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ZERO BASED INVARIANT SUBSPACES AND FRINGE OPERATORS OVER THE BIDISK

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Abstract. Let M be an invariant subspace of H^2 over the bidisk. Associated with M, we have the fringe operator F_z^M on $M \ominus wM$. It is studied the Fredholmness of F_z^M for (generalized) zero based invariant subspaces M. Also $\ker F_z^M$ and $\ker (F_z^M)^*$ are described.

1. Introduction

Let $H^2 = H^2(\mathbb{D}^2)$ be the Hardy space over the bidisk $\mathbb{D}^2$ with two variables z, w. We write $\|f\|$ the Hardy space norm of $f \in H^2$. We denote by T_z, T_w the multiplication operators on H^2 by z, w. A nonzero closed subspace M of H^2 is said to be invariant if $T_z M \subset M$ and $T_w M \subset M$. The structure of invariant subspaces of H^2 is fairly complicated and at this moment it seems to be out of reach (see [1, 3, 6, 7]). We have

$$M = \bigoplus_{n=0}^{\infty} w^n (M \ominus wM),$$

so the space $M \ominus wM$ contains many informations of an invariant subspace M. In [7], Yang studied the operator F_z^M on $M \ominus wM$ defined by

$$F_z^M f = P_{M \ominus wM} T_z f, \quad f \in M \ominus wM,$$

where P_A is the orthogonal projection from H^2 onto $A \subset H^2$, and he called F_z^M the fringe operator of M.

Let $N = H^2 \ominus M$. We set

$$\Omega(M) = M \ominus (zM + wM) \quad \text{and} \quad \widetilde{\Omega}(N) = N \ominus (T_z^* N + T_w^* N).$$

We have $\Omega(M) \neq \{0\}$,

$$\Omega(M) = \{f \in M : T_z^* f \in N, T_w^* f \in N\} \tag{1.1}$$

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and

$$\widetilde{\Omega}(N) = \{f \in N : T_z f \in M, T_w f \in M\}. \tag{1.2}$$

It is known that $\widetilde{\Omega}(N)$ may be an empty set. Generally, we do not know whether $zM + wM$ is closed or not. In [7], Yang pointed out that $zM + wM$ is closed if and only if F_z^M has closed range. Let $H^\infty = H^\infty(\mathbb{D}^2)$ be the space of bounded analytic functions on $\mathbb{D}^2$ with the supremum norm $\|\cdot\|_\infty$. In [7], Yang also showed that if there is $h \in M \cap H^\infty$ satisfying $h(0,0) \neq 0$, then $zM + wM$ is closed and $\Omega(M) = \mathbb{C} \cdot P_M 1$. A bounded linear operator T on a separable Hilbert space is called Fredholm if T has closed range, $\dim \ker T < \infty$ and $\dim \ker T^* < \infty$ (see [2]). In this case, $\operatorname{ind} T = \dim \ker T - \dim \ker T^*$ is called the Fredholm index of T. The Fredholmness is one of the important subjects in operator theory. In [7], Yang pointed out that

$$\ker F_z^M = w\widetilde{\Omega}(N) \quad \text{and} \quad \ker (F_z^M)^* = \Omega(M).$$

Hence if F_z^M is Fredholm, then $\operatorname{ind} F_z^M = \dim \widetilde{\Omega}(N) - \dim \Omega(M)$.

We shall study the following questions in this paper.

- (Q1) How to prove the closedness of $zM + wM$?
- (Q2) How to describe the elements in $\Omega(M)$?
- (Q3) How to describe the elements in $\widetilde{\Omega}(N)$?

It is difficult to answer these questions completely. In this paper, we study these questions for the zero based invariant subspaces of H^2. Let E be a nonvoid subset $\mathbb{D}^2$ and

$$I(E) = \{f \in H^2 : f = 0 \text{ on } E\}.$$

Then $I(E)$ is an invariant subspace and $I(E)$ is called a zero based invariant subspace for E. We may assume that $I(E) \neq \{0\}$ and

$$E = Z(I(E)) := \big\{\lambda \in \mathbb{D}^2 : f(\lambda) = 0 \text{ for every } f \in I(E)\big\}.$$

In Section 2, we shall study the above questions for $I(E)$. We shall answer (Q3) for $M = I(E)$.

Let M be an invariant subspace of H^2 with $M \subset I(E)$ and $Z(M) = E$. We write $\mathbb{N} = \{0, 1, 2, \ldots\}$ and

$$D_z^n D_w^m = \frac{\partial^n}{\partial z^n} \frac{\partial^m}{\partial w^m}, \quad (n,m) \in \mathbb{N}^2,$$

where $D_z^0 D_w^m = D_w^m$, $D_z^n D_w^0 = D_z^n$ and $D_z^0 D_w^0 = 1$. For each $\lambda \in E$, let

$$A_M(\lambda) = \big\{(n,m) \in \mathbb{N}^2 : (D_z^n D_w^m f)(\lambda) = 0 \text{ for every } f \in M\big\}.$$

Since $Z(M) = E$, $(0,0) \in A_M(\lambda) \subsetneqq \mathbb{N}^2$ for every $\lambda \in E$. We have

$$I(E) = \bigcap_{\lambda \in E} \big\{f \in H^2 : (D_z^n D_w^m f)(\lambda) = 0 \text{ for every } (n,m) \in A_{I(E)}(\lambda)\big\}.$$

Let

$$\widetilde{M} = \bigcap_{\lambda \in E} \big\{ f \in H^2 : (D_z^n D_w^m f)(\lambda) = 0 \text{ for every } (n,m) \in A_M(\lambda) \big\}.$$

Then $\widetilde{M}$ is an invariant subspace. Since $A_{I(E)}(\lambda) \subset A_M(\lambda)$ for every $\lambda \in E$, we have that $M \subset \widetilde{M} \subset I(E)$ and $E \subset Z(\widetilde{M}) \subset Z(M) = E$. Hence $Z(\widetilde{M}) = E$. Since $I(E) = \widetilde{I}(E)$, as a generalization of a zero based invariant subspace $I(E)$ we assume that $M = \widetilde{M}$.

Let

$$M_0 = \bigcap_{\lambda \in E \setminus \{(0,0)\}} \big\{ f \in H^2 : (D_z^n D_w^m f)(\lambda) = 0 \text{ for every } (n,m) \in A_M(\lambda) \big\}.$$

Then M_0 is an invariant subspace, $M = \widetilde{M} \subset M_0$, and if $(0,0) \notin E$, then $\widetilde{M} = M_0$. In this paper, M_0 plays an important role. In Section 3, we shall study questions (Q1), (Q2) and (Q3).

In Section 4, we shall study the special cases. Let $\Lambda = \{(a,a) : a \in \mathbb{D}\}$. Then $I(\Lambda) = [z-w]$, where $[L]$ is the smallest invariant subspace containing $L \subset H^2$. Let M be an invariant subspace satisfying that $M \subsetneqq [z-w]$, $Z(M) = \Lambda$, $M = \widetilde{M}$ and $M_0 = [z-w]$. We shall show that F_z^M is Fredholm and $\operatorname{ind} F_z^M = -1$. We shall also describe $\widetilde{\Omega}(N)$ and $\Omega(M)$ completely.

We have a conjecture that if $\dim \Omega(M) < \infty$, then F_z^M is Fredholm and $\operatorname{ind} F_z^M = -1$. Our results in this paper support that this conjecture is true (see [4, 5, 7, 8, 9, 10, 11]).

2. Zero based invariant subspaces

Let M be an invariant subspace of H^2 and $N = H^2 \ominus M$. In [7], Yang pointed out the following facts.

Lemma 2.1. $\ker F_z^M = w\widetilde{\Omega}(N)$ *and* $\ker (F_z^M)^* = \Omega(M)$.

Lemma 2.2. *$zM + wM$ is closed if and only if F_z^M has closed range.*

Lemma 2.3. *If there is $h \in M \cap H^\infty$ satisfying $h(0,0) \neq 0$, then $zM + wM$ is closed and $\Omega(M) = \mathbb{C} \cdot P_M 1$.*

Actually he showed that $zM + wM = M \cap (zH^2 + wH^2)$ under the assumption in Lemma 2.3. Using the same idea, we have the following.

Proposition 2.4. *If there is $h \in M \cap H^\infty$ satisfying $h(0,0) \neq 0$, then F_z^M is Fredholm and* $\operatorname{ind} F_z^M = -1$.

Proof. We shall show $\widetilde{\Omega}(N) = \{0\}$. We may assume that $h(0,0) = 1$ and write $h = 1 + zh_1(z) + wh_2$ for some $h_1(z), h_2 \in H^\infty$. Let $f \in \widetilde{\Omega}(N)$. We have

$$f = f(h - zh_1(z) - wh_2) = fh - zfh_1(z) - wfh_2.$$

By (1.2), $zf \in M$ and $wf \in M$. So $zfh_1(z) + wfh_2 \in M$. Since $h \in M \cap H^\infty$, we have $fh \in M$, so by the above we have $f \in M$. Since $f \perp M$, we have $f = 0$. Thus $\widetilde{\Omega}(N) = \{0\}$. By Lemmas 2.1–2.3, we get the assertion. □

The following is a well known fact.

