J. Korean Math. Soc. **53** (2016), No. 4, pp. 835–846
http://dx.doi.org/10.4134/JKMS.j150296
pISSN: 0304-9914 / eISSN: 2234-3008

ON SOLUTIONS TO SOME NONLINEAR DIFFERENCE AND DIFFERENTIAL EQUATIONS

Wei Chen, Pei-Chu Hu, and Yingying Zhang

Abstract. In this paper, we study entire solutions of some nonlinear difference equations and transcendental meromorphic solutons of some nonlinear differential equations. Our results generalize the results due to [11], [17].

1. Introduction and main result

We assume that the reader is familiar with the standard notations and fundamental results in Nevanlinna theory. For example, we use the following notations in value distribution such as $T(r, f)$, $m(r, f)$, $N(r, f)$, $S(r, f)$, where as usual $S(r, f)$ denotes any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ outside a possible exceptional set of finite logarithmic measure. We refer the reader to the books [3, 6], and [7]. For an element η in complex plane $\mathbb{C}$, we will use $f(z+\eta)$ and $\Delta_\eta f(z) := f(z+\eta) - f(z)$ to denote the shift and difference of $f(z)$ respectively.

As we know, Nevanlinna theory is an efficient tool in the research of complex differential theory. It is interesting to use the Nevanlinna theory to study complex equation of various types. Many results about complex difference equations (cf. [1, 2, 4, 5]), complex differential equations (cf. [14]) or complex differential-difference equations (cf. [10], [12] and [15]) were rapidly obtained, respectively.

In 2004, Yang and Li [15] studied some certain types of nonlinear equations, and proved the following results.

Theorem A ([15])**.** *Take a positive integer n. Let $a, b_0, b_1, \ldots, b_{n-1}$ be polynomials, and let b_n be a nonzero constant. Set $L(f) = \sum_{k=0}^{n} b_k f^{(k)}$. If $a(z) \not\equiv 0$, then a transcendental meromorphic solution of the following equation*

$$f^2 + (L(f))^2 = a \tag{1.1}$$

Received May 14, 2015; Revised November 23, 2015.
2010 *Mathematics Subject Classification.* 30D35, 39B32, 34M05.
Key words and phrases. entire solutions, meromorphic solutions, difference equations, differential equations.
This paper is supported by Nature Science Foundation of China(No. 11461070, 11271227) and PCSIRT (IRT1264).

must have the form $f(z) = \frac{1}{2}(P(z)e^{R(z)} + Q(z)e^{-R(z)})$, *where* P, Q, R *are polynomials with* $PQ = a$.

Theorem B ([15])**.** *Let* a_1, a_2 *and* a_3 *be nonzero meromorphic functions. Then a necessary condition for the differential equation*

$$a_1 f^2 + a_2 (f')^2 = a_3 \tag{1.2}$$

to have a transcendental meromorphic solution satisfying $T(r, a_k) = S(r, f)$, $k = 1, 2, 3$, *is* $a_1/a_3 \equiv$ *constant.*

In the same paper, Yang and Li conjectured that the equation

$$P_1 f^2 + P_2 (f')^2 = P_3$$

has no transcendental meromorphic solution when P_1/P_3 is a nonzero constant and P_2/P_3 is not the square of any rational function, where P_1, P_2, P_3 are nonzero polynomials. Later, Tang and Liao [13] found that the answer to this conjecture was negative by some examples. Moreover, they had studied the differential equation as following

$$f^2(z) + P(z)(f')^2(z) = Q(z), \tag{1.3}$$

where $P(z)$ and $Q(z)$ are rational functions. Recently, Zhang and Liao [17] improved the results of Tang and Liao [13], and found all forms of transcendental meromorphic solutions of the differential equation (1.3) by proving the following result:

Theorem C ([17])**.** *If the differential equation*

$$f^2(z) + R(z)(f')^2(z) = Q(z) \tag{1.4}$$

admits a transcendental meromorphic solution, then we have $Q(z) \equiv C$, *where* C *is a constant, the multiplicity of zero of* $R(z)$ *is no greater than 2 and* $f(z) = \sqrt{C}\cos\alpha(z)$, $\alpha(z)$ *is a primitive function of* $\frac{1}{\sqrt{R(z)}}$ *such that* $\sqrt{C}\cos\alpha(z)$ *is a transcendental meromorphic function.*

In this paper, we consider the following nonlinear differential equation

$$f(z)^2 + R(z)(f')^2(z) = Q(z)e^{\alpha(z)}, \tag{1.5}$$

where $Q(z)$, $R(z)$ are rational functions such that $R(z)$ has a square root, and $\alpha(z)$ is a polynomial. We get the following theorem.

