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EXTENDING HYPERELLIPTIC K3 SURFACES, AND GODEAUX SURFACES WITH $\pi_1 = \mathbb{Z}/2$

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Abstract. We construct the extension of a hyperelliptic K3 surface to a Fano 6-fold with extraordinary properties in moduli. This leads us to a family of surfaces of general type with $p_g = 1$, $q = 0$, $K^2 = 2$ and hyperelliptic canonical curve, each of which is a weighted complete intersection inside a Fano 6-fold. Finally, we use these hyperelliptic surfaces to determine an 8-parameter family of Godeaux surfaces with $\pi_1 = \mathbb{Z}/2$.

1. Introduction

In the 1930s Godeaux constructed the first example of a surface of general type with $p_g = 0$, $K^2 = 1$, and surfaces with these invariants are now named after him. Godeaux surfaces have cyclic torsion group of order ≤ 5, and the components of the moduli space with torsion $\mathbb{Z}/5$, $\mathbb{Z}/4$, $\mathbb{Z}/3$ were constructed in [8]. In each case the moduli space is irreducible, unirational, and 8-dimensional, which is the expected dimension. The first simply connected example appeared in [2], and recently another simply connected Godeaux surface was constructed in [6] using $\mathbb{Q}$-Gorenstein smoothing theory. In Theorem 1, we give an 8-parameter family of Godeaux surfaces with topological fundamental group $\mathbb{Z}/2$. Previous constructions of Godeaux surfaces with $\pi_1 = \mathbb{Z}/2$ were given in [1], [12] but these give only 4-dimensional subvarieties of the moduli space.

To construct the $\mathbb{Z}/2$-Godeaux surface, we first consider the universal cover Y, which is a surface of general type with $p_g = 1$, $K^2 = 2$. Such surfaces are studied very thoroughly in [3], giving an almost complete classification of the moduli space. It is known (see [3], Theorem 6.1 or Lemma 5.1) that if Y is the universal cover of a Godeaux surface, then the canonical curve section must be hyperelliptic. For brevity, we call Y a *hyperelliptic surface of general type*. Such surfaces are special in the moduli space. Moreover Y is closely related to the symmetric determinantal quartic K3 surface with ten nodes. This is our starting point.

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The symmetric determinantal quartic K3 surface $T_4 \subset \mathbb{P}^3$ has a hyperelliptic degeneration to $T_{2,4} \subset \mathbb{P}(1^4, 2)$, a double cover of the nonsingular quadric surface with ten nodes, arising from nodes on the branch curve. Now $T_{2,4}$ is equipped with a Weil divisor A such that $\mathcal{O}_{T_{2,4}}(2A) = \mathcal{O}_{T_{2,4}}(1)$. Thus $T_{2,4}$ has a model T in weighted projective space $\mathbb{P}(2^4, 3^4, 4)$ given by Proj of the graded ring

$$R(T_{2,4}, A) = \bigoplus_{n \geq 0} H^0(T_{2,4}, \mathcal{O}_T(nA)),$$

where the ten nodes become $10 \times \frac{1}{2}(1,1)$ points. We use the shorthand $\frac{1}{2}$ point to mean the image of the origin in the quotient of $\mathbb{C}^n$ by $\mathbb{Z}/2$ acting by -1 on all coordinates.

The K3 surface T is the "elephant" hyperplane section of a Fano 3-fold $W \subset \mathbb{P}(1, 2^4, 3^4, 4)$ with $10 \times \frac{1}{2}$ points. In terms of graded rings, there is an element $a \in R(W, -K_W)$ of degree 1 such that

$$R(T, A) = R(W, -K_W)/(a).$$

We call this 3-fold an *extension* of T, and iterate this process to obtain further extensions up to a Fano 6-fold $W^6 \subset \mathbb{P}(1^4, 2^4, 3^4, 4)$, containing T as the intersection of four hyperplane sections of weight 1. We obtain a one-to-one correspondence of moduli between the surface T and the 6-fold W^6.

Main Theorem. *Let $T \subset \mathbb{P}(2^4, 3^4, 4)$ be a quasismooth hyperelliptic K3 surface with $10 \times \frac{1}{2}$ points. Then T determines (and is uniquely determined by) a unique extension to a quasismooth Fano 6-fold $W \subset \mathbb{P}(1^4, 2^4, 3^4, 4)$ with $10 \times \frac{1}{2}$ points and such that*

$$T = W \cap H_1 \cap H_2 \cap H_3 \cap H_4,$$

where the H_i are hyperplanes of weight 1 in the projective space $\mathbb{P}(1^4, 2^4, 3^4, 4)$.

The point of this extension is that, given such a Fano 6-fold W, it is easy to construct hyperelliptic surfaces of general type with $p_g = 1$, $q = 0$, $K^2 = 2$.

Corollary 1. *There is a 15-parameter family of hyperelliptic surfaces Y of general type with $p_g = 1$, $q = 0$, $K^2 = 2$, each of which is topologically simply connected, and a complete intersection of type $(1, 1, 1, 2)$ in a Fano 6-fold $W \subset \mathbb{P}(1^4, 2^4, 3^4, 4)$ with $10 \times \frac{1}{2}$ points.*

In fact this gives the canonical model of Y, and so the genus 3 hyperelliptic curve $D \in |K_Y|$ is the unique hyperplane section of weight 1 in Y. A construction for these surfaces was outlined by Catanese and Debarre in [3], but our method has the advantage of being more explicit, and thus suitable for the study of Godeaux surfaces. Indeed, we go on to construct a new family of Godeaux surfaces with fundamental group $\mathbb{Z}/2$.

Theorem 1. *There is an 8-parameter family of Godeaux surfaces X with $\pi_1 = \mathbb{Z}/2$, where each X is obtained as a $\mathbb{Z}/2$-quotient of some hyperelliptic surface Y constructed in Corollary 1.*

We outline the contents of this article. Sections 2 and 3 treat graded rings over hyperelliptic curves and K3 surfaces respectively, working relative to the hyperelliptic double cover. The Main Theorem is proved in Section 4, using a projection–unprojection construction for T and for W^6. This Theorem is a sequel to one of the results in [5]. See also [7] and [10] for details on these methods. Finally, in Section 5 we first prove Corollary 1, before undergoing a careful study of $\mathbb{Z}/2$ actions on our configuration. This results in an 8-parameter family of $\mathbb{Z}/2$-Godeaux surfaces, proving Theorem 1.

We do not prove that the moduli space of $\mathbb{Z}/2$-Godeaux surfaces is irreducible. This is a hard problem. Let $\mathcal{U}$ be the family constructed in Theorem 1. It remains to show that $\mathcal{U}$ covers an open set of the moduli space of Godeaux surfaces by studying deformations of surfaces in $\mathcal{U}$, and to study the closure of $\mathcal{U}$ by studying one parameter limits (degenerations) of surfaces in $\mathcal{U}$.

Remark 1.1. The arguments in Section 4 involve some explicit equations for the 6-fold W, and a posteriori, many of the results can be checked simply by verifying the stated formulae. We remind the reader of this at the relevant points. All stated equations have also been checked using computer algebra software.

2. Graded rings over hyperelliptic curves

In this section we briefly review hyperelliptic curves and their graded rings, using the double cover of $\mathbb{P}^1$. This is well known material contained for example in Section 4 of [9]. We include this section for completeness since we generalise to hyperelliptic K3 surfaces in Section 3.

Let D be a hyperelliptic curve of genus 3. Then the canonical linear system $|K_D|$ defines a double cover of $\mathbb{P}^1$ embedded as a plane conic, branched in eight points $Q_1, \dots, Q_8$. The corresponding ramification (or Weierstrass) points on D are labelled $P_1, \dots, P_8$. The double cover $\pi\colon D \to \mathbb{P}^1$ determines and is determined by the g^1_2. Moreover, we have $2P_i \sim g^1_2$ and $K_D \sim 2g^1_2$. There is a natural hyperelliptic involution h on D which swaps the two sheets of the double cover, and π is the quotient map of this involution.

Choose generators s_1, s_2 of $H^0(D, g^1_2)$. Then $D_8 \subset \mathbb{P}(1,1,4)$ is a model of D, with equation $t^2 = F_8(s_1, s_2)$, where F_8 vanishes at $Q_1 + \cdots + Q_8$. By considering rational functions on D, we have

$$4g^1_2 \sim P_1 + \cdots + P_8,$$

or equivalently,

$$B_1 + (8-a)g^1_2 \sim B_2 + 4g^1_2,$$

where $B_1 = P_1 + \cdots + P_a$, $B_2 = P_{a+1} + \cdots + P_8$. Since the B_i are effective Cartier divisors on D we can choose constant sections

$$u\colon \mathcal{O}_D \to \mathcal{O}_D(B_1), \quad v\colon \mathcal{O}_D \to \mathcal{O}_D(B_2).$$

Now u^2, uv, v^2 are sections of ag_2^1, $4g_2^1$, $(8-a)g_2^1$ respectively, so we have two relations $u^2 = f(s_1, s_2)$, $v^2 = g(s_1, s_2)$ and the identity $t = uv$. Here $f(s_1, s_2)$ is a homogeneous function of degree a on $\mathbb{P}^1$ with zeros at $Q_1, \ldots, Q_a$, similarly $g(s_1, s_2)$, so that $F = fg$.

Clearly every h-invariant divisor class A on D can be written in the form

$$A \sim P_1 + \cdots + P_a + bg_2^1 \sim P_{a+1} + \cdots + P_8 + (a + b - 4)g_2^1.$$

For such A, the graded ring

$$R(D, A) = \bigoplus_{n \geq 0} H^0(D, \mathcal{O}_D(nA))$$

can be studied relative to the base $\mathbb{P}^1$ via the double cover π. We quote the following proposition from [9] for D of genus 3, although the proposition and subsequent graded ring calculations work for any genus with only minor alterations.