Lemma 2.5. *Let M be an invariant subspace of H^2. Then $\Omega(M) \neq \{0\}$. Moreover* $\dim \Omega([f]) = 1$ *for every nonzero f in H^2.*

Let E be a nonvoid subset of $\mathbb{D}^2$. We assume that

$$I(E) \neq \{0\} \quad \text{and} \quad Z(I(E)) = E.$$

We write

$$N(E) = H^2 \ominus I(E).$$

Lemma 2.6. *Suppose that $(0,0) \notin E$. Then $\widetilde{\Omega}(N(E)) = \{0\}$.*

Proof. Let $f \in \widetilde{\Omega}(N(E))$. By (1.2), $(az + bw)f \in I(E)$ for every $a, b \in \mathbb{C}$. Since $(0,0) \notin E$, we have $f = 0$ on E, so $f \in I(E)$. Since $f \perp I(E)$, we get $f = 0$. □

Similarly, we have the following.

Lemma 2.7. *Suppose that $(0,0) \in E$ and $E \neq \{(0,0)\}$. If $I(E)$ contains all $f \in H^2$ satisfying $f = 0$ on $E \setminus \{(0,0)\}$, then $\widetilde{\Omega}(N(E)) = \{0\}$.*

Proof. Let $f \in \widetilde{\Omega}(N(E))$. By (1.2), $(az+bw)f \in I(E)$ for every $a, b \in \mathbb{C}$. Then $f = 0$ on $E \setminus \{(0,0)\}$. By the assumption, we have $f \in I(E)$. Since $f \perp I(E)$, we get $f = 0$. □

Proposition 2.8. *Suppose that $(0,0) \in E$ and $E \neq \{(0,0)\}$. If there is $f \in H^2$ such that $f = 0$ on $E \setminus \{(0,0)\}$ and $f(0,0) \neq 0$, then*

$$\widetilde{\Omega}(N(E)) = \mathbb{C} \cdot (f - P_{I(E)}f) \neq \{0\}.$$

Proof. Since $f \notin I(E)$, $f - P_{I(E)}f \neq 0$ and $f - P_{I(E)}f \in N(E)$. Since $f = 0$ on $E \setminus \{(0,0)\}$, we have

$$z(f - P_{I(E)}f), \;\; w(f - P_{I(E)}f) \in I(E).$$

By (1.2), $f - P_{I(E)}f \in \widetilde{\Omega}(N(E))$.

We may assume that $f(0,0) = 1$. Let $g \in \widetilde{\Omega}(N(E))$ and $g \neq 0$. As the proof of Lemma 2.7, $g = 0$ on $E \setminus \{(0,0)\}$ and $g(0,0) \neq 0$. We may assume that $g(0,0) = 1$. Hence $(f - P_{I(E)}f) - g \in I(E)$. Since $(f - P_{I(E)}f) - g \in \widetilde{\Omega}(N(E))$, we get $g = f - P_{I(E)}f$. □

Example 2.9. Let $\alpha \in \mathbb{D}$ with $\alpha \neq 0$ and

$$E = \{(0,0), (0,\alpha), (\alpha,0), (\alpha,\alpha)\}.$$

We write $b_\alpha(z) = (z - \alpha)/(1 - \overline{\alpha}z)$. One may checks that $I(E) = zb_\alpha(z)H^2 + wb_\alpha(w)H^2$. Let $f = b_\alpha(z)b_\alpha(w)$. Then $f(0,\alpha) = f(\alpha,0) = f(\alpha,\alpha) = 0$ and

$f(0,0) = \alpha^2 \neq 0$, so by Proposition 2.8 $\dim \widetilde{\Omega}(N(E)) = 1$. We have $f \perp I(E)$ and $\widetilde{\Omega}(N(E)) = \mathbb{C} \cdot f$. □

In the same way as the one by Yang [7], we may prove the following.

Theorem 2.10. *Suppose that* $(0,0) \in E$ *and* $E \neq \{(0,0)\}$*. If there is* $h \in H^\infty$ *satisfying* $h = 0$ *on* $E \setminus \{(0,0)\}$ *and* $h(0,0) \neq 0$*, then* $zI(E) + wI(E)$ *is closed and* $\Omega(I(E)) = \mathbb{C} \cdot P_{I(E)}z + \mathbb{C} \cdot P_{I(E)}w$*. Moreover* $F_z^{I(E)}$ *is Fredholm and* $\operatorname{ind} F_z^{I(E)} = -1$.

Proof. We may assume that $h(0,0) = 1$. Then there are $h_1(z)$ and h_2 in H^∞ such that $h = 1 + zh_1(z) + wh_2$. We write

$$H_0 = \{f \in H^2 : f \perp 1, f \perp z, f \perp w\}.$$

We shall show that

$$zI(E) + wI(E) = I(E) \cap H_0. \tag{2.1}$$

Let $f \in I(E) \cap H_0$. We have

$$f = fh - zfh_1(z) - wfh_2.$$

Since $f \in I(E)$, we have $zfh_1(z) + wfh_2 \in zI(E) + wI(E)$. Since $H_0 = z^2H^2 + zwH^2 + w^2H^2$, we may write $f = z^2f_1 + zwf_2 + w^2f_3$ for some $f_1, f_2, f_3 \in H^2$. Since $h = 0$ on $E \setminus \{(0,0)\}$, we have that $zf_1h, wf_2h, wf_3h \in I(E)$. Hence

$$fh = z(zf_1h + wf_2h) + w(wf_3h) \in zI(E) + wI(E),$$

so $f \in zI(E) + wI(E)$. Thus we get $I(E) \cap H_0 \subset zI(E) + wI(E)$.

Let $g \in zI(E) + wI(E)$. Then $g = zg_1 + wg_2$ for some $g_1, g_2 \in I(E)$. Since $(0,0) \in E$, $I(E) \subset zH^2 + wH^2$. Hence for each $i = 1, 2$, $g_i = zg_{i,1} + wg_{i,2}$ for some $g_{i,1}, g_{i,2} \in H^2$. We have

$$g = z^2 g_{1,1} + zw(g_{1,2} + g_{2,1}) + w^2 g_{2,2} \in H_0.$$

Thus $zI(E) + wI(E) \subset I(E) \cap H_0$, so we get (2.1). Since H_0 is closed, $zI(E) + wI(E)$ is closed.

Since $zh, wh \in I(E)$ and $h(0,0) = 1$, we have $P_{I(E)}z \neq 0$ and $P_{I(E)}w \neq 0$. Let $g \in I(E) \ominus (\mathbb{C} \cdot P_{I(E)}z + \mathbb{C} \cdot P_{I(E)}w)$. Then $g \perp 1, g \perp z$ and $g \perp w$. Hence $g \in H_0$, so $g \in I(E) \cap H_0$. Thus by (2.1),

$$I(E) \ominus (\mathbb{C} \cdot P_{I(E)}z + \mathbb{C} \cdot P_{I(E)}w) \subset zI(E) + wI(E).$$

Since $P_{I(E)}z, P_{I(E)}w \perp zI(E) + wI(E)$, we have

$$I(E) = (zI(E) + wI(E)) \oplus (\mathbb{C} \cdot P_{I(E)}z + \mathbb{C} \cdot P_{I(E)}w).$$

Hence

$$\Omega(I(E)) = \mathbb{C} \cdot P_{I(E)}z + \mathbb{C} \cdot P_{I(E)}w.$$

Since $P_{I(E)}z \perp wh$ and $P_{I(E)}w \not\perp wh$, we have $\mathbb{C} \cdot P_{I(E)}z \neq \mathbb{C} \cdot P_{I(E)}w$. Hence $\dim \Omega(I(E)) = 2$.

By Lemmas 2.1, 2.2 and Proposition 2.8, we conclude the assertion. □

Let $\Lambda = \{(a,a) : a \in \mathbb{D}\}$. Then $I(\Lambda) = [z-w]$. It is known that $F_z^{[z-w]}$ is Fredholm and $\operatorname{ind} F_z^{[z-w]} = -1$ (see [7]). The following is a generalization of this fact.

Theorem 2.11. *Let $\varphi(z)$ be an inner function with $\varphi(0) = 0$ and $g \in H^\infty$ with $g \neq 0$. Then $F_z^{[\varphi(z)-wg]}$ is Fredholm and* $\operatorname{ind} F_z^{[\varphi(z)-wg]} = -1$.