Theorem 1.1. *If the differential equation* (1.5) *admits a transcendental meromorphic solution* f, *we get either*

(i) *if* α *is a constant or* f *is of infinite order and if* R *has no zeros or all its zeros are of multiplicity 2, then* $f(z) = \sqrt{C}\cos\gamma(z)$, *where* C *is a constant such that* $C = Q(z)e^{\alpha(z)}$, *and* $\gamma(z)$ *is a primitive function of* $\frac{1}{\sqrt{R(z)}}$ *such that* $\sqrt{C}\cos\gamma(z)$ *is a transcendental meromorphic function*; *or*

(ii) *if* f *is of finite order, then* $f(z) = H(z)e^{h(z)}$, *where* $2h(z) = \alpha(z)$, $H^2(z) + (H'(z) + H(z)h'(z))R(z) = Q(z)$.

Now, we give the following examples to show that case (ii) above does exist.

Example 1. Consider the equation

$$f^2(z) + \frac{1}{z^2}(f'(z))^2 = (z^6 + z + 3)e^{2z}.$$

We can see that $f(z) = z^3 e^z$ is a finite order solution.

Example 2. Consider the equation

$$f^2(z) + z(f'(z))^2 = \frac{z^4 - 2z^3 + z}{(z-1)^4}e^{2z}.$$

We can see that $f(z) = \frac{z}{(z-1)^2}e^z$ is a finite order.

Remark 1.2. Obviously, 1.1 generalizes Theorem C. It is easy to check that a similar result can be got for the following equation

$$f^2(z) + R(z)e^{\beta(z)}(f')^2(z) = Q(z)e^{\alpha(z)},$$

where $\beta(z)$ is a polynomial.

Generally, we suggest the following question:

Question 1.1. What forms do the transcendental meromorphic solutions have for the differential equation $f^2(z) + R(z)(f^{(k)})^2(z) = Q(z)e^{\alpha(z)}$?

Next, we will consider similar questions on difference equations.

Based on Theorem A, we know that the transcendental meromorphic solutions of following equation

$$f^2 + (f')^2 = 1 \tag{1.6}$$

must satisfy $f(z) = \frac{1}{2}\Big(Pe^{\lambda z} + \frac{1}{P}e^{-\lambda z}\Big)$, where P, λ are nonzero constants. In 2009, Liu [9] replaced f' in above equation by $f(z+\eta)$ and considered the entire solutions of following equation

$$f^2(z) + f^2(z+\eta) = 1, \tag{1.7}$$

by proving that the transcendental entire solutions of finite order of equation (1.7) have the form $f(z) = \frac{1}{2}(h_1(z) + h_2(z))$, where $h_1(z), h_2(z)$ satisfy $\frac{h_1(z+\eta)}{h_1(z)} = i$, $\frac{h_2(z+\eta)}{h_2(z)} = -i$ and $h_1(z)h_2(z) = 1$. Later, Liu et al. [10] proved a precise result as follows:

Theorem D ([10])**.** *If difference equation* (1.7) *admits a transcendental entire solution* $f(z)$ *of finite order, then* $f(z)$ *must assume the form that* $f(z) = sin(Az + B)$, *where* $A = \frac{(4k+1)\pi}{2c}$, *and* B *is a constant,* k *is an integer.*

Recently, Liu and Yang [11] generalized above result by proving the following results.

Theorem E ([11])**.** *Let $P(z), Q(z)$ be two non-zero polynomials. If difference equation*

$$f^2(z) + P^2(z)f^2(z+\eta) = Q(z) \tag{1.8}$$

admits a transcendental entire solution $f(z)$ of finite order, then $P(z) \equiv \pm 1$ and $Q(z)$ reduces to a constant q. Moreover, $f(z) = \sqrt{q}\sin(Az+B)$, where $A = \frac{(4k+1)\pi}{2c}$, and B is a constant, k is an integer.

Theorem F ([11])**.** *Let $P(z), Q(z)$ be two non-zero polynomials. Then the following difference equation*

$$f^2(z) + P^2(z)\Delta_\eta^2 f(z) = Q(z) \tag{1.9}$$

has no transcendental entire solutions of finite order.

