converse of Lemma 3.7.

Proposition 3.12. *If $\alpha \in \rho(L)$, then $e^{\alpha\tau} \in \rho(V(0)) \cup \sigma_c(V(0))$.*

Proof. If $\alpha \in \rho(L)$, then by Lemma 3.1 we see that $(e^{\alpha t} I - V(0))^{-1}$ exists. Since $R(\alpha I - L) = P_\tau(X)$, it follows from Lemma 3.8 that $B_\alpha R(\alpha I - L)$ is dense in X. Moreover, since $B_\alpha R(\alpha I - L) \subset R(e^{\alpha t} I - V(0))$ by Corollary 3.11, the range $R(e^{\alpha t} I - V(0))$ is also dense in X. □

Corollary 3.13. *If $V(0)$ is a compact operator, then $\alpha \in \rho(L)$ if and only if $e^{\alpha\tau} \in \rho(V(0))$.*

Next we consider the continuous spectrum and the residual spectrum for $V(0)$ and L.

Proposition 3.14. *If $\alpha \in \sigma_c(L)$, then $e^{\alpha\tau} \in \sigma_c(V(0))$.*

Proof. If $\alpha \in \sigma_c(L)$, then $(\alpha I - L)^{-1}$ exists and $R(\alpha I - L)$ is dense in $P_\tau(X)$. It follows from Lemma 3.9 that $B_\alpha R(\alpha I - L)$ is dense in X. This implies that the range $R(e^{\alpha\tau} I - V(0))$ is also dense in X. Hence $e^{\alpha\tau} \in \rho(V(0)) \cup \sigma_c(V(0))$. If $e^{\alpha\tau} \in \rho(V(0))$, then $\alpha \in \rho(L)$ by Lemma 3.7. This yields a contradiction since $\alpha \in \sigma_c(L)$. Therefore $e^{\alpha\tau} \in \sigma_c(V(0))$. □

Combining Lemma 3.12 and Proposition 3.14 we obtain the following result.

Corollary 3.15. *If $\alpha \in \rho(L) \cup \sigma_c(L)$, then $e^{\alpha\tau} \in \rho(V(0)) \cup \sigma_c(V(0))$.*

Proposition 3.16. *If $e^{\alpha\tau} \in \sigma_r(V(0))$, then $\alpha \in \sigma_r(L)$.*

Proof. If $e^{\alpha\tau} \in \sigma_r(V(0))$, then $(e^{\alpha\tau} I - V(0))^{-1}$ exists, as a result, $(\alpha I - L)^{-1}$ exists. Moreover, since $R(e^{\alpha\tau} I - V(0))$ is not dense in X and $B_\alpha R(\alpha I - L) \subset R(e^{\alpha t} I - V(0))$, the range $R(\alpha I - L)$ is not also dense in $P_\tau(X)$ by Lemma 3.9. This proves the proposition. □

4. An example

Let $X = L^2([0,\pi], \mathbb{C})$. Then, X is a Hilbert space with the usual inner product $\langle , \rangle$ given by

$$\langle w, z \rangle = \int_0^\pi w(x)\overline{z(x)}dx, \ w, z \in X.$$

Let us consider a partial differential equation of the form

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + (\alpha(t) - \gamma)u(t,x), \quad 0 \le x \le \pi, \ t \ge 0 \tag{15}$$

$$u(t,0) = u(t,\pi) = 0, \ t \ge 0, \tag{16}$$

where $\gamma \in \mathbb{R}$ and $\alpha(t)$ is a π-periodic, continuous scalar-valued function. Define a linear operator A_0 by

$$A_0 u = \frac{d^2u}{dx^2} \quad \text{for} \quad u \in D(A_0),$$

where

$$\begin{aligned} D(A_0) = \{u \in X \mid\ & u \text{ is continuously differentiable and} \\ & u' \text{ is abosolutely continuous}, \\ & u'' \in X, u(0) = u(\pi) = 0\}. \end{aligned}$$

Then A_0 is a closed linear operator with dense domain in X and A_0 is self-adjoint. It is well-known that

$$\sigma(A_0) = \sigma_p(A_0) = \{\lambda_m := -m^2 \mid m \in \mathbb{N}\},$$

$\eta(\lambda_m I - A_0) = 1$ for all $m \in \mathbb{N}$, and $N_{\lambda_m}(A_0) = N(\lambda_m I - A_0) = \text{span}\{z_m\}$, where $z_m(x) = \sqrt{\frac{2}{\pi}} \sin mx$.