Proof. Put $M = [\varphi(z) - wg]$. We shall show that

$$zM + wM = M \cap (z\varphi(z)H^2 + wH^2). \tag{2.2}$$

Since $M \subset \varphi(z)H^2 + wH^2$, we have

$$zM + wM \subset M \cap (z\varphi(z)H^2 + wH^2).$$

Let $f \in M \cap (z\varphi(z)H^2 + wH^2)$. We may write $f = z\varphi(z)f_1 + wf_2$ for some $f_1, f_2 \in H^2$. Put $h = \varphi(z) - wg$. Then $M = [h]$ and

$$f = z(h + wg)f_1 + wf_2 = zhf_1 + w(zgf_1 + f_2). \tag{2.3}$$

Since $h \in M \cap H^\infty$, we have $hf_1 \in M$. Hence $zhf_1 \in zM$ and

$$w(zgf_1 + f_2) = f - zhf_1 \in M,$$

so there is a sequence of polynomials $\{p_n\}_n$ such that

$$(\varphi(z) - wg)p_n = hp_n \to w(zgf_1 + f_2)$$

in H^2 as $n \to \infty$. Putting $w = 0$, we have $\|\varphi(z)p_n(z,0)\| \to 0$, so $\|p_n(z,0)\| \to 0$. Hence

$$\begin{aligned} &\big\|h(p_n - p_n(z,0)) - w(zgf_1 + f_2)\big\| \\ &\le \|hp_n - w(zgf_1 + f_2)\| + \|h\|_\infty \|p_n(z,0)\| \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Since $p_n - p_n(z,0) = wq_n$ for some polynomial q_n, we have

$$h(p_n - p_n(z,0)) = whq_n \in w[h] = wM.$$

Hence $w(zgf_1 + f_2) \in wM$. Therefore by (2.3), $f \in zM + wM$. Thus we get (2.2).

Since $z\varphi(z)H^2 + wH^2$ is closed, by (2.2) $zM + wM$ is closed. By Lemma 2.2, F_z^M has closed range. Let $f \in \widetilde{\Omega}(N)$. Then $wf \in M$. Similarly as the last paragraph, we have $wf \in wM$, so $f \in M$. Hence $f = 0$. By Lemma 2.1, we have $\ker F_z^M = \{0\}$. By Lemma 2.5, we have $\dim \Omega(M) = 1$, so by Lemma 2.1 we have $\dim \ker (F_z^M)^* = 1$. Thus we get the assertion. □

Corollary 2.12. *Let $h \in H^\infty$ satisfy $|h(e^{i\theta}, 0)| > \delta > 0$ for almost every $e^{i\theta} \in \partial\mathbb{D}$. Then $F_z^{[h]}$ is Fredholm and* $\operatorname{ind} F_z^{[h]} = -1$.

Proof. We may write $h = h_1(z) + wh_2$ for some $h_1(z), h_2 \in H^\infty$. If $h_1(0) \neq 0$, then by Proposition 2.4 we have the assertion. So we assume that $h_1(0) = 0$. Let $h_1(z) = \varphi(z)f(z)$ be an inner-outer factorization of $h_1(z)$. We have $\varphi(0) = 0$. By the assumption, $f(z)$ is invertible in H^∞. Then we have

$$[h] = [f(z)(\varphi(z) + wf^{-1}(z)h_2)] = [\varphi(z) + wf^{-1}(z)h_2].$$

If $h_2 = 0$, then $[h] = \varphi(z)H^2$, so we get the assertion. If $h_2 \neq 0$, then by Theorem 2.11 we get the assertion. □

Example 2.13. By Theorem 2.11, for the following M we have that F_z^M is Fredholm and $\operatorname{ind} F_z^M = -1$;

$$M = [z - w], \quad M = [(z - w)^2], \quad M = [z^2 - w^3].$$

3. Generalizations

Let M be an invariant subspace of H^2 satisfying that $M \subset I(E)$ and $Z(M) = E$. We have $A_{I(E)}(\lambda) \subset A_M(\lambda)$ for every $\lambda \in E$,

$$T_z^*\big\{0, z^n w^m : (n,m) \in A_M(\lambda)\big\} \subset \big\{0, z^n w^m : (n,m) \in A_M(\lambda)\big\} \tag{3.1}$$

and

$$T_w^*\big\{0, z^n w^m : (n,m) \in A_M(\lambda)\big\} \subset \big\{0, z^n w^m : (n,m) \in A_M(\lambda)\big\}. \tag{3.2}$$

We recall that

$$\widetilde{M} = \bigcap_{\lambda \in E} \big\{f \in H^2 : (D_z^n D_w^m f)(\lambda) = 0 \text{ for every } (n,m) \in A_M(\lambda)\big\}. \tag{3.3}$$

Then $M \subset \widetilde{M} \subset I(E)$ and $E \subset Z(\widetilde{M}) \subset Z(M) = E$. Hence $Z(\widetilde{M}) = E$. Since $I(E) = \widetilde{I}(E)$, as a generalization of zero based invariant subspaces we assume that

$$M = \widetilde{M}. \tag{3.4}$$

Put $N = H^2 \ominus M$. We shall study about $\widetilde{\Omega}(N)$, $\Omega(M)$ and the Fredholmness of F_z^M under the above situation.

Lemma 3.1. *If* $(0,0) \notin E$, *then* $\widetilde{\Omega}(N) = \{0\}$.

Proof. Let $f \in \widetilde{\Omega}(N)$. By (1.2), $(az + bw)f \in M$ for every $a, b \in \mathbb{C}$. Since $(0,0) \notin E$, $(D_z^n D_w^m f)(\lambda) = 0$ for every $\lambda \in E$ and $(n,m) \in A_M(\lambda)$. By (3.3) and (3.4), we have $f \in M$. Since $M \perp \widetilde{\Omega}(N)$, we have $f = 0$. □

Lemma 3.2. *Suppose that* $M \subset z^n w^m H^2$ *for some* $(n,m) \in \mathbb{N}^2$ *with* $(n,m) \neq (0,0)$. *If* $f \in \widetilde{\Omega}(N)$, *then* $f \in z^n w^m H^2$.

Proof. Let $f \in \widetilde{\Omega}(N)$. Suppose that $f \notin z^n w^m H^2$. Then we may write $f = f_1 \oplus f_2$ for some $f_1 \in z^n w^m H^2$ and $f_2 \in H^2 \ominus z^n w^m H^2$. Since $f_2 \neq 0$, either $zf \notin z^n w^m H^2$ or $wf \notin z^n w^m H^2$. So either $zf \notin M$ or $wf \notin M$. By (1.2), $f \notin \widetilde{\Omega}(N)$. This is a contradiction. Thus we get $f \in z^n w^m H^2$. □

Corollary 3.3. *Suppose that* $M \subset z^n w^m H^2$ *for some* $(n,m) \in \mathbb{N}^2$ *with* $(n,m) \neq (0,0)$. *Let* $N_1 = H^2 \ominus \overline{z}^n \overline{w}^m M$. *Then* $\widetilde{\Omega}(N) = z^n w^m \widetilde{\Omega}(N_1)$.

By Corollary 3.3, to study $\widetilde{\Omega}(N)$ we may assume that $M \not\subset zH^2$ and $M \not\subset wH^2$.

Lemma 3.4. *Suppose that* $(0,0) \in E$, $M \not\subset zH^2$ *and* $M \not\subset wH^2$. *Then there are* $n_1, n_2, \dots, n_k, m_1, m_2, \dots, m_k \in \mathbb{N}$ *such that* $0 \le n_1 < n_2 < \cdots < n_k$, $0 \le m_k < m_{k-1} < \cdots < m_1$ *and*

$$A_M(0,0) = \bigcup_{j=1}^{k} \big\{(n,m) \in \mathbb{N}^2 : 0 \le n \le n_j, 0 \le m \le m_j\big\}.$$

Proof. Since $M \not\subset zH^2$ and $M \not\subset wH^2$, $(n,0) \notin A_M(0,0)$ and $(0,m) \notin A_M(0,0)$ for some $n, m \in \mathbb{N}$. By (3.1) and (3.2), we get the assertion. □

Suppose that $(0,0) \in E$ and $E \neq \{(0,0)\}$. Let

$$M_0 = \bigcap_{\lambda \in E \setminus \{(0,0)\}} \big\{f \in H^2 : (D_z^n D_w^m f)(\lambda) = 0 \text{ for every } (n,m) \in A_M(\lambda)\big\}.$$

Then by (3.3) and (3.4), we have $M \subset M_0$.

Lemma 3.5. *Suppose that* $(0,0) \in E$ *and* $E \neq \{(0,0)\}$. *If* $M = M_0$, *then* $\widetilde{\Omega}(N) = \{0\}$.

Proof. Let $g \in \widetilde{\Omega}(N)$. Then $(az + bw)g \in M$ for every $a, b \in \mathbb{C}$, so $g \in M_0$. By the assumption, we have $g \in M$. Thus we get the assertion. □

We may rewrite $A_M(0,0)$ as follows;

$$A_M(0,0) = \big\{(n,m) \in \mathbb{N}^2 : z^n w^m \perp M\big\}. \tag{3.5}$$

Lemma 3.6. *Suppose that* $(0,0) \in E$, $E \neq \{(0,0)\}$, $M \not\subset zH^2$ *and* $M \not\subset wH^2$. *If* $M \neq M_0$, *then* $\widetilde{\Omega}(N) \neq \{0\}$.