Example 3. $f(z) = ze^z$ is a solution of entire function with finite order, which satisfies

$$f^2(z) + f^2(z+\pi i) = [z^2 + (z+\pi i)^2]e^{2z},$$

and

$$f^2(z) + [f(z+2\pi i) - f(z)]^2 = [z^2 - 4\pi^2]e^{2z}.$$

From above example, we can see that if the polynomial Q in (1.8) and (1.9) is replaced by Qe^α, then we may get different results from Theorem E and Theorem F. Now, we consider this problem, and obtain the following results.

Theorem 1.3. *Let $P(z), Q(z)$ be two non-zero polynomials and let $\alpha(z)$ be a polynomial. If the following difference equation*

$$f^2(z) + P^2(z)f^2(z+\eta) = Q(z)e^{\alpha(z)} \tag{1.10}$$

admits a transcendental entire solution $f(z)$ of finite order, then $f(z)$ and α must be one of the following two cases:

(i) *$f(z) = \frac{d_1e^{A_1z+B_1}+d_2e^{A_2z+B_2}}{2}$, $\alpha(z) = (A_1+A_2)z + B_1 + B_2$, and $P(z)$, $Q(z)$ reduce to constants such that $Q(z) = d_1d_2$, $P^2(z) = \frac{1}{e^{(A_1+A_2)\eta}}$, where A_1, A_2, B_1, B_2, d_1, d_2 are constants.*

(ii) *$f(z) = \beta(z)e^{Az}$, $\alpha(z) = 2Az$, where A is a nonzero constant and $\beta(z)$ is a polynomial satisfies $\beta^2(z) + e^{2A\eta}\beta^2(z+\eta)P^2(z) = Q(z)$.*

Remark 1.4. If $\alpha = 0$ in Theorem 1.3, then $A_1 + A_2 = 0$ or $A = 0$. If $A = 0$, then f will be a polynomial, which contradicts with the assumption that f is a transcendental entire function. Hence, $P^2 = 1$. Then by Theorem D, Theorem E follows from Theorem 1.3.

Theorem 1.5. *Suppose $P(z)$, $Q(z)$ are two non-zero polynomials. Let $\alpha(z)$ be a polynomial. If the following difference equation*

$$f^2(z) + P^2(z)\Delta_\eta^2 f(z) = Q(z)e^{\alpha(z)} \tag{1.11}$$

admits a transcendental entire solution $f(z)$ of finite order, then $f(z)$ and α must be one of the following two cases:

(i) $f(z)=\frac{d_1e^{A_1z+B_1}+d_2e^{A_2z+B_2}}{2}$, $\alpha(z)=(A_1+A_2)z+B_1+B_2$, *and* $P(z),Q(z)$ *reduce to constants such that* $Q(z)=c_1c_2, P^2(z)(e^{(A_1+A_2)\eta}-1)=1$, *where* $A_1,A_2,B_1,\ B_2,d_1,d_2$ *are constants.*

(ii) $f(z)=\beta(z)e^{Az}$, $\alpha(z)=2Az$, *where* A *is a nonzero constant and* $\beta(z)$ *satisfies* $\beta^2(z)+e^{2A\eta}(\beta(z+\eta)-\beta(z))^2P^2(z)=Q(z)$.

Remark 1.6. If $\alpha=0$ in Theorem 1.5, then $A_1+A_2=0$ or $A=0$. Moreover, if $A=0$, then f will be a polynomial, which contradicts with the assumption that f is a transcendental entire function. Therefore, $A_1+A_2=0$, which implies that $e^{(A_1+A_2)\eta}=1$. Then there doesn't exit $P(z)$ satisfy $P^2(z)(e^{(A_1+A_2)\eta}-1)=1$. Hence, differential equation (1.11) has no solution. Hence Theorem 1.5 generalizes Theorem F.

2. Some lemmas

In order to prove our results, we will need the following lemmas.

Lemma 2.1 ([16])**.** *Let* $f(z)$ *be nonconstant meromorphic function. Then*

$$m\Big(r,\frac{f'}{f}\Big)=O(\log r),\quad r\to\infty,$$

if f *is of finite order, and*

$$m\Big(r,\frac{f'}{f}\Big)=O(\log(rT(r,f))),\quad r\to\infty,$$

possibly outside a set E *of* r *with finite linear measure if* f *is of infinite order.*

Lemma 2.2 ([16])**.** *Suppose that* $f_1(z),f_2(z),\dots,f_n(z)$ $(n\geq 3)$ *are meromorphic functions which are not constants except for* $f_n(z)$. *Furthermore, let*

$$\sum_{j=1}^{n}f_j=1.$$

If $f_n(z)\not\equiv 0$ *and*

$$\sum_{j=1}^{n}N(r,\frac{1}{f_j})+(n-1)\sum_{j=1}^{n}\overline{N}(r,f_j)<(\lambda+o(1))T(r,f_k),$$

where $\lambda<1$ *and* $k=1,2,\dots,n-1$, *then* $f_n(z)\equiv 1$.