Proposition 2.1. *Let D be a hyperelliptic curve of genus 3 with Weierstrass points $P_1, \ldots, P_8$, and write $\pi\colon D \to \mathbb{P}^1$ for the natural quotient by the hyperelliptic involution h. Then*

(1) $\pi_*\mathcal{O}_D = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-4)$;
(2) $\pi_*\mathcal{O}_D(g_2^1) = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$;
(3) $\pi_*\mathcal{O}_D(P_1 + \cdots + P_a) = \mathcal{O}_{\mathbb{P}^1}u \oplus \mathcal{O}_{\mathbb{P}^1}(a-4)v$;

where in each case the first summand is invariant under h and the second is anti-invariant.

Remark 2.2. Note that $\pi_*\mathcal{O}_D$ is a sheaf of $\mathcal{O}_{\mathbb{P}^1}$-algebras, where the multiplication

$$\mathcal{O}_{\mathbb{P}^1}(-4) \otimes \mathcal{O}_{\mathbb{P}^1}(-4) \to \mathcal{O}_{\mathbb{P}^1}$$

is defined via $t^2 = F(s_1, s_2)$.

2.1. The ineffective theta characteristic

An ineffective theta characteristic is a divisor class A on D such that $h^0(D, A) = 0$ and $2A \sim K_D$. Consider the ineffective theta characteristic

$$A \sim P_1 + \cdots + P_4 - g_2^1 \sim P_5 + \cdots + P_8 - g_2^1$$

on D. Using Proposition 2.1, we see that $R(D, A)$ is generated by monomials in s_1, s_2, u, v:

n	$H^0(D, \mathcal{O}_D(nA))$	$H^0(\mathbb{P}^1, \pi_*\mathcal{O}_D(nA))$
0	1	1
1	ϕ	ϕ
2	$y_1,\ y_2,\ y_3$	$s_1^2,\ s_1s_2,\ s_2^2$
3	$z_1,\ z_2,\ z_3,\ z_4$	$s_1u,\ s_2u,\ s_1v,\ s_2v$
4	t	uv

The relations between these generators are either of the trivial monomial kind, or derived from

$$u^2 = f_4(s_1, s_2),\ v^2 = g_4(s_1, s_2).$$

For example, it is clear that $z_1^2 = s_1^2 u^2 = y_1 f(y_1, y_2, y_3)$, where $f(y_1, y_2, y_3)$ is a *rendering* of $f_4(s_1, s_2)$ in the quadratic monomials s_1^2, $s_1 s_2$, s_2^2. In fact, we can present all the equations as

$$\operatorname{rank} \left(\begin{array}{cc|cc} y_1 & y_2 & z_1 & z_3 \\ y_2 & y_3 & z_2 & z_4 \\ \hline z_1 & z_2 & f_2 & t \\ z_3 & z_4 & t & g_2 \end{array} \right) \leq 1,$$

where f_2 and g_2 are quadrics in y_1, y_2, y_3. Taking $\operatorname{Proj} R(D, A)$ gives

$$D \subset \mathbb{P}(2^3, 3^4, 4),$$

and the double cover of $\mathbb{P}^1$ is the conic defined by the first 2×2 minor of the matrix.

3. Graded rings over hyperelliptic K3 surfaces

In this section we study the hyperelliptic degeneration of the symmetric determinantal quartic K3 surface. First we generalise the methods of Section 2 to hyperelliptic K3 surfaces, and in 3.1 we construct a hyperelliptic K3 surface T that extends the hyperelliptic curve D of genus 3 from Section 2.1. We go on to give alternative descriptions for D and T in Section 3.2, which is used in the proof of the Main Theorem.

A hyperelliptic K3 surface T is a K3 surface together with a complete linear system L such that $|L|$ contains an irreducible hyperelliptic curve D of arithmetic genus $g = h^0(T, \mathcal{O}_T(L)) - 1$. Then L determines a 2-to-1 map $\pi\colon T \to F$ where F is a surface of degree $g-1$ in $\mathbb{P}^g$, and the branch locus of π is some divisor in $|-2K_F|$. See [11] for further details on the hyperelliptic dichotomy for K3 surfaces. Del Pezzo classified surfaces F of minimal degree as rational scrolls or the Veronese surface. Since both have very simple explicit descriptions, we can analyse graded rings over any hyperelliptic K3 surface by calculating relative to the base F in the same way as for elliptic curves. For brevity we treat only the case $g = 3$, but more general examples are contained in [4].

We assume that $F = Q_2 \subset \mathbb{P}^3$ is a quadric of rank 4 and the double cover $\pi\colon T \to F$ is branched in a curve C of bidegree $(4, 4)$, which splits into two components $C_1 + C_2$ of bidegree $(3, 1)$ and $(1, 3)$ respectively. The components of the branch curve intersect one another transversally in ten points which are nodes of T. This is the hyperelliptic degeneration of the symmetric determinantal quartic K3 surface. As usual there is a hyperelliptic involution h on T exchanging the two sheets of the double cover, and π is the quotient map of h. Let H_1, H_2 be the generators of $\operatorname{Pic} Q$, then we omit π^* to write $\pi^* H_i = H_i$ on T, and $\pi^* C_i = 2D_i$.

Let s_1, s_2 be generators of $H^0(T, H_1)$, similarly t_1, t_2 for $H^0(T, H_2)$. Then there is an equation $F_{4,4}(s_1, s_2, t_1, t_2)$ defining the branch curve C on Q. This equation factors as $F = f_{3,1}(s_i, t_i)g_{1,3}(s_i, t_i)$, which gives the splitting $C = C_1 + C_2$. The double cover T is given by $t^2 = F$, and we have $2D_1 \sim 3H_1 + H_2$ and $2D_2 \sim H_1 + 3H_2$ on T. Considering the rational function $t/(t_1^2 s_1^2)$ on T, we find

$$2(H_1 + H_2) \sim D_1 + D_2.$$

By analogy with Section 2 we write down graded rings

$$R(T, A) = \bigoplus_{n \geq 0} H^0(T, \mathcal{O}_T(nA)),$$

where A is a divisor class which is invariant under h. Any such A can be written in the form

$$A \sim D_1 + n_1 H_1 + n_2 H_2 \sim D_2 + (n_1 + 1)H_1 + (n_2 - 1)H_2.$$

The following proposition is a natural extension of Proposition 2.1, which allows us to describe $R(T, A)$ relative to $R(Q, \pi_* A)$.

Proposition 3.1. *Let T be a hyperelliptic K3 surface double cover of the rank 4 quadric $Q \subset \mathbb{P}^3$, with ramification properties as described above. Choose constant sections $u\colon \mathcal{O}_T \to \mathcal{O}_T(D_1)$ and $v\colon \mathcal{O}_T \to \mathcal{O}_T(D_2)$ for the components D_i of the ramification curve. Clearly we have $u^2 = f_{3,1}(s_i, t_i)$, $uv = t$ and $v^2 = g_{1,3}(s_i, t_i)$, where $F = fg$. Moreover,*

(1) $\pi_* \mathcal{O}_T = \mathcal{O}_Q \oplus \mathcal{O}_Q(-2, -2)$;
(2) $\pi_* \mathcal{O}_T(H_1) = \mathcal{O}_Q(1, 0) \oplus \mathcal{O}_Q(-1, -2)$;
(3) $\pi_* \mathcal{O}_T(D_1) = \mathcal{O}_Q u \oplus \mathcal{O}_Q(1, -1)v$;

with similar results for H_2, D_2 respectively.

Remark 3.2. Once again we note the $\mathcal{O}_Q$-algebra structure on $\pi_* \mathcal{O}_T$. The multiplication map

$$\mathcal{O}_Q(-2, -2) \otimes \mathcal{O}_Q(-2, -2) \to \mathcal{O}_Q$$

is defined via the equation $t^2 = F_{4,4}(s_1, s_2, t_1, t_2)$.

3.1. Construction of the K3 surface T

Write $A \sim D_1 - H_1 \sim D_2 - H_2$, which is an h-invariant Weil divisor class on T, satisfying $H^0(T, \mathcal{O}_T(A)) = 0$ and $\mathcal{O}_T(2A) = \pi^* \mathcal{O}_Q(1)$. Note that A is the analogue of the ineffective theta characteristic in Section 2.1. We can describe

the ring $R(T,A)$ using Proposition 3.1. The generators for $R(T,A)$ are:

n	$H^0(T,\mathcal{O}_T(nA))$	$H^0(Q,\pi_*\mathcal{O}_T(nA))$
0	1	1
1	0	0
2	$y_1,\ y_2,\ y_3,\ y_4$	$s_1t_1,\ s_2t_1,\ s_1t_2,\ s_2t_2$
3	$z_1,\ z_2,\ z_3,\ z_4$	$t_1u,\ t_2u,\ s_1v,\ s_2v$
4	t	uv