Proof. Take $f_0 \in M_0 \ominus M$ with $f_0 \neq 0$. By (3.3) and (3.4), $(D_z^i D_w^j f_0)(0,0) \neq 0$ for some $(i,j) \in A_M(0,0)$. Here we use the notations given in Lemma 3.4. Since $z^i w^j \not\perp f_0$, there is $(s,t) \in \mathbb{N}^2$ such that $z^{n_\ell} w^{m_\ell} \not\perp z^s w^t f_0$ for some $1 \le \ell \le k$,

$$z^n w^m \perp z^{s+1} w^t f_0 \quad \text{and} \quad z^n w^m \perp z^s w^{t+1} f_0$$

for every $(n,m) \in A_M(0,0)$. By (3.3) and (3.4), we have $z^s w^t f_0 \notin M$ and $z^{s+1} w^t f_0, z^s w^{t+1} f_0 \in M$. Let $f_1 = z^s w^t f_0 - P_M z^s w^t f_0$. Then $f_1 \in N$ and $f_1 \neq 0$. Moreover we have $zf_1, wf_1 \in M$. By (1.2), we have $f_1 \in \widetilde{\Omega}(N)$. □

Proposition 3.7. *Suppose that* $(0,0) \in E$ *and* $E \neq \{(0,0)\}$. *Let* M *be an invariant subspace of* H^2 *such that* $M \subsetneqq I(E)$, $Z(M) = E$ *and* $M = \widetilde{M}$. *Moreover we assume that* $M \not\subset zH^2$ *and* $M \not\subset wH^2$. *Then* $\widetilde{\Omega}(N) \neq \{0\}$ *if and only if* $M \subsetneqq M_0$.

Proof. The necessity follows from Lemma 3.5. The reverse implication follows from Lemma 3.6. □

Under the condition $M \subsetneqq M_0$, we shall study about $\dim \widetilde{\Omega}(N)$.

Theorem 3.8. *Suppose that* $(0,0) \in E$ *and* $E \neq \{(0,0)\}$. *Let* M *be an invariant subspace of* H^2 *such that* $M \subsetneqq I(E)$, $Z(M) = E$, $M \subsetneqq M_0$ *and* $M = \widetilde{M}$. *Moreover we assume that* $M \not\subset zH^2$ *and* $M \not\subset wH^2$. *Let* $n_1, n_2, \ldots, n_k, m_1, m_2, \ldots, m_k \in \mathbb{N}$ *satisfy the conditions given in Lemma 3.4. Let*

$$\Sigma = \big\{(n_j, m_j) : 1 \leq j \leq k\big\} \subset A_M(0,0)$$

and

$$M_1 = \big\{f \in M_0 : f \perp z^n w^m \text{ for every } (n,m) \in A_M(0,0) \setminus \Sigma\big\}.$$

Then $\widetilde{\Omega}(N) = M_1 \ominus M$ *and* $1 \leq \dim \widetilde{\Omega}(N) \leq k$.

Proof. Since $M \subsetneqq M_0$, there is $f \in M_0 \ominus M$ with $f \neq 0$. Since $M = \widetilde{M}$, $f \not\perp z^i w^j$ for some $(i,j) \in A_M(0,0)$. By considering $z^s w^t f$ for $(s,t) \in \mathbb{N}^2$, we have $M \subsetneqq M_1 \subset M_0$.

Let $h \in \widetilde{\Omega}(N)$. Then $zh, wh \in M$. Since $M = \widetilde{M}$, we have $h \in M_0$. For any $(n,m) \in A_M(0,0) \setminus \Sigma$, either $(n+1,m) \in A_M(0,0)$ or $(n,m+1) \in A_M(0,0)$. If $(n+1,m) \in A_M(0,0)$, then $0 = \langle zh, z^{n+1}w^m \rangle = \langle h, z^n w^m \rangle$. If $(n,m+1) \in A_M(0,0)$, then $0 = \langle wh, z^n w^{m+1} \rangle = \langle h, z^n w^m \rangle$. Hence $h \in M_1$. Thus we get $\widetilde{\Omega}(N) \subset M_1 \ominus M$.

Let $f \in M_1 \ominus M$ and $(n,m) \in A_M(0,0)$. Then $f \in M_0$ and $\langle zf, z^n w^m \rangle = \langle f, z^{n-1} w^m \rangle = 0$. Hence $zf \in \widetilde{M} = M$. Similarly $wf \in M$. Hence $M_1 \ominus M \subset \widetilde{\Omega}(N)$. Thus we get the assertion. □

Theorem 3.9. *Suppose that* $(0,0) \in E$ *and* $E \neq \{(0,0)\}$. *Let* M *be an invariant subspace of* H^2 *such that* $M \subsetneqq I(E)$, $Z(M) = E$ *and* $M = \widetilde{M}$. *Moreover we assume that* $M \not\subset zH^2$ *and* $M \not\subset wH^2$. *Let* $n_1, n_2, \ldots, n_k, m_1, m_2, \ldots, m_k \in \mathbb{N}$ *satisfy the conditions given in Lemma 3.4. If* $(0,0) \notin Z(M_0)$, *then* $\dim \widetilde{\Omega}(N) = k$.

Proof. By the assumption, there is $f_0 \in M_0$ such that $f_0(0,0) = 1$. For each $1 \leq j \leq k$, we have $\langle z^{n_j} w^{m_j}, z^{n_j} w^{m_j} f_0 \rangle \neq 0$. By Lemma 3.4 and (3.5), we have $z^{n_j} w^{m_j} f_0 \notin M$. Let

$$f_j = z^{n_j} w^{m_j} f_0 - P_M(z^{n_j} w^{m_j} f_0).$$

Then $f_j \in N$ and $f_j \neq 0$. Since $M = \widetilde{M}$, it is not so difficult to show that $zf_j, wf_j \in M$ for every $1 \leq j \leq k$. Hence $f_j \in \widetilde{\Omega}(N)$ for every $1 \leq j \leq k$. Suppose that $\sum_{j=1}^k c_j f_j = 0$ for some $c_1, c_2, \ldots, c_k \in \mathbb{C}$. Since $(n_i, m_i) \in A_M(0,0)$ for every $1 \leq i \leq k$ and $f_0(0,0) = 1$, we have

$$0 = \Big\langle \sum_{j=1}^k c_j f_j, z^{n_i} w^{m_i} \Big\rangle = \Big\langle \sum_{j=1}^k c_j z^{n_j} w^{m_j} f_0, z^{n_i} w^{m_i} \Big\rangle$$

$$= c_i\langle z^{n_i}w^{m_i}f_0, z^{n_i}w^{m_i}\rangle = c_i.$$

Therefore $\dim \sum_{j=1}^k \mathbb{C}\cdot f_j = k$. By Theorem 3.8, we get $\dim \widetilde{\Omega}(N) = k$. □

We shall show an example satisfying conditions in Theorem 3.9.

Example 3.10. For $\alpha \in \mathbb{D}$, let $b_\alpha(z) = (z-\alpha)/(1-\overline{\alpha}z)$. For each $\ell \geq 1$, let

$$M = b_\alpha(z)b_\alpha(w)\sum_{j=0}^{\ell} z^{\ell-j}w^jH^2$$

and $E = Z(M)$. Then

$$E = (\{\alpha\}\times\mathbb{D})\cup(\mathbb{D}\times\{\alpha\})\cup\{(0,0)\},$$

$M \subsetneqq I(E)$, $M \not\subset zH^2$, $M \not\subset wH^2$ and $M = \widetilde{M}$. Moreover we have that $M_0 = b_\alpha(z)b_\alpha(w)H^2$, $Z(M_0) = (\{\alpha\}\times\mathbb{D})\cup(\mathbb{D}\times\{\alpha\})$ and

$$A_M(0,0) = \bigcup_{i=1}^{\ell}\{(i-1,0),(i-1,1),\ldots,(i-1,\ell-i)\}.$$

So in Lemma 3.4, we have

$$(n_1,m_1) = (0,\ell-1),(n_2,m_2) = (1,\ell-2),\ldots,(n_\ell,m_\ell) = (\ell-1,0)$$

and $k = \ell$. By Theorem 3.9, we have $\dim \widetilde{\Omega}(N) = \ell$. □

Example 3.11. Let $M = [z(z-w), w(z-w)]$. Then we have $M_0 = [z-w]$ and $Z(M) = Z(M_0) = \{(a,a) : a \in \mathbb{D}\}$, $\widetilde{M} = M$ and $M_0 \ominus M = \mathbb{C}\cdot(z-w)$. Hence $\widetilde{\Omega}(N) = \mathbb{C}\cdot(z-w)$ and $\dim\widetilde{\Omega}(N) = 1$. Moreover

$$A_M(0,0) = \{(0,0),(0,1),(1,0)\},$$

so in Lemma 3.4 we have $(n_1,m_1) = (0,1),(n_2,m_2) = (1,0)$ and $k = 2$. Hence $\dim\widetilde{\Omega}(N) = 1 < 2 = k$. □

In Theorem 3.8, we have $\dim\widetilde{\Omega}(N) \leq k$. In Example 3.11, we showed an example of M satisfying $\dim\widetilde{\Omega}(N) < k$. In Theorem 3.9, if $(0,0) \notin Z(M_0)$, then $\dim\widetilde{\Omega}(N) = k$. In the following, we shall show an example of M satisfying that $(0,0) \in Z(M_0)$ and $\dim\widetilde{\Omega}(N) = k$.