On the other hand, A_0 is the generator of a compact C_0-semigroup $T_0(t)$ on X such that $\|T_0(t)\| = e^{-t}$ for $t \geq 0$, cf. [2, 14]. Since A_0 is a self-adjoint operator, $T(t)$ is also a self-adjoint operator. Note that $\{z_m\}_{m=1}^{\infty}$ is an orthonormal basis in X. Since $T_0(t)z_m = e^{\lambda_m t} z_m, m = 1, 2, \ldots$, and $f = \sum_{m=1}^{\infty} \langle f, z_m \rangle z_m$ for every $f \in X$, we obtain

$$T_0(t)f = \sum_{m=1}^{\infty} e^{\lambda_m t} \langle f, z_m \rangle z_m.$$

Hence $0 \notin \sigma_p(T_0(t))$, $\sigma_p(T_0(t)) = \{e^{\lambda_m t} \mid m \in \mathbb{N}\},\ t > 0$ and $\eta(e^{\lambda_m t} I - T_0(t)) = 1$. Then we obtain the following result.

Proposition 4.1. *The following relations hold*:

$$\sigma_p(A_0) = \sigma_n(A_0), \quad \sigma_p(T_0(t)) = \sigma_n(T_0(t)), \quad \sigma_n(T_0(t)) = e^{t\sigma_n(A_0)},\ t > 0.$$

Furthermore, we define a closed linear operator A by

$$Au = \frac{d^2u}{dx^2} - \gamma u \quad \text{for} \quad u \in D(A) = D(A_0),$$

which is the generator of the C_0-semigroup $T(t) = e^{-\gamma t} T_0(t)$ on X. If we set $A(t) = A + \alpha(t)I$ for $t \in \mathbb{R}$, then the equation (15) is represented as

$$\frac{d}{dt} u(t) = A(t)u(t). \tag{17}$$

Set

$$a(t) = \int_0^t \alpha(r)dr.$$

Then, $a(t+\pi) = a(t) + a(\pi)$. The solution operator $U(t,s)$ of the equation (17) is represented as

$$U(t,s) = e^{a(t)-a(s)}T(t-s) = e^{-\gamma(t-s)}e^{a(t)-a(s)}T_0(t-s),\ t \geq s,$$

and hence,

$$V(0) := U(\pi,0) = e^{-\gamma\pi}e^{a(\pi)}T_0(\pi). \tag{18}$$

Clearly, $\sigma_p(V(0)) = \sigma_n(V(0))$. It is easy to show that $\{U(t,s)\}_{t\geq s}$ is a π-periodic evolutionary process on X. Note that $U(t,s), t > s$ is a compact operator and so is $V(0)$. Let L be the generator of the evolution semigroup $\{T^h\}_{h\geq 0}$ on $P_\pi(X)$ associated with $\{U(t,s)\}_{t\geq s}$. Then the operator L has the following properties.

Proposition 4.2.

$$\sigma_p(L) = \sigma_n(L),\ \sigma_n(V(0)) = e^{\pi\sigma_n(L)},$$

$$\sigma_n(L) = \{\alpha_m + 2ki \mid m \in \mathbb{N}, k \in \mathbb{Z}\},\quad \alpha_m = \frac{a(\pi)}{\pi} - (\gamma + m^2)$$

and

$$\eta((\alpha_m + 2ki)I - L) = 1$$

for all $m \in \mathbb{N}, k \in \mathbb{Z}$.

Proof. Since $\sigma_p(T_0(\pi)) = \sigma_n(T_0(\pi))$ and $\sigma_n(T_0(\pi)) = \{e^{-m^2\pi} \mid m \in \mathbb{N}\}$, it follows from (18) that

$$\sigma_p(V(0)) = \sigma_n(V(0)) = \{e^{\pi\alpha_m} \mid m \in \mathbb{N}\}.$$

Thus Corollary 3.2 implies that $\sigma_p(V(0)) = e^{\pi\sigma_p(L)}$. Let $\alpha \in \sigma_p(L)$. Then $e^{\pi\alpha} \in \sigma_p(V(0))$, and hence $e^{\pi\alpha} \in \sigma_n(V(0))$. By Theorem 3.4 we obtain $\alpha \in \sigma_n(L)$, that is, $\sigma_p(L) = \sigma_n(L)$. As a result, $\sigma_n(V(0)) = e^{\pi\sigma_n(L)}$. Moreover, Corollary 3.6 implies that $\sigma_n(L) = \{\alpha_m + 2ki \mid m \in \mathbb{N}, k \in \mathbb{Z}\}$. Furthermore, since $\eta(e^{-m^2\pi}I - T_0(\pi)) = 1$, we have $\eta(e^{\pi\alpha_m}I - V(0)) = 1$. Theorem 3.4 implies that $\eta((\alpha_m + 2ki)I - L) = 1$. □

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