3. Proofs of main results

Proof of Theorem 1.1. Suppose that (1.5) admits a transcendental meromorphic solution f. We can easily deduce that f has finitely many poles from (1.5), and hence $N(r,f)=S(r,f)=O(\log r)$. If α is a constant, it follows from Theorem B that Qe^{α} is a constant. Then, from the subcase (ii) of Theorem 1 of [8], we can get the first conclusion of Theorem 1.1. Next we assume α is not a constant.

If f is of infinite order, we know that Qe^{α} is a constant from Theorem B. Then, the first conclusion of Theorem 1.1 follows easily from the subcase (ii) of Theorem 1 of [8].

If f is of finite order. Differentiating (1.5), we have

$$2ff' + R'(f')^2 + 2Rf'f'' = Q\Big(\frac{Q'}{Q} + \alpha'\Big)e^{\alpha}. \tag{3.1}$$

We will distinguish the following two cases:

(I) If $\frac{Q'}{Q} + \alpha' \equiv 0$; (II) If $\frac{Q'}{Q} + \alpha' \not\equiv 0$.

(I) If $\frac{Q'}{Q} + \alpha' \equiv 0$, then $\alpha = -\ln Q + C_1$, where C_1 is a constant. We can deduce that Q is a constant. Otherwise, we will get α is not a polynomial, which contradicts with the condition that α is a polynomial. Then Q is a constant, which implies that α is also a constant, a contraction.

(II) If $\frac{Q'}{Q} + \alpha' \not\equiv 0$, then from (1.5) and (3.1), we get

$$\Big(\frac{Q'}{Q} + \alpha'\Big)f^2 + \Big[R\Big(\frac{Q'}{Q} + \alpha'\Big) - R'\Big](f')^2 = 2ff' + 2Rf'f''. \tag{3.2}$$

We distinguish two subcases:

(II-1) If f has infinitely many zeros. Let z_0 be a zero of f which is not a zero and pole of R and Q. Then, by (1.5) and (3.2), we have

$$(f')^2(z_0) = \frac{Q(z_0)}{R(z_0)}e^{\alpha(z_0)} \neq 0,$$
$$[R(z_0)(\frac{Q'(z_0)}{Q(z_0)} + \alpha'(z_0)) - R'(z_0)](f'(z_0))^2 = 2R(z_0)f'(z_0)f''(z_0).$$

It follows that z_0 is a simple zero of f and zero of $[R(\frac{Q'}{Q} + \alpha') - R']f' - 2Rf''$. Therefore, $\beta(z) = \frac{[R(\frac{Q'}{Q}+\alpha')-R']f'-2Rf''}{f}$ has only finitely many poles. Noting that f is of finite order, by Lemma 2.1, we have

$$\begin{aligned} &T\Big(r, \frac{[R(\frac{Q'}{Q} + \alpha') - R']f' - 2Rf''}{f}\Big) \\ &= m\Big(r, \frac{[R(\frac{Q'}{Q} + \alpha') - R']f' - 2Rf''}{f}\Big) + O(\log r) = O(\log r). \end{aligned} \tag{3.3}$$

Hence, $\beta(z)$ is a rational function. If $\beta(z) \equiv 0$, then

$$2\frac{f''}{f'} = \frac{Q'}{Q} + \alpha' - \frac{R'}{R}, \tag{3.4}$$

so $(f')^2 = \frac{C_2Qe^{\alpha}}{R}$, where C_2 is a constant. By substituting it into (1.5), we have

$$f^2 = (1 - C_2)Qe^{\alpha}, \tag{3.5}$$

it contradicts with the assumption that f has infinitely many zeros.

Therefore, $\beta(z) \not\equiv 0$, and

$$f'' = \frac{1}{2}\Big[\Big(\frac{Q'}{Q} + \alpha'\Big) - \frac{R'}{R}\Big]f' - \frac{\beta}{2R}f. \tag{3.6}$$

By substituting (3.6) into (3.2), we get

$$(2-\beta)f' = \Big(\frac{Q'}{Q} + \alpha'\Big)f. \tag{3.7}$$

Obviously, $\beta(z) \not\equiv 2$. Substituting (3.7) into (1.5), we obtain that

$$f^2 = \Bigg[\frac{(2-\beta)^2 Q}{(2-\beta)^2 + R\Big(\frac{Q'}{Q} + \alpha'\Big)^2}\Bigg]e^{\alpha}. \tag{3.8}$$

It implies that f has only finitely many zeros, a contradiction.