Example 3.12. Let

$$M = \big\{f \in [z-w] : f \perp z, z^2, w, zw, z^2w, w^2, w^3\big\}.$$

Then $M_0 = [z-w]$ and

$$A_M(0,0) = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(2,0),(2,1)\}.$$

Note that $(n_1,m_1) = (0,3),(n_2,m_2) = (2,1)$ and $k = 2$ in Lemma 3.4. Moreover

$$M = \big[z(z^2-w^2), z^3(z-w), z^2w(z-w), zw^2(z-w), w^3(z-w)\big]$$

and $\widetilde{M} = M$. In Theorem 3.8, we have $\Sigma = \{(0,3),(2,1)\}$ and

$$M_1 = \big[z^2(z-w), zw(z-w), w^2(z-w)\big].$$

We have

$$M_1 \ominus M = \mathbb{C} \cdot w(z^2 - w^2) \oplus \mathbb{C} \cdot (z^3 - z^2 w + zw^2 - w^3).$$

Then by Theorem 3.8, $\dim \widetilde{\Omega}(N) = 2 = k$. □

Suppose that $(0,0) \in E$ and $E \neq \{(0,0)\}$. Let M be an invariant subspace of H^2 such that $M \subsetneqq I(E)$, $Z(M) = E$ and $M = \widetilde{M}$. Moreover we assume that $M \not\subset zH^2$ and $M \not\subset wH^2$. To describe $\Omega(M)$, we set

$$B_M(0,0) = \mathbb{N}^2 \setminus A_M(0,0).$$

Let $n_1, n_2, \ldots, n_k, m_1, m_2, \ldots, m_k \in \mathbb{N}$ satisfy the conditions given in Lemma 3.4. Put

$$(s_1,t_1) = (0, m_1+1),\ \ (s_2,t_2) = (n_1+1, m_2+1),\ \ \ldots,$$

$$(s_k,t_k) = (n_{k-1}+1, m_k+1),\ \ (s_{k+1},t_{k+1}) = (n_k+1, 0).$$

Then $0 = s_1 < s_2 < \cdots < s_{k+1}$, $0 = t_{k+1} < t_k < \cdots < t_1$ and

$$B_M(0,0) = \bigcup_{j=1}^{k+1} \{(s_j+n, t_j+m) : (n,m) \in \mathbb{N}^2\}. \tag{3.6}$$

Let $1 \le \sigma_1 < \sigma_2 < \cdots < \sigma_q$ be the integers such that for each $1 \le i \le q$ there is $1 \le j \le k+1$ satisfying $s_j + t_j = \sigma_i$ and

$$\{(s_j,t_j) : 1 \le j \le k+1\} = \bigcup_{i=1}^{q} \{(s_j,t_j) : 1 \le j \le k+1, s_j + t_j = \sigma_i\}.$$

Set

$$\Gamma = \{(s_j,t_j) : 1 \le j \le k+1\}$$

and

$$\Gamma_i = \{(s_j,t_j) : 1 \le j \le k+1, s_j + t_j = \sigma_i\}. \tag{3.7}$$

Then $\sum_{i=1}^{q} \#\Gamma_i = \#\Gamma = k+1$, where $\#\Gamma$ denotes the number of elements in Γ.

Lemma 3.13. *$P_M z^{s_j} w^{t_j} \neq 0$ and $P_M z^{s_j} w^{t_j} \in \Omega(M)$ for every $1 \le j \le k+1$.*

Proof. Since $(s_j,t_j) \notin A_M(0,0)$, we have $z^{s_j} w^{t_j} \not\perp M$. Then $P_M z^{s_j} w^{t_j} \neq 0$,

$$z^{s_j} w^{t_j} = P_M z^{s_j} w^{t_j} \oplus (z^{s_j} w^{t_j} - P_M z^{s_j} w^{t_j})$$

and $z^{s_j} w^{t_j} - P_M z^{s_j} w^{t_j} \in N$. Since $T_z^* z^{s_j} w^{t_j}, T_w^* z^{s_j} w^{t_j} \in N$, by (1.1) we have $P_M z^{s_j} w^{t_j} \in \Omega(M)$. □

Corollary 3.14. $\dim \sum_{j=1}^{k+1} \mathbb{C} \cdot P_M z^{s_j} w^{t_j} \le \dim \Omega(M)$.

Example 3.15. Let

$$M = \big[z(z^3 + z^2w + zw^2 + w^3), w(z^3 + z^2w + zw^2 + w^3)\big].$$

Then $M = \widetilde{M}$, $M \not\subset zH^2$ and $M \not\subset wH^2$. We have

$$B_M(0,0) = \bigcup_{j=0}^{4} \big((4-j, j) + \mathbb{N}^2\big)$$

and $k = 4$. We also have

$$\begin{aligned}\sum_{j=0}^{4} \mathbb{C} \cdot P_M z^{4-j} w^j &= \mathbb{C} \cdot z(z^3 + z^2w + zw^2 + w^3) + \mathbb{C} \cdot w(z^3 + z^2w + zw^2 + w^3) \\ &= \Omega(M)\end{aligned}$$

and

$$\widetilde{\Omega}(N) = \mathbb{C} \cdot (z^3 + z^2w + zw^2 + w^3).$$

Theorem 3.16. *Suppose that $(0,0) \in E$ and $E \neq \{(0,0)\}$. Let M be an invariant subspace of H^2 such that $M \subsetneqq I(E)$, $Z(M) = E$ and $M = \widetilde{M}$. Moreover we assume that $M \not\subset zH^2$ and $M \not\subset wH^2$. If there is $h \in M_0 \cap H^\infty$ satisfying $h(0,0) \neq 0$, then F_z^M is Fredholm and $\operatorname{ind} F_z^M = -1$.*

Proof. First, we shall show that

$$zM + wM = M \cap \sum_{j=1}^{k+1} z^{s_j} w^{t_j} (zH^2 + wH^2). \tag{3.8}$$

Let $s_1, s_2, \ldots, s_{k+1}, t_1, t_2, \ldots, t_{k+1} \in \mathbb{N}$ satisfy the conditions given above Lemma 3.13. Since $M \subset \sum_{j=1}^{k+1} z^{s_j} w^{t_j} H^2$, we have

$$zM + wM \subset M \cap \sum_{j=1}^{k+1} z^{s_j} w^{t_j} (zH^2 + wH^2).$$

Let

$$f \in M \cap \sum_{j=1}^{k+1} z^{s_j} w^{t_j} (zH^2 + wH^2).$$

We may assume that $h(0,0) = 1$ and write $h = 1 + zh_1(z) + wh_2$ for some $h_1(z), h_2 \in H^\infty$. Then

$$f = fh - zfh_1(z) - wfh_2.$$

Since $f \in M$, we have $zfh_1(z) + wfh_2 \in zM + wM$. We may also write

$$f = \sum_{j=1}^{k+1} z^{s_j} w^{t_j} (zf_j + wg_j), \quad f_j, g_j \in H^2.$$

We have

$$fh = z\Big(\sum_{j=1}^{k+1} z^{s_j} w^{t_j} f_j h\Big) + w\Big(\sum_{j=1}^{k+1} z^{s_j} w^{t_j} g_j h\Big).$$

Since $h \in M_0 \cap H^\infty$, we have $f_j h, g_j h \in M_0$. By (3.6), we have

$$\sum_{j=1}^{k+1} z^{s_j} w^{t_j} f_j h, \ \sum_{j=1}^{k+1} z^{s_j} w^{t_j} g_j h \perp z^n w^m$$

for every $(n, m) \in A_M(0, 0)$. Since $M = \widetilde{M}$, we get

$$\sum_{j=1}^{k+1} z^{s_j} w^{t_j} f_j h, \ \sum_{j=1}^{k+1} z^{s_j} w^{t_j} g_j h \in M.$$

Hence $fh \in zM + wM$, so $f \in zM + wM$ and

$$M \cap \sum_{j=1}^{k+1} z^{s_j} w^{t_j} (zH^2 + wH^2) \subset zM + wM.$$

Thus we get (3.8).

It is not difficult to see that $\sum_{j=1}^{k+1} z^{s_j} w^{t_j} (zH^2 + wH^2)$ is closed, so $zM + wM$ is closed.

By Theorem 3.9, we have $\dim \widetilde{\Omega}(N) = k$. By Lemma 3.13, we also have $P_M z^{s_j} w^{t_j} \neq 0$ and

$$\sum_{j=1}^{k+1} \mathbb{C} \cdot P_M z^{s_j} w^{t_j} \subset \Omega(M).$$

Suppose that $\sum_{j=1}^{k+1} c_j P_M z^{s_j} w^{t_j} = 0$ for some $\{c_j\}_{j=1}^{k+1} \subset \mathbb{C}$. Since $h \in M_0$, we have $z^{s_j} w^{t_j} h \in \widetilde{M} = M$ for every $1 \leq j \leq k+1$. Since $h(0, 0) = 1$, for each $1 \leq i \leq k+1$ we have

$$0 = \Big\langle \sum_{j=1}^{k+1} c_j P_M z^{s_j} w^{t_j}, z^{s_i} w^{t_i} h \Big\rangle = \sum_{j=1}^{k+1} c_j \langle z^{s_j} w^{t_j}, z^{s_i} w^{t_i} h \rangle = c_i.$$

Hence $\{P_M z^{s_j} w^{t_j}\}_{j=1}^{k+1}$ is linearly independent, so by Corollary 3.14 $k + 1 \leq \dim \Omega(M)$.