The relations are again mostly trivial monomial relations, together with those derived from $u^2 = f_{3,1}$ and $v^2 = g_{1,3}$. For example, $z_1^2 = t_1^2 f_{3,1}$, and we can render the bihomogeneous expression $t_1^2 f_{3,1}$ as a cubic in the variables $y_1,\dots,y_4$. In fact, since all terms of the right hand side are divisible by either s_1t_1 or s_2t_1, we have

$$z_1^2 = p_1y_1 + q_1y_2,$$

where p_1, q_1 are suitable quadrics in $y_1,\dots,y_4$. A slightly more intricate calculation gives

$$z_1t = t_1u^2v = t_1vf_{3,1} = s_1vq_{2,2} + s_2vq'_{2,2} = p_1z_3 + q_1z_4,$$

where p_1 and q_1 are the same quadrics as above. The trick here is to make $f_{3,1}$ bihomogeneous by incorporating the factor t_1 into f and simultaneously taking out the excess in s_1, s_2. Clearly, there are certain choices involved in writing down p_1 and q_1. Fortunately, these do not matter, as any discrepancy is accounted for by the rank condition (1) below. We present all the relations of $R(T,A)$ as follows

$$\text{(1)}\qquad \operatorname{rank}\left(\begin{array}{cc|c} y_1 & y_2 & z_1 \\ y_3 & y_4 & z_2 \\ \hline z_3 & z_4 & t \end{array}\right) \le 1,$$

$$\begin{aligned} z_1^2 &= p_1y_1 + q_1y_2 & z_3^2 &= p_3y_1 + q_3y_3 \\ z_1z_2 &= p_1y_3 + q_1y_4 = p_2y_1 + q_2y_2 & z_3z_4 &= p_3y_2 + q_3y_4 = p_4y_1 + q_4y_3 \\ z_2^2 &= p_2y_3 + q_2y_4 & z_4^2 &= p_4y_2 + q_4y_4 \\ z_1t &= p_1z_3 + q_1z_4 & z_3t &= p_3z_1 + q_3z_2 \\ z_2t &= p_2z_3 + q_2z_4 & z_4t &= p_4z_1 + q_4z_2 \end{aligned}$$

$$t^2 = F(y_i) = p_1p_3 + p_2q_3 + p_4q_1 + q_2q_4,$$

where the p_i, q_i are quadrics in $y_1,\dots,y_4$, as described above for the case $i = 1$. Then $\operatorname{Proj} R(T,A)$ gives us the K3 surface

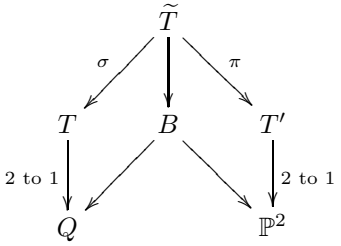
$$T \subset \mathbb{P}(2^4, 3^4, 4),$$

which has $10 \times \frac{1}{2}(1,1)$ points. Note that the curve D of Section 2.1 is obtained by taking a hyperplane section of weight 2 in T, avoiding the $\frac{1}{2}$ points.

3.2. Projection of T to a complete intersection

Let $Q \subset \mathbb{P}^3$ be the quadric of rank 4 and consider the projection $Q \dashrightarrow \mathbb{P}^2$ of del Pezzo surfaces. This map is obtained by taking the blow up B of a point P on Q, then contracting the two (-1)-curves on B arising from the rulings of Q to get $\mathbb{P}^2$. Now, suppose we have a curve C on Q of type $(4,4)$ which splits as $C = C_1 + C_2$ where $C_1 \in (3,1)$, $C_2 \in (1,3)$ so that C has ten nodes. If the centre of projection P is chosen to be one of these nodes then the two components C_1, C_2 are projected to nodal plane cubics, and P itself is projected to the line L through these two nodes.

Now suppose we have a hyperelliptic $K3$ surface T which is a double cover of Q branched in C. The classical projection $Q \dashrightarrow \mathbb{P}^2$ lifts to the double cover as illustrated by the diagram below:



where $\sigma\colon \widetilde{T} \to T$ is the blowup of P in T and we write $E \cong \mathbb{P}^1$ for the exceptional divisor. The image T' of the projection is a double cover of $\mathbb{P}^2$ branched over the two nodal cubics. The centre of projection P in T is projected to a rational curve of arithmetic genus 2, double covering L away from the two nodes, and branched over the residual intersection with C.

Now this diagram can also be recast as a projection–unprojection operation in the sense of [7], [10].

Proposition 3.3. *The projection from a $\frac{1}{2}$ point of $T \subset \mathbb{P}(2^4, 3^4, 4)$ is a complete intersection $T'_{6,6} \subset \mathbb{P}(2,2,2,3,3)$ with equations*

$$\begin{aligned} z_1^2 &= L_1(y_1,y_2,y_3)y_2^2 + l_4 y_2 f g + y_1 f^2, \\ z_2^2 &= M_1(y_1,y_2,y_3)y_2^2 + m_4 y_2 f g + y_3 g^2, \end{aligned} \tag{2}$$

where L_1, M_1 are linear in y_1, y_2, y_3 but do not involve y_4, and l_4, m_4 are scalars. The image of the exceptional curve E is defined by $y_2 = 0$.

Proof. Start from $T \subset \mathbb{P}(2^4, 3^4, 4)$ with $10 \times \frac{1}{2}$ points and polarising divisor A, as described in Section 3.1. Choose a $\frac{1}{2}$ point P in T, and write $\sigma\colon \widetilde{T} \to T$ for the $(1,1)$-weighted blowup of P, whose exceptional curve is E. According to [7], [10], the projected surface $T'_{6,6} \subset \mathbb{P}(2,2,2,3,3)$ is calculated as

$$T' = \operatorname{Proj} R(\widetilde{T}, \sigma^* A - \tfrac{1}{2}E),$$

where certain functions (to be determined) on $\widetilde{T}$ are eliminated by the projection, because they do not vanish appropriately along E.

We express this elimination explicitly as an operation in commutative algebra. Assume the centre of projection is a $\frac{1}{2}$ point at the coordinate point P_{y_4}, with local coordinates z_3, z_4. Then adjusting the coordinates from Section 3.1 slightly, we write down the determinantal relations

$$\operatorname{rank}\begin{pmatrix} y_2 & f & z_1 \\ g & y_4 & z_3 \\ z_2 & z_4 & t \end{pmatrix} \leq 1, \tag{3}$$

where we reserve the right to choose f, g later. These equations are a subset of those for $T \subset \mathbb{P}(2^4, 3^4, 4)$ after a trivial change of coordinates. The remaining equations for T are completely determined by

$$z_1^2 = L_1 y_2^2 + L_2 y_2 f + L_3 f^2,$$
$$z_2^2 = M_1 y_2^2 + M_2 y_2 g + M_3 g^2,$$

where a priori L_i, M_i are linear in $y_1, \ldots, y_4$. Indeed, the equations we have written down so far are sufficient to determine the two components C_1 and C_2 of the branch curve, along with their defining equations $f_{3,1}$ and $g_{1,3}$. We can fill in the remaining equations of T using the procedure outlined in Section 3.1.

Moreover, since we fixed a $\frac{1}{2}$ point at P_{y_4}, the last equation for T can be written as

$$t^2 = a_2(y_1, y_3) y_4^2 + b_3(y_1, y_2, y_3) y_4 + c_4(y_1, y_2, y_3).$$

Now the tangent cone to P must factorise, because the branch curve C splits into two components, so we can choose coordinates

$$f = y_1 + \alpha y_3, \quad g = \beta y_1 + y_3$$

so that $a = y_1 y_3$. This in turn forces $L_3 = y_1$, $M_3 = y_3$ so that modulo the rank condition (3), the equations involving z_1^2 and z_2^2 take the form (2).

We are finally in a position to describe the projection centred at P_{y_4} in terms of explicit equations. The local coordinates near P are z_3, z_4 so we expect the projection to eliminate these variables along with y_4 (see [10], example 9.13). In fact the projection also eliminates t, and we are left with equations (2) defining a complete intersection

$$T'_{6,6} \subset \mathbb{P}(2, 2, 2, 3, 3).$$

The image of the exceptional curve E is defined by $y_2 = 0$. This is the hyperelliptic degeneration of the *totally tangent conic* configuration described in [5]. □

4. Extending hyperelliptic graded rings

In this section we consider extensions of the hyperelliptic K3 surface T constructed in Section 3.1, and prove the Main Theorem. As with the symmetric determinantal extensions of [5], the most convenient way to extend T is by using the projection construction of Section 3.2. We describe the reverse procedure to Proposition 3.3. Start from

$$\mathbb{P}^1 \xrightarrow{\varphi} T'_{6,6} \subset \mathbb{P}(2,2,2,3,3),$$

where T' is a double cover of $\mathbb{P}(2,2,2)$ branched in two nodal cubics. The image of φ is a curve of arithmetic genus 2, which is a double cover of the line joining the two nodes. Constructing φ and $T'_{6,6}$ is equivalent to constructing T itself, so we prove the theorem by extending φ and T'.

We assume that φ is a double cover of the line $(y_2 = 0) \subset \mathbb{P}(2,2,2)$ branched over the points $\varphi(1,0)$ and $\varphi(0,1)$. Then for general T' the map φ is

$$\varphi\colon \mathbb{P}^1 \to \mathbb{P}(2,2,2,3,3)$$

$$(u,v) \mapsto \left(u^2, 0, v^2, u(u^2+\alpha v^2), v(\beta u^2 + v^2)\right). \tag{4}$$

Rendering $\varphi^*(z_i^2)$ in terms of y_1, y_3 we see that the image of φ is defined by the equations

$$C_1\colon z_1^2 = y_1 f^2, \tag{5}$$

$$C_2\colon z_2^2 = y_3 g^2, \tag{6}$$

$$y_2 = 0, \tag{7}$$

where $f = y_1 + \alpha y_3$ and $g = \beta y_1 + y_3$. To define $T' \subset \mathbb{P}(2,2,2,3,3)$ we must choose two appropriate combinations of weight 6 in equations (5)–(7). Note that if we want the branch curves to be nondegenerate then we should ensure that both equations for T' involve y_2 nontrivially. Moreover, after incorporating y_2 into the equations we should check that there are still two bona fide nodes on the branch locus at $(-\alpha, 0, 1)$ and $(1, 0, -\beta)$. So, calculating the tangent cone to each curve at these points forces the equations of T' to take the form

$$\begin{aligned} &C_1 + l_1 Q_1 + l_2 Q_2 + l_3 Q_3 + l_4 Q_4, \\ &C_2 + m_1 Q_1 + m_2 Q_2 + m_3 Q_3 + m_4 Q_4, \end{aligned} \tag{8}$$

where l_i, m_i are scalar parameters and

$$Q_1 = f y_2^2, \quad Q_2 = y_2^3, \quad Q_3 = g y_2^2, \quad Q_4 = f g y_2.$$

One can check that (8) is equivalent to equations (2) of Proposition 3.3.