(II-2) If f has finitely many zeros, we can assume that $f(z) = H(z)e^{h(z)}$, where $H(z)$ is a rational function, and $h(z)$ is a nonconstant polynomial. Substituting it into (1.5), we get

$$\begin{aligned} &2h(z) = \alpha(z),\\ &H^2(z) + (H'(z) + H(z)h'(z))R(z) = Q(z). \end{aligned}$$

□

Proof of Theorem 1.3. Suppose that $f(z)$ is a transcendental entire solution of (1.10) with finite order, then

$$[f(z) + iP(z)f(z+\eta)][f(z) - iP(z)f(z+\eta)] = Q(z)e^{\alpha(z)}. \tag{3.9}$$

Under the assumption of $f(z)$, we know that $f(z) + iP(z)f(z+\eta)$ and $f(z) - iP(z)f(z+\eta)$ are all entire functions which having finitely many zeros from (3.9). By Hadamard factorization theorem , we may assume that

$$f(z) + iP(z)f(z+\eta) = Q_1(z)e^{\alpha_1(z)}$$

and

$$f(z) - iP(z)f(z+\eta) = Q_2(z)e^{\alpha_2(z)},$$

where $Q_1(z), Q_2(z)$ are two non-zero polynomials, and $\alpha_1(z), \alpha_2(z)$ are two polynomials.

From above two equations, we obtain

$$f(z) = \frac{Q_1(z)e^{\alpha_1(z)} + Q_2(z)e^{\alpha_2(z)}}{2} \tag{3.10}$$

and

$$f(z+\eta) = \frac{Q_1(z)e^{\alpha_1(z)} - Q_2(z)e^{\alpha_2(z)}}{2iP(z)}. \tag{3.11}$$

From (3.10), we can see that $\alpha_1(z)$, $\alpha_2(z)$ can not be constants at same time. Otherwise, $f(z)$ will be a polynomial, which contradicts with the assumption that f is a transcendental entire solution.

From (3.10) and (3.11), we have

$$f(z+\eta)=\frac{Q_1(z)e^{\alpha_1(z)}-Q_2(z)e^{\alpha_2(z)}}{2iP(z)} =\frac{Q_1(z+\eta)e^{\alpha_1(z+\eta)}+Q_2(z+\eta)e^{\alpha_2(z+\eta)}}{2}. \tag{3.12}$$

Rewriting (3.12), we get

$$\frac{iP(z)Q_1(z+\eta)}{Q_1(z)}e^{\alpha_1(z+\eta)-\alpha_1(z)}+\frac{iP(z)Q_2(z+\eta)}{Q_1(z)}e^{\alpha_2(z+\eta)-\alpha_1(z)} +\frac{Q_2(z)}{Q_1(z)}e^{\alpha_2(z)-\alpha_1(z)}=1. \tag{3.13}$$

Next, we will distinguish two cases:
(1) $e^{\alpha_2(z)-\alpha_1(z)}$ is not a constant; (2) $e^{\alpha_2(z)-\alpha_1(z)}$ is a constant.

(1) If $e^{\alpha_2(z)-\alpha_1(z)}$ is not a constant, which implies that $\alpha_2(z)-\alpha_1(z)$ is a nonconstant polynomial. Then we claim that $\frac{iP(z)Q_2(z+\eta)}{Q_1(z)}e^{\alpha_2(z+\eta)-\alpha_1(z)}$ and $\frac{Q_2(z)}{Q_1(z)}e^{\alpha_2(z)-\alpha_1(z)}$ are not constants. Otherwise, we may assume

$$\frac{iP(z)Q_2(z+\eta)}{Q_1(z)}e^{\alpha_2(z+\eta)-\alpha_1(z)}\equiv c_1 \tag{3.14}$$

or

$$\frac{Q_2(z)}{Q_1(z)}e^{\alpha_2(z)-\alpha_1(z)}\equiv c_2, \tag{3.15}$$

where c_1, c_2 are two non-zero constants. Hence

$$e^{\alpha_2(z+\eta)-\alpha_1(z)}\equiv\frac{c_1Q_1(z)}{iP(z)Q_2(z+\eta)} \tag{3.16}$$