To show $k + 1 = \dim \Omega(M)$, let $f \in \Omega(M)$ satisfy $f \perp P_M z^{s_j} w^{t_j}$ for every $1 \leq j \leq k+1$. Then $f \perp z^{s_j} w^{t_j}$ for every $1 \leq j \leq k+1$. Since $f \perp z^n w^m$ for every $(n, m) \in A_M(0, 0)$, we have

$$f \in M \cap \sum_{j=1}^{k+1} z^{s_j} w^{t_j} (zH^2 + wH^2).$$

By (3.8), we have $f \in zM + wM$, so $f = 0$. Thus we get the assertion. □

4. Special cases

Let $\Lambda = \{(a, a) : a \in \mathbb{D}\}$. Then $I(\Lambda) = [z - w]$ and $Z(I(\Lambda)) = \Lambda$. In this section, we shall study invariant subspaces M of H^2 satisfying $M \subsetneqq [z - w]$, $Z(M) = \Lambda$, $M \subset M_0 = [z - w]$ and $M = \widetilde{M}$. Moreover we assume that $M \not\subset zH^2$ and $M \not\subset wH^2$. Since $M_0 = [z - w]$ and $M = \widetilde{M}$, we have

$$M = \big\{f \in [z - w] : f \perp z^n w^m \text{ for every } (n, m) \in A_M(0, 0)\big\}.$$

For each positive integer n, let

$$[z - w]_n = \sum_{j=0}^{n-1} \mathbb{C} \cdot (z^{n-j} w^j - w^n). \tag{4.1}$$

Then

$$[z - w] = \bigoplus_{n=1}^{\infty} [z - w]_n. \tag{4.2}$$

Let

$$\mathcal{L}_n = \sum_{j=0}^{n} \mathbb{C} \cdot z^{n-j} w^j.$$

Then $[z - w]_n \subset \mathcal{L}_n$. We note that $P_{\mathcal{L}_n} f = P_{[z-w]_n} f$ for every $f \in [z - w]$.

Since $M_0 = [z - w]$, $A_M((a, a)) = \{(0, 0)\}$ for every $a \in \mathbb{D} \setminus \{0\}$. By Lemma 3.4, there are $n_1, n_2, \ldots, n_k, m_1, m_2, \ldots, m_k \in \mathbb{N}$ satisfying that $0 \le n_1 < n_2 < \cdots < n_k$, $0 \le m_k < m_{k-1} < \cdots < m_1$ and

$$A_M(0, 0) = \bigcup_{j=1}^{k} \{(n, m) \in \mathbb{N}^2 : 0 \le n \le n_j, 0 \le m \le m_j\}. \tag{4.3}$$

Since $Z(M) = \Lambda$ and $M \subsetneqq M_0 = [z - w]$, we have $A_M(0, 0) \ne \{(0, 0)\}$, so $n_j + m_j \ge 1$ for every $1 \le j \le k$. Hence there are integers $1 \le \ell_1 < \ell_2 < \cdots < \ell_p$ such that for each $1 \le i \le p$ there is $1 \le j \le k$ satisfying $n_j + m_j = \ell_i$ and

$$\Sigma = \bigcup_{i=1}^{p} \{(n_j, m_j) : 1 \le j \le k, n_j + m_j = \ell_i\}.$$

Set

$$\Sigma_i = \{(n_j, m_j) : 1 \le j \le k, n_j + m_j = \ell_i\}.$$

Then $\Sigma_i \ne \emptyset$ and $\Sigma_i \cap \Sigma_j = \emptyset$ for $i \ne j$. We have $\sum_{i=1}^{p} \#\Sigma_i = \#\Sigma = k$. Let

$$\Sigma^e = \bigoplus_{(n,m) \in \Sigma} \mathbb{C} \cdot z^n w^m \quad \text{and} \quad \Sigma_i^e = \bigoplus_{(n,m) \in \Sigma_i} \mathbb{C} \cdot z^n w^m.$$

Recall that $B_M(0, 0) = \mathbb{N}^2 \setminus A_M(0, 0)$ and

$$(s_1, t_1) = (0, m_1 + 1), \ (s_2, t_2) = (n_1 + 1, m_2 + 1), \ \ldots,$$
$$(s_k, t_k) = (n_{k-1} + 1, m_k + 1), \ (s_{k+1}, t_{k+1}) = (n_k + 1, 0).$$

Then by (4.3),

$$B_M(0,0) = \bigcup_{j=1}^{k+1} ((s_j, t_j) + \mathbb{N}^2). \tag{4.4}$$

Let $1 \le \sigma_1 < \sigma_2 < \cdots < \sigma_q$ be the integers such that for each $1 \le i \le q$ there is $1 \le j \le k+1$ satisfying $s_j + t_j = \sigma_i$ and

$$\{(s_j, t_j) : 1 \le j \le k+1\} = \bigcup_{i=1}^{q} \{(s_j, t_j) : 1 \le j \le k+1, s_j + t_j = \sigma_i\}.$$

Set

$$\Gamma = \{(s_j, t_j) : 1 \le j \le k+1\}$$

and

$$\Gamma_i = \{(s_j, t_j) : 1 \le j \le k+1, s_j + t_j = \sigma_i\}.$$

Then $\sum_{i=1}^{q} \#\Gamma_i = \#\Gamma = k+1$.

Lemma 4.1. (i) *$s + t \ge \sigma_1$ for every $(s,t) \in B_M(0,0)$.*

(ii) *If $(s,t) \in B_M(0,0)$ and $s + t = \sigma_1$, then $(s,t) \in \Gamma_1$.*

(iii) *For each $(s_1, t_1) \in B_M(0,0)$, we have*

$$\#\{(s,t) \in B_M(0,0) : s + t = s_1 + t_1\} \ge 2.$$

Proof. (i) and (ii) follow from (4.4).

(iii) Since $(s_1, t_1) \in B_M(0,0)$, there is $f \in M$ satisfying $z^{s_1} w^{t_1} \not\perp f$. Since $f \in [z - w]$, by (4.1) and (4.2)

$$M \ni P_{[z-w]_{s_1+t_1}} f = \sum_{j=0}^{s_1+t_1-1} c_j (z^{s_1+t_1-j} w^j - w^{s_1+t_1}) \ne 0.$$

This shows (iii). □

Theorem 4.2. *Let M be an invariant subspace of H^2 with $M \subsetneqq [z-w]$ such that $Z(M) = \Lambda$, $M \subset M_0 = [z-w]$ and $M = \widetilde{M}$. Moreover we assume that $M \not\subset zH^2$ and $M \not\subset wH^2$. Let $n_1, n_2, \ldots, n_k$, $m_1, m_2, \ldots, m_k \in \mathbb{N}$ satisfy the conditions given in Lemma 3.4. Then $\max\{k-1, 1\} \le \dim \widetilde{\Omega}(N) \le k$.*

Proof. Let $f \in \widetilde{\Omega}(N)$. By (1.2), $zf, wf \in M \subset [z-w]$, so $f \in [z-w]$. Recall that

$$M_1 = \big\{ f \in [z-w] : f \perp z^n w^m \text{ for every } (n,m) \in A_M(0,0) \setminus \Sigma \big\}.$$

Then we have $f \in M_1$. Hence $\widetilde{\Omega}(N) \subset M_1$. Since $zM_1 \subset M$ and $wM_1 \subset M$, we have

$$\widetilde{\Omega}(N) = M_1 \ominus M.$$

We have

$$M = \bigoplus_{n=1}^{\infty} M \cap [z-w]_n \quad \text{and} \quad M_1 = \bigoplus_{n=1}^{\infty} M_1 \cap [z-w]_n,$$

so

$$\widetilde{\Omega}(N) = \bigoplus_{i=1}^{p} \widetilde{\Omega}(N) \cap [z-w]_{\ell_i}.$$

Hence

$$\dim \widetilde{\Omega}(N) = \sum_{i=1}^{p} \dim \widetilde{\Omega}(N) \cap [z-w]_{\ell_i}. \tag{4.5}$$

For $2 \le i \le p$, there is $(s,t) \in B_M(0,0)$ such that $s+t=\ell_i$. Let

$$K_i = \{(s,t) \in B_M(0,0) : s+t = \ell_i\}.$$

By Lemma 4.1(iii), we have $\# K_i \ge 2$. For each $(n_j, m_j) \in \Sigma_i$, let

$$f_j = z^{n_j} w^{m_j} - \frac{1}{\# K_i} \sum_{(s,t)\in K_i} z^s w^t \in [z-w]_{\ell_i}.$$

It is not difficult to see that

$$f_j \in M_1 \ominus M = \widetilde{\Omega}(N), \quad (n_j, m_j) \in \Sigma_i,$$

so

$$\widetilde{\Omega}(N) \cap [z-w]_{\ell_i} = \sum_{(n_j,m_j)\in\Sigma_i} \mathbb{C} \cdot f_j.$$

Hence

$$\dim \widetilde{\Omega}(N) \cap [z-w]_{\ell_i} = \# \Sigma_i, \quad 2 \le i \le p.$$

We consider two cases for $i = 1$.