Strategy of proof of Main Theorem. Start with a quasismooth Fano 6-fold $W \subset \mathbb{P}(1^4, 2^4, 3^4, 4)$ with $10 \times \frac{1}{2}$ points, such that $T = W \cap H_1 \cap H_2 \cap H_3 \cap H_4$ is a hyperelliptic K3 surface. By the above discussion, the projection from a $\frac{1}{2}$ point P in T gives $T'_{6,6} \subset \mathbb{P}(2,2,2,3,3)$ together with $\varphi\colon \mathbb{P}^1 \to T'$. Let $\tilde{P}$ be the $\frac{1}{2}$ point on W corresponding to P on T. The projection from $\tilde{P}$ gives a Fano 6-fold

$W'_{6,6} \subset \mathbb{P}(1^4, 2^3, 3^2)$ containing the image of a map $\Phi\colon \mathbb{P}^5 \to \mathbb{P}(1^4, 2^3, 3^2)$, where $\Phi|_{\mathbb{P}(2^3,3^2)} = \varphi$. We prove that Φ and W' exist and are uniquely determined by φ and T'. Then the reverse projection shows that W exists and is uniquely determined by T.

We lay some groundwork for the proof, by computing a normal form for Φ. Define $\varphi\colon \mathbb{P}^1 \to \mathbb{P}(2,2,2,3,3)$ as in (4) and write $\varphi_0\colon \mathbb{P}^1 \to \mathbb{P}(2,2,2)$ for the map

$$\varphi_0^*(y_1) = u^2, \quad \varphi_0^*(y_2) = 0, \quad \varphi_0^*(y_3) = v^2.$$

Then writing u, v, a, b, c, d for the coordinates on $\mathbb{P}^5$, up to automorphisms of $\mathbb{P}^5$ and $\mathbb{P}(1^4, 2^3)$ the general extension of φ_0 to $\Phi_0\colon \mathbb{P}^5 \to \mathbb{P}(1^4, 2^3)$ is

$$\Phi_0^*(a) = a, \quad \Phi_0^*(b) = b, \quad \Phi_0^*(c) = c, \quad \Phi_0^*(d) = d,$$

$$\begin{aligned} \Phi_0^*(y_1) &= u^2 \qquad\quad +2av, \\ \Phi_0^*(y_2) &= 0 + \quad bu + cv, \\ \Phi_0^*(y_3) &= v^2 + 2du. \end{aligned} \tag{9}$$

We prove that there is a unique map $\Phi\colon \mathbb{P}^5 \to \mathbb{P}(1^4, 2^3, 3^2)$ which is a lift of Φ_0 and which extends $T'_{6,6}$ to $W'_{6,6}$.

Lemma 4.1. *The general forms of $\Phi^*(z_i)$ are*

$$\begin{aligned} \Phi^*(z_1) &= u(f + s_4) + s_2uv + s_5v, \\ \Phi^*(z_2) &= v(g + t_5) + t_2uv + t_4u. \end{aligned} \tag{10}$$

Proof. We begin by fixing some notation. Write M, R, S for the coordinate rings of $\mathbb{P}^5$, $\mathbb{P}(1^4, 2^3)$ and $\mathbb{P}(1^4, 2^3, 3^2)$ respectively. By equation (9), the map Φ_0^* induces a graded R-module structure on M with generators 1, u, v and uv. Similarly Φ^* makes M into a graded S-module with the same generators. The presentation of M as a module over R is

$$0 \leftarrow M \xleftarrow{(1,u,v,uv)} R \oplus 2R(-1) \oplus R(-2) \xleftarrow{A} R(-2) \oplus 2R(-3) \oplus R(-4),$$

where A is the matrix

$$\begin{pmatrix} -y_2 & by_1 & cy_3 & -2cdy_1 + 4ady_2 - 2aby_3 \\ b & -y_2 & -2cd & cy_3 \\ c & -2ab & -y_2 & by_1 \\ 0 & c & b & -y_2 \end{pmatrix}. \tag{11}$$

Since Φ is a lift of φ the general forms of $\Phi^*(z_i)$ are

$$\begin{aligned} \Phi^*(z_1) &= u^3 + \alpha uv^2 + s_1u^2 + s_2uv + s_3v^2 + s_4u + s_5v, \\ \Phi^*(z_2) &= \beta u^2v + v^3 + t_1u^2 + t_2uv + t_3v^2 + t_4u + t_5v, \end{aligned}$$

where the $s_i(a,b,c,d)$, $t_i(a,b,c,d)$ are homogeneous polynomials of degree 1 or 2 as appropriate. Then using the R-module structure of M, and coordinate

changes $z_1 \mapsto z_1 + s_1 y_1$ and similar to absorb the values of s_1, s_3, t_1, t_3 into z_1, z_2, we get the normal form given in equation (10). □

Theorem 4.2. (I) *The kernel of* $\Phi^*\colon S \to M$ *contains equations extending* (5), (6) *of the form*

$$z_1^2 - y_1 f^2 \in R + Rz_1 + Rz_2,$$
$$z_2^2 - y_3 g^2 \in R + Rz_1 + Rz_2$$

if and only if

$$s_2 = (1-\alpha\beta)a, \quad s_4 = \beta a^2 + \alpha^2 d^2, \quad s_5 = \alpha(\alpha\beta - 1)ad,$$
$$t_2 = (1-\alpha\beta)d, \quad t_4 = \beta(\alpha\beta - 1)ad, \quad t_5 = \beta^2 a^2 + \alpha d^2.$$

(II) *Given part* (*I*), *the equations are*

$$\begin{aligned} z_1^2 - y_1(f+s_4)^2 &= -4(f+s_4)s_2 a y_3 - 4s_2 s_5 d y_1 + s_2^2 y_1 y_3 + s_5^2 y_3 \\ &\quad + 2(1-\alpha\beta)a^2(3dz_1 - az_2) - 2\alpha a(f+s_4)z_2, \end{aligned} \tag{12}$$

$$\begin{aligned} z_2^2 - y_3(g+t_5)^2 &= -4(g+t_5)t_2 d y_1 - 4t_2 t_4 a y_3 + t_2^2 y_1 y_3 + t_4^2 y_1 \\ &\quad + 2(1-\alpha\beta)d^2(3az_2 - dz_1) - 2\beta d(g+t_5)z_1. \end{aligned} \tag{13}$$

Corollary 4.3. *Given the values of* s_i, t_i *stated in Theorem 4.2, the kernel of* Φ^* *also contains* (*nontrivial*) *equations extending* Q_i *for* $i = 1, \dots, 4$ *of the form*

$$f y_2^2,\ y_2^3,\ g y_2^2,\ f g y_2 \in R + Rz_1 + Rz_2$$

respectively.

Proof of Theorem 4.2. The "if" part of the theorem is proved by evaluating equations (12), (13) under Φ^* with s_i, t_i taking the values stated in the theorem. The remainder of the proof is for the "only if" part.

Using the graded module structure of $k[u,v]$ over $k[y_1,y_2,y_3]$ via φ_0^* we write

$$\varphi^*(z_1) = fu, \quad \varphi^*(z_2) = gv.$$

Then squaring each of these expressions and rendering u^2, v^2 as y_1, y_3 respectively, gives equations (5), (6) immediately. We attempt to do the same rendering calculation for the extended map Φ^*. Since

$$u^2 = \Phi^*(y_1) - 2av,$$
$$v^2 = \Phi^*(y_3) - 2du,$$

we can eliminate all terms involving u^2 or v^2 from $\Phi^*(z_i^2)$ to obtain

$$\Phi^*\left(z_1^2 - y_1(f+s_4)^2 + 4(f+s_4)s_2 a y_3 + 4s_2 s_5 d y_1 - s_2^2 y_1 y_3 - s_5^2 y_3\right) \equiv 0,$$
$$\Phi^*\left(z_2^2 - y_3(g+t_5)^2 + 4(g+t_5)t_2 d y_1 + 4t_2 t_4 a y_3 - t_2^2 y_1 y_3 - t_4^2 y_1\right) \equiv 0$$

modulo $(a,b,c,d)M$. The residual parts to these congruences are

$$K = K_u u + K_v v + K_{uv} uv,$$

$$L = L_u u + L_v v + L_{uv} uv$$

respectively, where

$$\begin{aligned} K_u &= 8(f+s_4)s_2 ad - 2s_5^2 d - 2s_2^2 dy_1 + 2s_2 s_5 y_3, \\ K_v &= -2(f+s_4)^2 a + 8s_2 s_5 ad + 2(f+s_4)s_2 y_1 - 2s_2^2 a y_3, \\ K_{uv} &= 2(f+s_4)s_5 + 4s_2^2 ad, \end{aligned} \tag{14}$$

and

$$\begin{aligned} L_u &= -2(g+t_5)^2 d + 8t_2 t_4 ad + 2(g+t_5)t_2 y_3 - 2t_2^2 dy_1, \\ L_v &= 8(g+t_5)t_2 ad - 2t_4^2 a - 2t_2^2 a y_3 + 2t_2 t_4 y_1, \\ L_{uv} &= 2(g+t_5)t_4 + 4t_2^2 ad. \end{aligned} \tag{15}$$

Now K, L are homogeneous expressions of degree 6 in $(a,b,c,d)M$, and we prove that if they are to be contained in the submodule $R + Rz_1 + Rz_2 \subset M$ then s_i, t_i must take the values stated in the theorem.