or

$$e^{\alpha_2(z)-\alpha_1(z)}\equiv\frac{c_2Q_1(z)}{Q_2(z)}. \tag{3.17}$$

We can see that the left side of (3.16) is a transcendental entire function, but the right side of it is a rational function, a contradiction. Therefore, $\frac{iP(z)Q_2(z+\eta)}{Q_1(z)}\times e^{\alpha_2(z+\eta)-\alpha_1(z)}$ is not a constant. Similarly, $\frac{Q_2(z)}{Q_1(z)}e^{\alpha_2(z)-\alpha_1(z)}$ also is not a constant. Then by Lemma 2.2, we have

$$iP(z)Q_1(z+\eta)e^{\alpha_1(z+\eta)-\alpha_1(z)}\equiv Q_1(z). \tag{3.18}$$

Hence, $\alpha_1(z)=A_1z+B_1$, where A_1 is a non-zero constant and B_1 is a constant. So, we obtain

$$iP(z)e^{A_1\eta}Q_1(z+\eta)\equiv Q_1(z). \tag{3.19}$$

Set

$$P(z)=a_kz^k+a_{k-1}z^{k-1}+\cdots+a_0,$$

and

$$Q_1(z) = b_s z^s + b_{s-1} z^{s-1} + \cdots + b_0,$$

where k is a nonnegative integer, $a_k(\neq 0), a_{k-1}, \ldots, a_0, b_s(\neq 0), b_{s-1}, \ldots, b_0$, are complex constants. Comparing the degree and the coefficients of highest degree of two sides of (3.19), we can deduce that $P(z)$ and $Q(z)$ are constants, we denote them by c and d_1 respectively. Moreover, $P(z) = c = \frac{1}{ie^{A_1\eta}}$. Then, combining (3.13) with (3.18), we have

$$icQ_2(z+\eta)e^{\alpha_2(z+\eta)} = -Q_2(z)e^{\alpha_2(z)}, \tag{3.20}$$

which implies that

$$e^{\alpha_2(z+\eta)-\alpha_2(z)} = -\frac{Q_2(z)}{icQ_2(z+\eta)}. \tag{3.21}$$

Thus $Q_2(z) = A_2 z + B_2$, where A_2 is a non-zero constant and B_2 is a constant. Then we get

$$ice^{A_2\eta}Q_2(z+\eta) \equiv -Q_2(z). \tag{3.22}$$

Set

$$Q_2(z) = c_l z^l + c_{l-1} z^{l-1} + \cdots + c_0,$$

where $l, c_l(\neq 0), c_{l-1}, \ldots, c_0$ are constants. Comparing the coefficients of highest degree of two sides of (3.22), we can obtain that $ice^{A_2\eta} = e^{(A_2-A_1)\eta} = -1$. Then we can deduce that $Q_2(z+\eta) = Q_2(z)$ is a constant, we may denote it by d_2. Hence, $f(z) = \frac{d_1 e^{A_1 z+B_1} + d_2 e^{A_2 z+B_2}}{2}$. From (3.18) and (3.20), we have $P^2(z)e^{(A_1+A_2)\eta} = 1$.

(2) If $e^{\alpha_2(z)-\alpha_1(z)}$ is a constant, which implies that $\alpha_2(z)-\alpha_1(z)$ is a constant. Then by (3.13), we deduce that $e^{\alpha_1(z+\eta)-\alpha_1(z)}$ is also a constant. Otherwise, we will get a contradiction easily. So we can deduce that $\alpha_1(z) = Az + B_1$ and $\alpha_2(z) = Az + B_2$, where A is a nonzero constant and B_1, B_2 are constants. Therefore, we have $f(z) = \beta(z)e^{Az}$, $\alpha(z) = 2Az$, where $\beta(z)$ satisfies $\beta^2(z) + e^{A\eta}\beta^2(z+\eta)P^2(z) = Q(z)$. Theorem 1.3 is proved. □

Proof of Theorem 1.5. Assume that $f(z)$ be a finite order transcendental entire solution of (1.11), then

$$[f(z) + iP(z)\Delta_\eta f(z)][f(z) - iP(z)\Delta_\eta f(z)] = Q(z)e^{\alpha(z)}. \tag{3.23}$$

Similar as in the proof of Theorem 1.3, we may assume that

$$f(z) + iP(z)\Delta_\eta f(z) = Q_1(z)e^{\alpha_1(z)}$$

and

$$f(z) - iP(z)\Delta_\eta f(z) = Q_2(z)e^{\alpha_2(z)},$$

where $Q_1(z), Q_2(z)$ are two non-zero polynomials, and $\alpha_1(z), \alpha_2(z)$ are two polynomials.