Case 1. Suppose that there is $(s,t) \in B_M(0,0)$ such that $s+t=\ell_1$. Similarly as above, we have $\dim \widetilde{\Omega}(N) \cap [z-w]_{\ell_1} = \# \Sigma_1$. Hence in this case, by (4.5) we have

$$\dim \widetilde{\Omega}(N) = \sum_{i=1}^{p} \# \Sigma_i = \# \Sigma = k.$$

Case 2. Suppose that $\{(s,t) \in B_M(0,0) : s+t = \ell_1\} = \emptyset$. In this case, take $(n_0, m_0) \in \Sigma_1$. Then

$$\widetilde{\Omega}(N) \cap [z-w]_{\ell_1} = \sum_{(n,m)\in\Sigma_1} \mathbb{C} \cdot (z^n w^m - z^{n_0} w^{m_0}),$$

so

$$\dim \widetilde{\Omega}(N) \cap [z-w]_{\ell_1} = \# \Sigma_1 - 1.$$

Hence

$$\begin{aligned} \dim \widetilde{\Omega}(N) &= \dim \widetilde{\Omega}(N) \cap [z-w]_{\ell_1} + \sum_{i=2}^{p} \dim \widetilde{\Omega}(N) \cap [z-w]_{\ell_i} \\ &= \# \Sigma_1 - 1 + \sum_{i=2}^{p} \# \Sigma_i = k-1. \end{aligned}$$

By Theorem 3.8, $1 \leq \dim \widetilde{\Omega}(N) \leq k$. Thus we get the assertion. □

Let M be an invariant subspace of H^2 with $M \subset [z-w]$ satisfying the conditions given in Theorem 4.2. Next, we shall study about $\Omega(M)$. In [5], the authors proved the following.

Lemma 4.3. *Let M_1 and M_2 be invariant subspaces of H^2 satisfying $M_2 \subsetneqq M_1$ and $\dim(M_1 \ominus M_2) < \infty$. Then $F_z^{M_1}$ is a Fredholm operator if and only if so is $F_z^{M_2}$. In this case, we have $\operatorname{ind} F_z^{M_1} = \operatorname{ind} F_z^{M_2}$.*

Corollary 4.4. *Let M be an invariant subspace of H^2 with $M \subset [z-w]$ such that $Z(M) = \Lambda$, $M \subsetneqq M_0 = [z-w]$ and $M = \widetilde{M}$. Moreover we assume that $M \not\subset zH^2$ and $M \not\subset wH^2$. Then F_z^M is Fredholm and $\operatorname{ind} F_z^M = -1$.*

Proof. By Example 2.13, $F_z^{[z-w]}$ is Fredholm and $\operatorname{ind} F_z^{[z-w]} = -1$. By Lemma 3.4, $\dim([z-w] \ominus M) < \infty$. Then by Lemma 4.3, we get the assertion. □

In the proof of Theorem 4.2, we described the elements in $\widetilde{\Omega}(N)$. By Lemma 2.1 and Corollary 4.4, we have $\dim \Omega(M) = \dim \widetilde{\Omega}(N) + 1$. We shall describe the elements in $\Omega(M)$. We shall use the same notations given above Lemma 3.13. Since $M \subsetneqq [z-w]$, we have $2 \leq \sigma_1$. We note that $n + m \geq \sigma_1$ for every $(n,m) \in B_M(0,0)$. Moreover if $(n,m) \in B_M(0,0)$ and $n + m = \sigma_1$, then $(n,m) \in \Gamma_1$.

Lemma 4.5. (i) *$\#\Gamma_1 \geq 2$ and if $(n,m) \in B_M(0,0)$, then $n + m = \sigma_1$ if and only if $(n,m) \in \Gamma_1$.*

(ii)

$$\dim \sum_{(s_j,t_j)\in\Gamma_1} \mathbb{C} \cdot P_M z^{s_j} w^{t_j} = \#\Gamma_1 - 1.$$

(iii) *For each $2 \leq i \leq q$, we have*

$$\dim \sum_{(s_j,t_j)\in\Gamma_i} \mathbb{C} \cdot P_M z^{s_j} w^{t_j} = \#\Gamma_i.$$

Proof. (i) By Lemma 4.1(ii) and (iii), we have $\#\Gamma_1 \geq 2$. The second assertion is already pointed out above Lemma 4.5.

(ii) Take $(s_{j_0}, t_{j_0}) \in \Gamma_1$. Since $M = \widetilde{M}$, for $(s,t) \in \Gamma_1$ we have $z^s w^t - z^{s_{j_0}} w^{t_{j_0}} \in M$ and

$$\sum_{(s,t)\in\Gamma_1} \mathbb{C} \cdot (z^s w^t - z^{s_{j_0}} w^{t_{j_0}}) \subset M.$$

By (i),

$$z^{s_j} w^{t_j} \perp M \ominus \sum_{(s,t)\in\Gamma_1} \mathbb{C} \cdot (z^s w^t - z^{s_{j_0}} w^{t_{j_0}})$$

for every $(s_j, t_j) \in \Gamma_1$. Hence

$$\sum_{(s_j,t_j)\in\Gamma_1} \mathbb{C} \cdot P_M z^{s_j} w^{t_j} \subset \sum_{(s,t)\in\Gamma_1} \mathbb{C} \cdot (z^s w^t - z^{s_{j_0}} w^{t_{j_0}}).$$

Let

$$g \in \Big(\sum_{(s,t)\in\Gamma_1} \mathbb{C}\cdot (z^s w^t - z^{s_{j_0}} w^{t_{j_0}})\Big) \ominus \Big(\sum_{(s_j,t_j)\in\Gamma_1} \mathbb{C}\cdot P_M z^{s_j} w^{t_j}\Big).$$

Then $g \perp z^{s_j} w^{t_j}$ for every $(s_j,t_j)\in\Gamma_1$, so $g=0$. Hence

$$\sum_{(s_j,t_j)\in\Gamma_1} \mathbb{C}\cdot P_M z^{s_j} w^{t_j} = \sum_{(s,t)\in\Gamma_1} \mathbb{C}\cdot (z^s w^t - z^{s_{j_0}} w^{t_{j_0}}).$$

Therefore we get (ii).

(iii) Since $2 \le i$, there is $(s,t)\in B_M(0,0)\setminus\Gamma$ such that $s+t=\sigma_i$. Let

$$\widetilde{\Gamma}_i = \{(s,t)\in B_M(0,0) : s+t=\sigma_i\}.$$

Then $\Gamma_i \subsetneqq \widetilde{\Gamma}_i$. Take $(s_0,t_0)\in\widetilde{\Gamma}_i\setminus\Gamma_i$. Since $M=\widetilde{M}$, for $(s,t)\in\widetilde{\Gamma}_i$ we have $z^s w^t - z^{s_0} w^{t_0}\in M$ and

$$z^{s_j} w^{t_j} \perp M \ominus \sum_{(s,t)\in\widetilde{\Gamma}_i} \mathbb{C}\cdot (z^s w^t - z^{s_0} w^{t_0})$$

for every $(s_j,t_j)\in\Gamma_i$. Hence

$$\sum_{(s_j,t_j)\in\Gamma_i} \mathbb{C}\cdot P_M z^{s_j} w^{t_j} \subset \sum_{(s,t)\in\widetilde{\Gamma}_i} \mathbb{C}\cdot (z^s w^t - z^{s_0} w^{t_0}) \subset M.$$

Let

$$h \in \Big(\sum_{(s,t)\in\widetilde{\Gamma}_i} \mathbb{C}\cdot (z^s w^t - z^{s_0} w^{t_0})\Big) \ominus \Big(\sum_{(s_j,t_j)\in\Gamma_i} \mathbb{C}\cdot P_M z^{s_j} w^{t_j}\Big).$$

Then $h \perp z^{s_j} w^{t_j}$ for every $(s_j,t_j)\in\Gamma_i$. Hence

$$h \in \sum_{(s,t)\in\widetilde{\Gamma}_i\setminus\Gamma_i} \mathbb{C}\cdot (z^s w^t - z^{s_0} w^{t_0}).$$

This shows that

$$\begin{aligned}\sum_{(s_j,t_j)\in\Gamma_i} \mathbb{C}\cdot P_M z^{s_j} w^{t_j} \quad &= \quad \Big(\sum_{(s,t)\in\widetilde{\Gamma}_i} \mathbb{C}\cdot (z^s w^t - z^{s_0} w^{t_0})\Big) \ominus \\ &\qquad \Big(\sum_{(s,t)\in\widetilde{\Gamma}_i\setminus\Gamma_i} \mathbb{C}\cdot (z^s w^t - z^{s_0} w^{t_0})\Big).\end{aligned}$$