From the definition of $\Phi^*(z_i)$ in (10), the submodule $R + Rz_1 + Rz_2$ is the image of the composite map

$$M \xleftarrow{(1,u,v,uv)} R \oplus 2R(-1) \oplus R(-2) \xleftarrow{B} R \oplus 2R(-3) \oplus R(-2) \oplus 2R(-3) \oplus R(-4)$$

where B is the matrix

$$\left(\begin{array}{ccc|ccc|c} 1 & 0 & 0 & -y_2 & by_1 & cy_3 & -2cdy_1 + 4ady_2 - 2aby_3 \\ 0 & f+s_4 & t_4 & b & -y_2 & -2cd & cy_3 \\ 0 & s_5 & g+t_5 & c & -2ab & -y_2 & by_1 \\ 0 & s_2 & t_2 & 0 & c & b & -y_2 \end{array}\right).$$

The first three columns of B are the generators 1, z_1, z_2 and the last four columns are the matrix A from (11), which is mapped to 0 under the composite.

Thus we seek vectors $\xi, \eta \in R \oplus 2R(-3) \oplus R(-2) \oplus 2R(-3) \oplus R(-4)$ such that

$$\begin{aligned} K &= \begin{pmatrix} 1, & u, & v, & uv \end{pmatrix} B\xi, \\ L &= \begin{pmatrix} 1, & u, & v, & uv \end{pmatrix} B\eta. \end{aligned} \tag{16}$$

Proposition 4.4. *The only solution to equation* (16) *is*

$$\begin{aligned} \xi_2 &= 6(1-\alpha\beta)a^2 d & \eta_2 &= -2\beta d(g+t_5) - 2(1-\alpha\beta)d^3 \\ \xi_3 &= -2\alpha a(f+s_4) - 2(1-\alpha\beta)a^3 & \eta_3 &= 6(1-\alpha\beta)ad^2, \end{aligned}$$

and the other $\xi_i = \eta_i = 0$.

We postpone the proof of this proposition to the end of the section. The extended equations (12), (13) are obtained by writing out the vectors ξ, η in

terms of the generators of $R + Rz_1 + Rz_2$

$$\begin{aligned} z_1^2 - y_1(f+s_4)^2 &= -4(f+s_4)s_2ay_3 - 4s_2s_5dy_1 + s_2^2y_1y_3 + s_5^2y_3 \\ &\quad + \xi_2 z_1 + \xi_3 z_2, \\ z_2^2 - y_3(g+t_5)^2 &= -4(g+t_5)t_2dy_1 - 4t_2t_4ay_3 + t_2^2y_1y_3 + t_4^2y_1 \\ &\quad + \eta_2 z_1 + \eta_3 z_2. \end{aligned}$$

This concludes the proof of Theorem 4.2. □

Proof of Corollary 4.3. First observe that the fourth column of B is equivalent to $y_2 = bu + cv$. Thus the extension of Q_1 is calculated by expressing $(f + s_4)y_2(bu+cv)$ in terms of the other columns of B. We have to find μ such that

$$(f+s_4)y_2(bu+cv) = \begin{pmatrix} 1, & u, & v, & uv \end{pmatrix} B\mu.$$

The solution to this linear algebra problem is

$$\begin{aligned} \mu_2 &= 2by_2 + 2(\beta ab - cd)c & \mu_3 &= 2(\alpha b^2 + c^2)a \\ \mu_4 &= -\beta a s_2 y_2 - 2ac(g+t_5) + 2(cd - \beta ab)s_5 & \mu_5 &= b(f+s_4) - \beta ab s_2 \\ \mu_6 &= -c(f+s_4) - \beta acs_2 + 2bs_5 & \mu_7 &= 2bs_2, \end{aligned}$$

where $\mu_1 = y_2\mu_4 - by_1\mu_5 - cy_3\mu_6 - (-2cdy_1 + 4ady_2 - 2aby_3)\mu_7$ uses the first column of B to remove any excess terms. Thus the equation extending Q_1 is

$$\widetilde{Q}_1 \colon (f+s_4)y_2^2 = \mu_1 + \mu_2 z_1 + \mu_3 z_2.$$

Similar linear algebra considerations give the equations extending Q_2, Q_3, Q_4, and we list the corresponding vectors below. The equation extending Q_2 is

$$\widetilde{Q}_2 \colon y_2^3 = \mu_1 + \mu_2 z_1 + \mu_3 z_2,$$

where

$$\begin{aligned} \mu_1 &= y_2\mu_4 - by_1\mu_5 - cy_3\mu_6 - (-2cdy_1 + 4ady_2 - 2aby_3)\mu_7, \\ \mu_2 &= \frac{2}{\alpha\beta - 1}(b^2 + \beta c^2)b, \\ \mu_3 &= \frac{2}{\alpha\beta - 1}(\alpha b^2 + c^2)c, \\ \mu_4 &= \frac{2}{1-\alpha\beta}\left(b^2(f+s_4) + c^2(g+t_5)\right) + (\beta ac + \alpha bd)y_2 + 2(2-\alpha\beta)abcd, \\ \mu_5 &= -by_2 + 2c^2d + (\beta ac + \alpha bd)b, \\ \mu_6 &= -cy_2 + 2ab^2 + (\beta ac + \alpha bd)c, \\ \mu_7 &= -2bc. \end{aligned}$$

The equation extending Q_3 is

$$\widetilde{Q}_3 \colon (g+t_5)y_2^2 = \mu_1 + \mu_2 z_1 + \mu_3 z_2,$$

where

$$\mu_1 = y_2\mu_4 - by_1\mu_5 - cy_3\mu_6 - (-2cdy_1 + 4ady_2 - 2aby_3)\mu_7$$

$$\begin{aligned}
&\mu_2 = 2(b^2 + \beta c^2)d && \mu_3 = 2cy_2 - 2(ab - \alpha cd)b \\
&\mu_4 = -\alpha dt_2y_2 - 2bd(f + s_4) + 2(ab - \alpha cd)t_4 && \mu_5 = -b(g + t_5) - \alpha bdt_2 + 2ct_4 \\
&\mu_6 = c(g + t_5) - \alpha cdt_2 && \mu_7 = 2ct_2.
\end{aligned}$$

Finally, Q_4 is extended by

$$\widetilde{Q}_4\colon (f + s_4)(g + t_5)y_2 = \mu_1 + \mu_2z_1 + \mu_3z_2,$$

where

$$\mu_1 = y_2\mu_4 - by_1\mu_5 - cy_3\mu_6 - (-2cdy_1 + 4ady_2 - 2aby_3)\mu_7$$

$$\begin{aligned}
&\mu_2 = b(g + t_5) + ct_4 - t_2y_2 && \mu_3 = c(f + s_4) + bs_5 - s_2y_2 \\
&\mu_4 = -s_5t_4 && \mu_5 = -t_2(f + s_4) - s_2t_4 \\
&\mu_6 = -s_2(g + t_5) - s_5t_2 && \mu_7 = -2s_2t_2.
\end{aligned}$$

This completes the proof of the corollary. □

Proof of Main Theorem. Given Theorem 4.2 and its Corollary, we can prove that there is a unique hyperelliptic Fano 6-fold $W'_{6,6} \subset \mathbb{P}(1^4, 2^3, 3^2)$ extending any given projected hyperelliptic K3 surface $T'_{6,6}$, as claimed in the Introduction. Simply take the combination of equations (12), (13) and $\widetilde{Q}_i$ corresponding to the choice (8) made in the definition of $T'_{6,6}$. According to our strategy, the unprojection of W' is the unique extension of T. This proves the Main Theorem. □

Proof of Proposition 4.4. In order to solve for ξ, η and consequently fix the values of s_i, t_i we stratify K, L according to degree in y_1, y_2, y_3. In other words, write

$$K = K^{(0)} + K^{(1)} + K^{(2)},$$
$$L = L^{(0)} + L^{(1)} + L^{(2)},$$

where $K^{(i)}$, $L^{(i)}$ have degree i in y_1, y_2, y_3 and similarly we write

$$\xi = \xi^{(0)} + \xi^{(1)},$$
$$\eta = \eta^{(0)} + \eta^{(1)}.$$

We begin with $K^{(2)}$, which is calculated from (14) as

$$K^{(2)} = 2f(y_1s_2 - fa)v.$$

We must find $\xi^{(1)}$ such that

$$K^{(2)} = \begin{pmatrix}1, & u, & v, & uv\end{pmatrix} B\xi^{(1)} + \text{ lower order terms.} \tag{17}$$

Comparing coefficients of y_1^2 and y_3^2, the only solution is

$$\xi_3^{(1)} = \frac{2}{\beta}(s_2 - a)y_1 - 2\alpha^2 a y_3,$$

with the other $\xi_i^{(1)} = 0$. Then the coefficient of $y_1 y_3$ in (17) dictates that

$$s_2 = (1 - \alpha\beta)a$$

and therefore $\xi_3^{(1)} = -2\alpha a f$. An exactly similar calculation with $L^{(2)}$ and $\eta_2^{(1)}$ yields

$$t_2 = (1 - \alpha\beta)d$$

and $\eta_2^{(1)} = -2\beta d g$.