Then

$$f(z) = \frac{Q_1(z)e^{\alpha_1(z)} + Q_2(z)e^{\alpha_2(z)}}{2} \tag{3.24}$$

and

$$\Delta_\eta f(z) = \frac{Q_1(z)e^{\alpha_1(z)} - Q_2(z)e^{\alpha_2(z)}}{2iP(z)}. \tag{3.25}$$

It follows form (3.24) that $\alpha_1(z)$, $\alpha_2(z)$ can not be constants at same time, otherwise, $f(z)$ will be a polynomial, a contradiction. By (3.24) and (3.25), we have

$$\begin{aligned} &f(z+\eta) - f(z) \\ =& \frac{Q_1(z)e^{\alpha_1(z)} - Q_2(z)e^{\alpha_2(z)}}{2iP(z)} \\ =& \frac{Q_1(z+\eta)e^{\alpha_1(z+\eta)} + Q_2(z+\eta)e^{\alpha_2(z+\eta)} - Q_1(z)e^{\alpha_1(z)} + Q_2(z)e^{\alpha_2(z)}}{2}. \end{aligned} \tag{3.26}$$

Rewriting (3.26), we get

$$\begin{aligned} \frac{iP(z)Q_1(z+\eta)}{(iP(z)+1)Q_1(z)}e^{\alpha_1(z+\eta)-\alpha_1(z)} + \frac{iP(z)Q_2(z+\eta)}{(iP(z)+1)Q_1(z)}e^{\alpha_2(z+\eta)-\alpha_1(z)} \\ + \frac{(1-iP(z))Q_2(z)}{(iP(z)+1)Q_1(z)}e^{\alpha_2(z)-\alpha_1(z)} = 1. \end{aligned} \tag{3.27}$$

Next, we will consider two cases:

(I) $e^{\alpha_2(z)-\alpha_1(z)}$ is not a constant; (II) $e^{\alpha_2(z)-\alpha_1(z)}$ is a constant.

(I) If $e^{\alpha_2(z)-\alpha_1(z)}$ is not a constant, which implies that $\alpha_2(z) - \alpha_1(z)$ is a nonconstant polynomial. Similar as in the proof of Theorem 1.3, we also get that $\frac{iP(z)Q_2(z+\eta)}{(iP(z)+1)Q_1(z)}e^{\alpha_2(z+\eta)-\alpha_1(z)}$ and $\frac{(1-iP(z))Q_2(z)}{(1+iP(z))Q_1(z)}e^{\alpha_2(z)-\alpha_1(z)}$ are not constants. Then by Lemma 2.2, we have

$$iP(z)Q_1(z+\eta)e^{\alpha_1(z+\eta)-\alpha_1(z)} = (iP(z)+1)Q_1(z) \tag{3.28}$$

and

$$iP(z)Q_2(z+\eta)e^{\alpha_2(z+\eta)-\alpha_1(z)} = (iP(z)-1)Q_2(z). \tag{3.29}$$

Hence, $\alpha_1(z) = A_1z+B_1$, where A_1 is a non-zero constant and B_1 is a constant, and $\alpha_2(z) = A_2z + B_2$, where A_2 is a non-zero constant and B_2 is a constant. So, we obtain

$$iP(z)Q_1(z+\eta)e^{A_1\eta} = (iP(z)+1)Q_1(z) \tag{3.30}$$

and

$$iP(z)Q_2(z+\eta)e^{A_2\eta} = (iP(z)-1)Q_2(z). \tag{3.31}$$

Moreover, we will divide into two subcases.

(I-1) If $P(z)$ is a constant, we use c to denote it, then we deduce that $P(z) = c = \frac{1}{i(e^{A_1\eta}-1)}$, where $e^{A_1\eta} \neq 1$ and $Q_1(z+\eta) = Q_1(z)$ is a constant, we may denote it by d_1. From (3.31), we have

$$ice^{A_2\eta}Q_2(z+\eta) \equiv (ic-1)Q_2(z). \tag{3.32}$$

Then, we can see that $ic(e^{A_2\eta}-1)=\frac{e^{A_2\eta}-1}{e^{A_1\eta}-1}=-1$, where $e^{A_2\eta}\neq 1$. Thus, $e^{A_1\eta}+e^{A_2\eta}=2$, and $Q_2(z+\eta)=Q_2(z)$ is a constant, we may denote it by d_2. Hence, $f(z)=\frac{d_1e^{A_1z+B_1}+d_2e^{A_2z+B_2}}{2}$. From (3.30) and (3.31), we have $P^2(z)(e^{(A_1+A_2)\eta}-1)=1$.