Hence

$$\dim \sum_{(s_j,t_j)\in\Gamma_i} \mathbb{C}\cdot P_M z^{s_j} w^{t_j} = (\#\widetilde{\Gamma}_i - 1) - (\#(\widetilde{\Gamma}_i\setminus\Gamma_i) - 1) = \#\Gamma_i.$$

We note that

$$z^{s_j} w^{t_j} - \frac{1}{\#(\Gamma_i\setminus\Gamma_i)} \sum_{(s,t)\in\widetilde{\Gamma}_i\setminus\Gamma_i} z^s w^t \in \mathbb{C}\cdot P_M z^{s_j} w^{t_j}, \quad (s_j,t_j)\in\Gamma_i.$$

□

Theorem 4.6. *Let M be an invariant subspace of H^2 with $M \subsetneqq [z-w]$ such that $Z(M) = \Lambda$, $M \subset M_0 = [z-w]$ and $M = \widetilde{M}$. Moreover we assume that $M \not\subset zH^2$ and $M \not\subset wH^2$. Let $n_1, n_2, \ldots, n_k, m_1, m_2, \ldots, m_k \in \mathbb{N}$ satisfy the conditions given in Lemma 3.4 and $\ell_1 = \min_{1\le j\le k} n_j + m_j$. Then we have the following.*

(i) *Suppose that $s+t \neq \ell_1$ for any $(s,t) \in B_M(0,0)$. Then*

$$\Omega(M) = \sum_{(s,t)\in\Gamma} \mathbb{C}\cdot P_M z^s w^t$$

and $\dim \Omega(M) = k$.

(ii) *Suppose that there is $(s,t) \in B_M(0,0)$ such that $s+t = \ell_1$. Let*

$$g = \sum_{(s,t)\in\Gamma_1} z^s w^t (z-w) \in M.$$

Then

$$\Omega(M) = \mathbb{C}\cdot g \oplus \sum_{(s,t)\in\Gamma} \mathbb{C}\cdot P_M z^s w^t$$

and $\dim \Omega(M) = k+1$.

Proof. (i) By the proof of Theorem 4.2, we have $\dim \widetilde{\Omega}(N) = k-1$. By Lemma 2.1 and Corollary 4.4, we have $\dim \Omega(M) = k$. By Lemma 3.13,

$$\sum_{(s,t)\in\Gamma} \mathbb{C}\cdot P_M z^s w^t \subset \Omega(M)$$

and

$$\begin{aligned}\dim \sum_{(s,t)\in\Gamma} \mathbb{C}\cdot P_M z^s w^t &= \sum_{i=1}^{q} \dim \sum_{(s,t)\in\Gamma_i} \mathbb{C}\cdot P_M z^s w^t \\ &= \#\Gamma_1 - 1 + \sum_{i=2}^{q} \#\Gamma_i \qquad \text{by Lemma 4.5} \\ &= \#\Gamma - 1 = k+1-1 = k.\end{aligned}$$

Thus we get (i).

(ii) In this case, by the proof of Theorem 4.2 we have $\dim \widetilde{\Omega}(N) = k$, so $\dim \Omega(M) = k+1$. In the same way as the one in (i), we have

$$\sum_{(s,t)\in\Gamma} \mathbb{C}\cdot P_M z^s w^t \subset \Omega(M)$$

and

$$\dim \sum_{(s,t)\in\Gamma} \mathbb{C}\cdot P_M z^s w^t = k.$$

By Lemma 4.5(i), $\#\Gamma_1 \ge 2$. Put

$$\Gamma_1 = \left\{(s_{j_1}, t_{j_1}), (s_{j_2}, t_{j_2}), \ldots, (s_{j_\gamma}, t_{j_\gamma})\right\} \subset B_M(0,0), \tag{4.6}$$

where $0 \le s_{j_1} < s_{j_2} < \cdots < s_{j_\gamma}$ and $\gamma \ge 2$. We have $\sigma_1 \le s + t$ for every $(s,t) \in B_M(0,0)$, and for $(s,t) \in B_M(0,0)$, $\sigma_1 = s+t$ if and only if $(s,t) \in \Gamma_1$. If $s_{j_{n+1}} - s_{j_n} = 1$, then $(s_{j_n}, t_{j_n} - 1) \in \Sigma$. Hence

$$\ell_1 \le s_{j_n} + t_{j_n} - 1 = \sigma_1 - 1 < \sigma_1 \le s + t$$

for every $(s,t) \in B_M(0,0)$. This contradicts with the assumption of (ii). Hence $s_{j_{n+1}} - s_{j_n} = t_{j_n} - t_{j_{n+1}} \ge 2$ for every $1 \le n \le \gamma - 1$. This shows that $(s_{j_n} + 1, t_{j_n} - 1) \in A_M(0,0)$ for every $1 \le n \le \gamma - 1$ and $(s_{j_n} - 1, t_{j_n} + 1) \in A_M(0,0)$ for every $2 \le n \le \gamma$. If $s_{j_1} \ge 1$, then we have $(s_{j_1} - 1, t_{j_1} + 1) \in A_M(0,0)$. For, if $(s_{j_1} - 1, t_{j_1} + 1) \in B_M(0,0)$, then $(s_{j_1} - 1, t_{j_1} + 1) \in \Gamma_1$ and this contradicts with (4.6). Similarly if $t_{j_\gamma} \ge 1$, then $(s_{j_\gamma} + 1, t_{j_\gamma} - 1) \in A_M(0,0)$.

Let

$$g = \sum_{n=1}^{\gamma} z^{s_{j_n}} w^{t_{j_n}} (z - w) \in M.$$

We have

$$\begin{aligned} P_M T_z^* g &= P_M\Big(\Big(\sum_{n=1}^{\gamma}(-z^{s_{j_n}-1} w^{t_{j_n}+1})\Big) + \Big(\sum_{n=1}^{\gamma} z^{s_{j_n}} w^{t_{j_n}}\Big)\Big) \\ &= P_M\Big(\sum_{n=1}^{\gamma} z^{s_{j_n}} w^{t_{j_n}}\Big). \end{aligned}$$

Since

$$M \cap \big(\mathbb{C} \cdot z^{\sigma_1} \oplus \mathbb{C} \cdot z^{\sigma_1 - 1} w \oplus \cdots \oplus \mathbb{C} \cdot w^{\sigma_1}\big) = \sum_{n=2}^{\gamma} \mathbb{C} \cdot (z^{s_{j_1}} w^{t_{j_1}} - z^{s_{j_n}} w^{t_{j_n}}),$$

we have

$$P_M\Big(\sum_{n=1}^{\gamma} z^{s_{j_n}} w^{t_{j_n}}\Big) = 0.$$

Hence $P_M T_z^* g = 0$. Similarly $P_M T_w^* g = 0$. Thus by (1.1), we get $g \in \Omega(M)$.

Since $g \perp z^s w^t$, we have $g \perp P_M z^s w^t$ for every $(s,t) \in \Gamma$. Hence

$$\mathbb{C} \cdot g \oplus \sum_{(s,t) \in \Gamma} \mathbb{C} \cdot P_M z^s w^t \subset \Omega(M)$$

and

$$\dim \Big(\mathbb{C} \cdot g \oplus \sum_{(s,t) \in \Gamma} \mathbb{C} \cdot P_M z^s w^t\Big) = k + 1.$$

Thus we get

$$\Omega(M) = \mathbb{C} \cdot g \oplus \sum_{(s,t) \in \Gamma} \mathbb{C} \cdot P_M z^s w^t.$$

□

We shall give an example satisfying $M \neq \widetilde{M}$.

Example 4.7. Let

$$M = [z^2 - w^2, z^3(z-w), z^2w(z-w), zw^2(z-w), w^3(z-w)].$$

Then $M_0 = [z-w]$, $A_M(0,0) = \{(0,0),(0,1),(1,0),(1,1)\}$ and

$$\widetilde{M} = \{f \in [z-w] : f \perp z, f \perp zw, f \perp w\}.$$

We have $zw(z-w) \in \widetilde{M}$ and $zw(z-w) \notin M$, so $M \neq \widetilde{M}$. We have $\Sigma = \{(1,1)\}$, so $M_1 = [z(z-w), w(z-w)]$. We have $z^2 - 2zw + w^2 \in M_1 \ominus M$ and $z(z^2 - 2zw + w^2) \notin M$. Hence $M_1 \ominus M \not\subset \widetilde{\Omega}(N)$ and compare with the assertion of Theorem 3.8. By calculation, we have

$$\widetilde{\Omega}(N) = \mathbb{C} \cdot \big((z^3 + zw^2) - (z^2w + w^3)\big)$$

and

$$\Omega(M) = \mathbb{C} \cdot (z^2 - w^2) + \mathbb{C} \cdot (2z^4 - 3z^3w + 2z^2w^2 - 3zw^3 + 2w^4).$$

By Example 2.13 and Lemma 4.3, F_z^M is Fredholm and $\operatorname{ind} F_z^M = -1$.

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