Proceeding to $K^{(1)}$, we must solve

(18)
$$K^{(1)} - \xi_3^{(1)}(t_4 u + t_5 v + t_2 uv) = \begin{pmatrix} 1, & u, & v, & uv \end{pmatrix} B\xi^{(0)} + \text{ lower order terms}$$

where the term involving $\xi_3^{(1)}$ is necessary to account for the lower order terms from equation (17). Now examining the coefficient of uv in (18), we see that

$$2f(s_5 + \alpha a t_2) = s_2\xi_2^{(0)} + t_2\xi_3^{(0)}.$$

However, $\xi^{(0)}$ has degree 0 in y_i by construction, so the left hand side must be identically 0. Hence

$$s_5 = -\alpha a t_2$$

and by considering the coefficient of uv in $L^{(1)}$ we find

$$t_4 = -\beta d s_2.$$

Comparing coefficients of u and v in equation (18) we obtain

$$6(1 - \alpha\beta)a^2 df = (f + s_4)\xi_2^{(0)} + t_4\xi_3^{(0)} + \text{ lower order terms},$$

$$2a(-s_4(f + \alpha g) + \alpha f t_5 - s_2^2 y_3) = s_5\xi_2^{(0)} + (g + t_5)\xi_3^{(0)} + \text{ lower order terms}.$$

Since $\xi^{(0)}$ has degree 0 in y_i we must have $\xi_2^{(0)} = 6(1 - \alpha\beta)a^2 d$. Moreover the coefficient of v must be divisible by g, which is equivalent to

$$\alpha t_5 - s_4 = -\beta(1 - \alpha\beta)a^2. \tag{19}$$

By considering the coefficients of u, v in $L^{(1)}$ in the same way we get $\eta_3^{(0)} = 6(1 - \alpha\beta)ad^2$ and a further restriction on s_4, t_5:

$$t_5 - \beta s_4 = \alpha(1 - \alpha\beta)d^2. \tag{20}$$

Solving equations (19), (20) simultaneously forces

$$s_4 = \beta a^2 + \alpha^2 d^2,$$
$$t_5 = \beta^2 a^2 + \alpha d^2,$$

which in turn means that

$$\xi_3^{(0)} = -2(1 - \alpha\beta)a^3 - 2\alpha a s_4,$$

$$\eta_2^{(0)} = -2(1-\alpha\beta)d^3 - 2\beta dt_5.$$

We can finally write out ξ and η in full, and we get the values stated in Proposition 4.4. It is necessary to check that ξ and η actually solve equations (16) when all the lower order terms are replaced, and this can be verified directly. □

5. Godeaux surfaces with $\pi_1 = \mathbb{Z}/2$

In this section we construct Godeaux surfaces with $\pi_1 = \mathbb{Z}/2$ by considering the Galois étale double cover. This double cover is a surface Y of general type with $p_g = 1$, $q = 0$, and $K^2 = 2$ with hyperelliptic canonical curve section, and we find such surfaces as complete intersections inside a key variety W constructed in the Main Theorem. The problem is to find those W having a $\mathbb{Z}/2$-action which restricts to an appropriate fixed point free $\mathbb{Z}/2$-action on some $Y \subset W$.

5.1. Covers of Godeaux surfaces

Let X be the canonical model of a surface of general type with $p_g = 0$, $K^2 = 1$. We call X a Godeaux surface, and we assume that the torsion subgroup $\operatorname{Tors} X \subset \operatorname{Pic} X$ has order 2. Write σ for the generator of $\operatorname{Tors} X$, and consider the Galois étale double cover $f\colon Y \to X$ induced by σ. The cover is constructed by taking $Y = \operatorname{Proj} R(Y, K_X, \sigma)$, or written out in full

$$Y = \operatorname{Proj} \bigoplus_{n\geq 0} \left(H^0(X, nK_X) \oplus H^0(X, nK_X + \sigma) \right).$$

The surface Y is the canonical model of a surface of general type with $p_g = 1$, $q = 0$, $K^2 = 2$, and the extra $\mathbb{Z}/2$-grading on the ring $R(Y, K_X, \sigma)$ determines a fixed point free $\mathbb{Z}/2$ group action on Y, where the first summand is invariant and the second is anti-invariant. The quotient by this group action is the map f.

An analysis of $R(Y, K_X, \sigma)$ reveals that the canonical curve of Y must be hyperelliptic.

Lemma 5.1. *If Y is the unramified double covering of a Godeaux surface with torsion $\mathbb{Z}/2$, then the canonical curve section D in $|K_Y|$ is hyperelliptic.*

This lemma was proved in [3], using a monomial counting proof. We use a Hilbert series approach which has some advantages, and yields slightly more information about the group action for use later.

Proof. Define the bigraded Hilbert series of the ring $R(Y, K_X, \sigma)$ by

$$P_Y(t, e) = \sum_{n\geq 0} \left(h^0(X, nK_X)t^n + h^0(X, nK_X + \sigma)t^n e \right),$$

where t keeps track of the degree, and e keeps track of the eigenspace, so that $e^2 = 1$. Then using the Riemann–Roch theorem,

$$P_Y(t, e) = 1 + et + 2t^2 + 2t^2 e + 4t^3 + 4t^3 e + \cdots$$

which can be written as the rational function

$$P_Y(t,e) = \frac{1+(e-1)t^4+(-2e-2)t^5+(-4e-6)t^6+(7e+8)t^8+\cdots}{(1-et)(1-t^2)(1-et^2)^2(1-t^3)^2(1-et^3)^2}.$$

Using well-known Hilbert series properties would normally indicate that Y is a subvariety of $\mathbb{P}(1,2^3,3^4)$. However, the first nontrivial coefficient in the numerator is not negative, due to the bigrading. Thus we must introduce an extra generator of degree 4 in the negative eigenspace, dividing $P_Y(t,e)$ by $(1-et^4)$ so that the numerator becomes

$$1-t^4+(-2e-2)t^5+(-4e-6)t^6+\cdots.$$

The extra $-t^4$ term in the numerator suggests that it is necessary to introduce a relation in degree 4, which does not eliminate the new generator of degree 4. Thus the canonical curve section of Y must be hyperelliptic, and is given by $D \subset \mathbb{P}(2^3,3^4,4)$. □

Now, we can construct hyperelliptic surfaces Y of general type using the key variety of the Main Theorem.

Theorem 5.2. *There is a* 15*-parameter family of hyperelliptic surfaces* Y *of general type with* $p_g = 1$, $q = 0$, $K^2 = 2$, *each of which is topologically simply connected, and a complete intersection of type* $(1,1,1,2)$ *in a Fano* 6*-fold* $W \subset \mathbb{P}(1^4,2^4,3^4,4)$ *with* $10 \times \frac{1}{2}$ *points.*

The proof is a simple application of adjunction, vanishing, and Riemann–Roch. The parameter count follows from the Main Theorem and by considering the number of additional parameters involved in choosing a complete intersection $(1,1,1,2)$. Note that we obtain the canonical model of Y using this construction. In order to find Godeaux surfaces with $\pi_1 = \mathbb{Z}/2$, we must determine which surfaces Y have an appropriate $\mathbb{Z}/2$-action. Our method is to examine the $\mathbb{Z}/2$-action on the hyperelliptic curve D, and extend it to the key variety W.

5.2. The canonical curve

Now let $f\colon Y \to X$ be the étale double cover of a Godeaux surface X and suppose D is a nonsingular curve in $|K_Y|$, similarly C in $|K_X+\sigma|$. By Lemma 5.1, D is a hyperelliptic curve and C has genus 2 so is automatically hyperelliptic too. Let $\pi_D\colon D \to Q \cong \mathbb{P}^1$ denote the quotient map of the hyperelliptic involution on D, similarly $\pi_C\colon C \to \mathbb{P}^1$. Since D is an unramified double cover of C via $f|_D$, this induces a double cover of $\operatorname{Im}\pi_C = \mathbb{P}^1$ by Q. We get the following picture:

$$\begin{array}{ccccc} E & \longrightarrow & D & \xrightarrow{f|_D} & C \\ & & \downarrow{\scriptstyle \pi_D} & & \downarrow{\scriptstyle \pi_C} \\ & & Q & \longrightarrow & \mathbb{P}^1 \end{array}$$

There is a fixed point free involution on D corresponding to the unramified double cover $f|_D$, which we call the Godeaux involution. We use the same notation for the Godeaux involution and the torsion element $\sigma \in \operatorname{Pic} X$.

Consider the ineffective theta characteristic $A_D = K_Y|_D$ on D which is determined by the surface Y. Now A_D is invariant under both σ and the hyperelliptic involution, and the Weierstrass points of D must be invariant under σ. Since

$$A_D \sim P_1 + P_2 + P_3 + P_4 - g_2^1 \sim P_5 + P_6 + P_7 + P_8 - g_2^1,$$

there are two possibilites: either $\{P_1, \dots, P_4\}$ is invariant under σ, so that A_D is a σ-invariant divisor, or $\{P_1, \dots, P_4\}$ and $\{P_5, \dots, P_8\}$ are interchanged by σ, in which case A_D is only a σ-invariant divisor class.