(I-2) If $P(z)$ is not a constant, from (3.30) and (3.31), we can see that $e^{A_1\eta}=e^{A_2\eta}=1$. Noting that $Q(z)=Q_1(z)Q_2(z)$, by (3.30) and (3.31), we obtain that

$$P^2(z)(Q(z+\eta)e^{(A_1+A_2)\eta}-Q(z))=Q(z).$$

Comparing the degree of above equations, we will get a contradiction.

(II) If $e^{\alpha_2(z)-\alpha_1(z)}$ is a constant, which implies that $\alpha_2(z)-\alpha_1(z)$ is a constant. Then by (3.27), we deduce that $e^{\alpha_1(z+\eta)-\alpha_1(z)}$ also is a constant. Otherwise, we will get a contradiction easily. So we can deduce that $\alpha_1(z)=Az+B_1$ and $\alpha_2(z)=Az+B_2$, where A,B_1,B_2 are constants . Therefore, we have $f(z)=\beta(z)e^{Az}$, $\alpha(z)=2Az$, where $\beta(z)$ satisfies $\beta^2(z)+e^{A\eta}(\beta(z+\eta)-\beta(z))^2P^2(z)=Q(z)$. □

Acknowledgments. The author would like to thank the reviewers for their helpful comments and suggestions which have greatly improved the manuscript.

References

[1] Z. X. Chen, *Growth and zeros of meromorphic solution of some linear difference equations*, J. Math. Anal. Appl. **373** (2011), no. 1, 235–241.

[2] ———, *Zeros of entire solutions to complex linear difference equations*, Acta Math. Sci. Ser. B Engl. Ed. **32** (2012), no. 3, 1141–1148.

[3] W. Cherry and Z. Ye, *Nevanlinna's Theory of Value Distribution*, Berlin-Heidelberg: Springer-Verlag, 2001.

[4] Y. M. Chiang and S. J. Feng, *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane*, Ramanujan J. **16** (2008), no. 1, 105–129.

[5] ———, *On the growth of logarithmic differences, difference quotients and logarithmic derivatives of meromorphic functions*, Trans. Amer. Math. Soc. **361** (2009), no. 7, 3767–3791.

[6] W. Hayman, *Meromorphic Functions*, Oxford: Clarendon Press, 1975.

[7] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, Berlin-New York, Walter de Gruyter, 1993.

[8] B. Q. Li, *On certain non-linear differential equations in complex domains*, Arch. Math. (Basel) **91** (2008), no. 4, 344–353.

[9] K. Liu, *Meromorphic functions sharing a set with applications to difference equations*, J. Math. Anal. Appl. **359** (2009), no. 1, 384–393.

[10] K. Liu, T. B. Cao, and H. Z. Cao, *Entire solutions of Fermat type differential-difference equations*, Arch. Math. (Basel) **99** (2012), no. 2, 147–155.

[11] K. Liu and L. Z. Yang, *On entire solutions of some differential-difference equations*, Comput. Methods Funct. Theory **13** (2013), no. 3, 433–447.

[12] P. Montel, *Lecons sur les familles normales de fonctions analytiques et leurs applications*, Gauthier-Villars, Paris, (1927), 135–136 (French).

[13] J. F. Tang and L. W. Liao, *The transcendental meromorphic solutions of a certain type of nonlinear differential equations*, J. Math. Anal. Appl. **334** (2007), no. 1, 517–527.

[14] C. C. Yang, *On entire solutions of a certain type of nonlinear differential equations*, Bull. Aust. Math. Soc. **64** (2001), no. 3, 377–380.

[15] C. C. Yang and P. Li, *On the transcendental solutions of a certain type of nonlinear differential equations*, Arch. Math. (Basel) **82** (2004), no. 5, 442–448.
[16] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers, Dordrecht, 2003.
[17] X. Zhang and L. W. Liao, *On a certain type of nonlinear differential equations admitting transcendental meromorphic solutions*, Sci. China Math. **56** (2013), no. 10, 2025–2034.

Wei Chen
Department of Mathematics
Shandong University
Jinan, Shandong, 250100, P. R. China
E-mail address: weichensdu@126.com

Pei-Chu Hu
Department of Mathematics
Shandong University
Jinan, Shandong, 250100, P. R. China
E-mail address: pchu@sdu.edu.cn

Yingying Zhang
School of Mathematical Sciences
Xiamen University
Xiamen, Fujian, 361005, P. R. China
E-mail address: 450793707@qq.com