We constructed the graded ring $R(D, A_D)$ in Section 2.1. Furthermore by the adjunction formula, it is clear that $2g_2^1 \sim 2A_D \sim K_D$. However, these two divisor classes A_D and g_2^1 are distinct, because the g_2^1 is effective whereas A_D is ineffective. Thus we have a 2-torsion class

$$\tau = A_D - g_2^1$$

on D, which corresponds to a genus 5 unramified double cover E of D, where

$$E = \operatorname{Proj} R(D, A_D, \tau) = \operatorname{Proj} \bigoplus_{n \geq 0} \left(H^0(D, nA_D) \oplus H^0(D, nA_D + \tau) \right).$$

We outline the construction of the bigraded ring $R(D, A_D, \tau)$. Using the notation of Section 2, write s_1, s_2 for the sections of the g_2^1 and

$$u \colon \mathcal{O}_D \to \mathcal{O}_D(P_1 + \dots + P_4), \quad v \colon \mathcal{O}_D \to \mathcal{O}_D(P_5 + \dots + P_8).$$

We can very quickly write down generators and relations for $R(D, A_D, \tau)$:

n	$H^0(D, nA_D)$	$H^0(D, nA_D + \tau)$
0	k	ϕ
1	ϕ	s_1, s_2
2	$s_1^2, s_1 s_2, s_2^2$	u, v
3	$\dots$	$\dots$

Thus E is a complete intersection

$$E_{4,4} \subset \mathbb{P}(1, 1, 2, 2),$$

defined by equations $u^2 = f_4(s_1, s_2)$ and $v^2 = g_4(s_1, s_2)$. The polynomials f and g are functions on $\mathbb{P}^1$ whose vanishing determines the splitting of the Weierstrass points of D into two sets of four.

The curve E comes bundled at no extra cost with the fixed point free involution $\tau \colon E \to E$ associated to the torsion τ of D. We recover the restricted algebra $R(D, K_Y|_D)$ of Section 2.1 by taking the τ-invariant subring of $R(D, A_D, \tau)$:

$$R(D, A_D) = R(D, A_D, \tau)^{\langle \tau \rangle}.$$

For future reference, we write out the action of τ on E using the above eigenspace table:

$$s_1 \mapsto -s_1, \quad s_2 \mapsto -s_2, \quad u \mapsto -u, \quad v \mapsto -v.$$

Now, we claim that E completely determines the Godeaux involution σ on D. First observe that D is a quotient of E, and that this cover only exists because D is the curve section of $|K_Y|$. Thus σ lifts to the curve E and should be compatible with the involution τ on E, so that $\sigma^2 = 1$ or τ on E.

Proposition 5.3. *The action of σ on E is given by*

$$s_1 \mapsto is_1, \quad s_2 \mapsto -is_2, \quad u \mapsto iv, \quad v \mapsto iu,$$

so that $\sigma^2 = \tau$ and the group $\langle \sigma, \tau \rangle$ is isomorphic to $\mathbb{Z}/4$. Moreover, σ has no fixed points on D, and interchanges $\{P_1, \dots, P_4\}$ with $\{P_5, \dots, P_8\}$.

Proof. The Hilbert series of Lemma 5.1 gives the eigenspace decomposition of σ on D, which we must abide by. In particular, $R(D, A_D)$ should have only one invariant generator in degree 2, and the generator in degree 4 should be anti-invariant. This forces $\sigma^2 = \tau$, so that the group $\langle \sigma, \tau \rangle$ acting on E is $\mathbb{Z}/4$ rather than $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Now, there are two possibilities for σ depending on whether A_D is an invariant divisor or only invariant as a divisor class. The correct choice is

$$s_1 \mapsto is_1, \quad s_2 \mapsto -is_2, \quad u \mapsto iv, \quad v \mapsto iu,$$

which corresponds to the second possibility. Indeed, the alternative

$$s_1 \mapsto is_1, \quad s_2 \mapsto -is_2, \quad u \mapsto iu, \quad v \mapsto iv,$$

is obliged to have two fixed points on D at the coordinate points P_{y_1} and P_{y_3}, and so can not possibly be the Godeaux involution. Hence the corresponding action on $R(D, A_D)$ is given by

$$\operatorname{rank} \left(\begin{array}{cc|cc} y_1 & y_2 & z_1 & z_3 \\ y_2 & y_3 & z_2 & z_4 \\ \hline z_1 & z_2 & f_2 & t \\ z_3 & z_4 & t & g_2 \end{array} \right) \leq 1 \quad \mapsto \quad \operatorname{rank} \left(\begin{array}{cc|cc} -y_1 & y_2 & -z_3 & -z_1 \\ y_2 & -y_3 & z_4 & z_2 \\ \hline -z_3 & z_4 & -g_2 & -t \\ -z_1 & z_2 & -t & -f_2 \end{array} \right) \leq 1,$$

where

$$\begin{aligned} f_2 &= \alpha_1 y_1^2 + \alpha_2 y_1 y_2 + \alpha_3 y_1 y_3 + \alpha_4 y_2^2 + \alpha_5 y_2 y_3 + \alpha_6 y_3^2 \\ g_2 &= -\alpha_1 y_1^2 + \alpha_2 y_1 y_2 - \alpha_3 y_1 y_3 - \alpha_4 y_2^2 + \alpha_5 y_2 y_3 - \alpha_6 y_3^2 \end{aligned}$$

and the involution has no fixed points as long as α_1 and α_6 are not zero. This proves the proposition. □

5.3. Involution on the K3 surface

Moving one step up the tower, we lift the involution σ on the canonical curve section D to the hyperelliptic K3 surface $T \subset \mathbb{P}(2^4, 3^4, 4)$, which contains D as a quadric section. The whole argument becomes quite transparent when viewed in terms of commutative algebra. The graded ring $R(T, A_T)$ is described explicitly in Section 3.1, and after eliminating one of the generators in degree 2, we obtain $R(D, A_D)$.

Proposition 5.4. *Let D be the unramified double cover of a Godeaux curve C with its involution $\sigma\colon D \to D$ from Proposition 5.3. Then there is at least one K3 surface T containing the curve D such that the involution σ on D has a unique lift to T. Moreover, such a lift $\sigma\colon T \to T$ has four fixed points which are $\frac{1}{2}(1,1)$ points of T. We call σ the Godeaux involution on T.*

Remark 5.5. This is surprising because the involution is assumed to be fixed point free on Y, so one might expect that the involution on the K3 surface T is also free.

Proof. Step (1) *Determining the character of σ.* Temporarily choose coordinates on T so that $D = T \cap (y_4 = 0)$, where y_4 must be semi-invariant under any putative involution. Then the determinantal equations (1), which partially define T, take the general form

$$\operatorname{rank} \begin{pmatrix} y_1 + \alpha y_4 & y_2 + \beta y_4 & z_1 \\ y_2 + \gamma y_4 & y_3 + \delta y_4 & z_2 \\ z_3 & z_4 & t \end{pmatrix} \leq 1,$$

where $\alpha, \beta, \gamma, \delta$ are scalars. Now if σ lifts to T, then our choice of coordinates means that the action of σ on T is predetermined by $\sigma|_D$ of Proposition 5.3, excepting the new variable y_4. Since T is a double cover of a quadric $Q \subset \mathbb{P}^3$ of rank 4, the determinantal equations imply y_4 is anti-invariant, and the signature of σ on Q is $(1,3)$. We now recalibrate the coordinate system so that the determinantal equations and involution on T are

$$\operatorname{rank} \left(\begin{array}{cc|c} y_1 & y_2 & z_1 \\ y_3 & y_4 & z_2 \\ \hline z_3 & z_4 & t \end{array}\right) \leq 1 \quad \mapsto \quad \operatorname{rank} \left(\begin{array}{cc|c} -y_1 & y_3 & -z_3 \\ y_2 & -y_4 & z_4 \\ \hline -z_1 & z_2 & -t \end{array}\right) \leq 1,$$

where the original curve D is obtained from T by taking the anti-invariant quadric section $y_2 = y_3$.

Step (2) *Fixed points of σ.* The involution on T swaps the two branch curves, and also swaps the sheets of the hyperelliptic double cover $\pi\colon T \to Q$. Thus any fixed points of σ lie on both components of the branch curve, and so must be $\frac{1}{2}$ points of T.

For a $\frac{1}{2}$ point of T to be fixed under σ, one of two things must happen:

$$y_1 = y_4 = y_2 - y_3 = 0, \text{ or } y_2 + y_3 = 0.$$

The only case we need to worry about is when $y_2 + y_3 = 0$ since the other case reduces to considerations on the curve D, on which σ is fixed point free by hypothesis. Now recall that $Q = \mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $y_1 = s_1t_1$, $y_2 = s_2t_1$, $y_3 = s_1t_2$, $y_4 = s_2t_2$ and to ensure C_1 and C_2 are interchanged under σ, their equations must be of the form

$$\begin{aligned} f_{3,1} &= \alpha_1 s_1^3 t_1 + \alpha_2 s_1^2 s_2 t_1 + \alpha_3 s_1 s_2^2 t_1 + \alpha_4 s_2^3 t_1 \\ &\quad + \beta_1 s_1^3 t_2 + \beta_2 s_1^2 s_2 t_2 + \beta_3 s_1 s_2^2 t_2 + \beta_4 s_2^3 t_2, \\ g_{1,3} &= -\alpha_1 s_1 t_1^3 + \alpha_2 s_1 t_1^2 t_2 - \alpha_3 s_1 t_1 t_2^2 + \alpha_4 s_1 t_2^3 \\ &\quad + \beta_1 s_2 t_1^3 - \beta_2 s_2 t_1^2 t_2 + \beta_3 s_2 t_1 t_2^2 - \beta_4 s_2 t_2^3. \end{aligned} \tag{21}$$

Note that there is more than one possible choice of $f_{3,1}$, $g_{1,3}$ for which $T \cap (y_2 = y_3) = D$, so we can not claim that T is unique in the statement of the proposition. Since P_{y_1}, P_{y_4} are not contained in T we may assume $y_2 = 1$, $y_3 = -1$, and then for a general choice of branch curve there are four fixed points on T. These are $(\lambda, 1, -1, -1/\lambda, 0, 0, 0, 0, 0)$, where λ is a root of the quartic equation derived from evaluating equations (21)

$$\alpha_1\lambda^4 + (\alpha_2 - \beta_1)\lambda^3 + (\alpha_3 - \beta_2)\lambda^2 + (\alpha_4 - \beta_3)\lambda - \beta_4,$$

which proves the proposition. □

5.4. Involution on the Fano 6-fold

We extend the involution on the K3 surface T to the Fano 6-fold W constructed in the Main Theorem. To do this we use a $\mathbb{Z}/2$-equivariant form of the projection–unprojection construction described in Section 3.2.

We begin with a $\mathbb{Z}/2$-equivariant unprojection construction for the K3 surface T with a Godeaux involution. Let P_{y_4} be one of the fixed points of the Godeaux involution from Proposition 5.4, and project from P to obtain

$$\mathbb{P}^1 \xrightarrow{\varphi} T'_{6,6} \subset \mathbb{P}(2, 2, 2, 3, 3).$$

There is an induced involution on T', which swaps the two branch cubics and leaves the image of φ invariant. We call this a $\mathbb{Z}/2$-*equivariant projection*.

Following Section 3.2, the determinantal equations for T are:

$$\operatorname{rank} \begin{pmatrix} y_2 & f & z_1 \\ g & y_4 & z_3 \\ z_2 & z_4 & t \end{pmatrix} \leq 1,$$

where here $f = y_1 + \alpha y_3$, $g = \alpha y_1 + y_3$ because the two branch curves are interchanged by σ. Proposition 5.4 fixes the involution on T as

$$\begin{gathered} f \mapsto g, \quad y_2 \mapsto -y_2, \quad g \mapsto f, \quad z_1 \mapsto -z_2, \quad z_2 \mapsto -z_1, \\ y_4 \mapsto -y_4, \quad z_3 \mapsto z_4, \quad z_4 \mapsto z_3, \quad t \mapsto -t \end{gathered} \tag{22}$$

and we note that this implies $\sigma(y_1) = y_3$, $\sigma(y_3) = y_1$. Hence referring to equations (2) of Section 3.2, $T'_{6,6} \subset \mathbb{P}(2, 2, 2, 3, 3)$ must be defined by equations

of the form

$$\begin{aligned} z_1^2 &= y_1 f^2 + y_2^2(l_1 f + l_2 y_2 + l_3 g) + l_4 y_2 f g \\ z_2^2 &= y_3 g^2 + y_2^2(l_3 f - l_2 y_2 + l_1 g) - l_4 y_2 f g, \end{aligned} \tag{23}$$

where the l_i are scalars. The remaining equations of T can be calculated from those of T' using the unprojection procedure outlined in Section 3.2.

There are three isolated fixed points on T' when $z_1 = z_2 = y_1 + y_3 = 0$, which correspond to three of the $9 \times \frac{1}{2}$ points as expected. Moreover, T' has two fixed points on the image of φ which arise because the centre of projection P was itself a fixed point. Indeed, suppose we have a local orbifold chart for the $\frac{1}{2}(1,1)$ point P in T. We choose coordinates u, v on $\mathbb{C}^2$ so that P is the $\mathbb{Z}/2$-quotient acting by -1 on both u and v. Then by equation (22), σ lifts to the chart as

$$u \mapsto -iv, \quad v \mapsto -iu.$$

The $(1,1)$ weighted blowup of T at P introduces the ratio $(u : v)$ as the exceptional $\mathbb{P}^1$, which is then mapped into T' by φ. Thus the induced action of σ on the image of φ inside T' has two fixed points at $\varphi(1,1)$ and $\varphi(-1,1)$.

Remark 5.6. The $\mathbb{Z}/2$-equivariant projection–unprojection construction for T relies on the choice of $\frac{1}{2}$ point P. As such we can no longer assume there is a canonical choice for the curve $D \subset T$ defined by setting $f = g$, as we did in the proof of Proposition 5.4. Instead D is defined by any anti-invariant quadric section of T which avoids the $10 \times \frac{1}{2}$ points. Indeed, the quadric $f = g$ contains the point P and so is no longer a valid choice.

Now, we claim that the involution on T can be extended to the Fano 6-fold W at the top of the tower.

Proposition 5.7. *Suppose $T \subset \mathbb{P}(2^4, 3^4, 4)$ is a K3 surface with $10 \times \frac{1}{2}$ points and $\sigma\colon T \to T$ is a Godeaux involution lifted from some quadric section $D \subset T$. Then there is a lift of σ to the unique Fano 6-fold $W \subset \mathbb{P}(1^4, 2^4, 3^4, 4)$ extending T which was constructed in the Main Theorem. Moreover the involution $\sigma\colon W \to W$ has fixed locus consisting of four isolated $\frac{1}{2}$ points.*

Proof. Project from one of the fixed $\frac{1}{2}$ points on T to get

$$\varphi\colon \mathbb{P}^1 \to T'_{6,6} \subset \mathbb{P}(2^3, 3^2).$$

Following the extension procedure outlined in the proof of the Main Theorem, the extended map

$$\Phi\colon \mathbb{P}^5 \to W'_{6,6} \subset \mathbb{P}(1^4, 2^3, 3^2)$$

must be

$$\Phi\colon (a, b, c, d, u, v) \mapsto (a, b, c, d, u^2 + 2av, bu + cv, v^2 + 2du, h_1, h_2),$$

where

$$h_1 = u\left(f + \alpha(a^2 + \alpha d^2)\right) + (1 - \alpha^2)auv + \alpha(\alpha^2 - 1)adv,$$

$$h_2 = v\left(g + \alpha(\alpha a^2 + d^2)\right) + (1-\alpha^2)duv + \alpha(\alpha^2-1)adu.$$

To make Φ compatible with the lift of $\sigma\colon T \to T$ defined by equation (22), the action on $\mathbb{P}^5$ must be

$$u \mapsto -v, \quad v \mapsto -u, \quad a \mapsto -d, \quad b \mapsto c, \quad c \mapsto b, \quad d \mapsto -a.$$

Thus Φ is σ-equivariant, so the equations defining the image of Φ are invariant and consequently $W' \subset \mathbb{P}(1^4, 2^3, 3^2)$ can be chosen to be invariant. Alternatively, a direct calculation following the proof of the Main Theorem demonstrates explicitly that the equations of the image of Φ are invariant. Hence by $\mathbb{Z}/2$-equivariant unprojection, the involution lifts to the 6-fold W.

Now outside the image of Φ, there are just three isolated points on W' that are fixed under σ. These are the same $\frac{1}{2}$ points that were fixed under $\sigma|_{T'}$. On the image of Φ itself there are two copies of $\mathbb{P}^2 \subset \mathbb{P}^5$ whose image under Φ are fixed by σ. These are defined by

$$\mathbb{P}^5 \cap (u = v, a = d, b = -c), \quad \mathbb{P}^5 \cap (u = -v, a = -d, b = c),$$

and they are the analogue of the two fixed points on $\varphi(\mathbb{P}^1) \subset T'$. These nonisolated fixed loci are contracted to the centre of projection P on W, so that σ fixes just four isolated $\frac{1}{2}$ points there. This proves the proposition. □

5.5. Godeaux surfaces with $\pi_1 = \mathbb{Z}/2$

Let D be a double cover of a Godeaux curve C as described in Section 5.2, and construct a hyperelliptic tower $D \subset T \subset W$, where W is the unique Fano 6-fold extending the K3 surface T. Now further suppose that the tower is constructed so that the Godeaux involution σ on D lifts to T and therefore W, as described in Propositions 5.4 and 5.7. Write A for the hyperplane class on W so that $\mathcal{O}_W(A) = \mathcal{O}_W(1)$, and $-K_W = 4A$. Then σ induces a $\mathbb{Z}\oplus\mathbb{Z}/2$-bigrading on the ring $R(W, A)$ according to eigenspace:

n	$H^0(W, nA)^+$	$H^0(W, nA)^-$
1	$a-d, b+c$	$a+d, b-c$
2	y_1+y_3	y_1-y_3, y_2, y_4
3	z_1-z_2, z_3+z_4	z_1+z_2, z_3-z_4
4		t

Now by Theorem 5.2, we obtain a topologically simply connected surface Y of general type with $p_g = 1$, $q = 0$, $K^2 = 2$ as a complete intersection inside W as long as Y avoids the $\frac{1}{2}$ points of W, which is an open condition. Referring to the above table and the eigenspace decomposition on Y given by Lemma 5.1, if we take Y to be a complete intersection of type $(1^+, 1^+, 1^-, 2^-)$ inside W then $\sigma|_Y$ will be the fixed point free Godeaux involution. Hence we have:

Theorem 5.8. *There is an* 8 *parameter family of Godeaux surfaces with* $\pi_1 = \mathbb{Z}/2$.

The parameter count is a matter of calculating the moduli of W using the Main Theorem, Section 5.4 and then counting the number of free parameters involved in choosing the complete intersection $(1^+, 1^+, 1^-, 2^-)$.

Remark 5.9. From Section 5.2 onwards, we assumed that D is nonsingular. A posteriori, we see that this assumption is not necessary. For example, we can choose the degree 2^- equation defining Y to be $y_4 = 0$. Then D is a double cover of the rank 2 conic $fg = 0$ in $\mathbb{P}^2$ and so D is singular. On the other hand, we used the computer to check that we can still choose a nonsingular surface Y containing D.

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