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CONTINUOUS HAMILTONIAN DYNAMICS AND AREA-PRESERVING HOMEOMORPHISM GROUP OF D^2

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Abstract. The main purpose of this paper is to propose a scheme of a proof of the nonsimpleness of the group $\mathrm{Homeo}^\Omega(D^2, \partial D^2)$ of area preserving homeomorphisms of the 2-disc D^2. We first establish the existence of Alexander isotopy in the category of Hamiltonian homeomorphisms. This reduces the question of extendability of the well-known Calabi homomorphism $\mathrm{Cal} : \mathrm{Diff}^\Omega(D^1, \partial D^2) \to \mathbb{R}$ to a homomorphism $\overline{\mathrm{Cal}} : \mathrm{Hameo}(D^2, \partial D^2) \to \mathbb{R}$ to that of the vanishing of the basic phase function $f_{\underline{\mathbb{F}}}$, a Floer theoretic graph selector constructed in [9], that is associated to the graph of the topological Hamiltonian loop and its normalized Hamiltonian $\underline{F}$ on S^2 that is obtained via the natural embedding $D^2 \hookrightarrow S^2$. Here $\mathrm{Hameo}(D^2, \partial D^2)$ is the group of Hamiltonian homeomorphisms introduced by Müller and the author [18]. We then provide an evidence of this vanishing conjecture by proving the conjecture for the special class of *weakly graphical* topological Hamiltonian loops on D^2 via a study of the associated Hamiton-Jacobi equation.

1. Introduction and statements of main results

1.1. Calabi homomorphism on D^2

Denote by $\mathrm{Diff}^\Omega(D^2, \partial D^2)$ the group of area-preserving diffeomorphisms supported in the interior of D^2 with respect to the standard area form $\Omega = dq \wedge dp$ on $D^2 \subset \mathbb{R}^2$. For any $\phi \in \mathrm{Diff}^\Omega(D^2, \partial D^2)$

$$\phi^*\Omega = \Omega$$

by definition. Write $\Omega = d\alpha$ for some choice of α. Then this equation leads to the statement that $\phi^*\alpha - \alpha$ is closed. Furthermore since ϕ is supported in the interior, the one-form

$$\phi^*\alpha - \alpha$$

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vanishes near ∂D^2 and so defines a de Rham cohomology class in $H^1(D^2, \partial D^2)$. Since the latter group is trivial, we can find a function $h_{\phi,\alpha}$ supported in the interior such that

$$dh_{\phi,\alpha} = \phi^*\alpha - \alpha. \tag{1.1}$$

Then the following is the well-known definition of Calabi invariant [3].

Definition 1.1 (Calabi invariant)**.** We define

$$\mathrm{Cal}(\phi) = \frac{1}{2}\int_{D^2} h_{\phi,\alpha}.$$

One can show that this value does not depend on the choice of the one-form α but depends only on the diffeomorphism. We will fix one such form α and so suppress the dependence α from our notation, and just denote $h_\phi = h_{\phi,\alpha}$.

Another equivalent definition does not involve the choice of one-form α but uses the 'past history' of the diffeomorphism in the setting of Hamiltonian dynamics [1]. More precisely, this definition implicitly relies on the following three facts:

(1) Ω on two dimensional surface is a symplectic form and hence
$$\mathrm{Diff}^\Omega(D^2, \partial D^2) = Symp_\omega(D^2, \partial D^2),$$
where $\omega = \Omega$.

(2) D^2 is simply connected, which in turn implies that any symplectic isotopy is a Hamiltonian isotopy.

(3) The group $\mathrm{Diff}^\Omega(D^2, \partial D^2)$ is contractible. (For this matter, finiteness of $\pi_1(\mathrm{Diff}^\Omega(D^2, \partial D^2), id) \cong \{0\}$ is enough.)

It is well-known (see [8], [14] for example) and easy to construct a sequence $\phi_i \in \mathrm{Diff}^\Omega(D^2, \partial D^2)$ such that $\phi_i \to id$ in C^0 topology but

$$\mathrm{Cal}(\phi_i) = 1$$

for all i's. This implies that Cal cannot be continuously extended to the full group $\mathrm{Homeo}^\Omega(D^2, \partial D^2)$ of area-preserving homeomorphisms.

However here is the main conjecture of the paper concerning nonsimpleness of the group $\mathrm{Homeo}^\Omega(D^2, \partial D^2)$. The author learned from A. Fathi in our discussion on the group $\mathrm{Hameo}(D^2, \partial D^2)$ [7] that the following will be important in relation to the study of nonsimpleness conjecture. (We refer to [13] for the argument needed to complete this nonsimpleness proof out of this conjecture.)

Conjecture 1.2. *Let* $\mathrm{Hameo}(D^2, \partial D^2) \subset \mathrm{Homeo}^\Omega(D^2, \partial D^2)$ *be the subgroup of Hamiltonian homeomorphisms on the two-disc. Then the Calabi homomorphism* $\mathrm{Cal} : \mathrm{Diff}^\Omega(D^2, \partial D^2) \to \mathbb{R}$ *extends continuously to* $\mathrm{Hameo}(D^2, \partial D^2)$ *in Hamiltonian topology in the sense of* [18].

For the study towards this conjecture, as we did in [13], we first define a homomorphism on the path spaces

$$\mathrm{Cal}^{path}(\lambda) : \mathcal{P}^{ham}(Symp(D^2, \partial D^2), id) \to \mathbb{R}$$

by

$$\mathrm{Cal}^{path}(\lambda) = \int_0^1 \int_{D^2} H(t,x)\,\Omega\,dt. \tag{1.2}$$

We will also denote this average by $\mathrm{Cal}(H)$ depending on the circumstances. Based on these facts (1) and (2), we can represent $\phi = \phi_H^1$ for the time-one map ϕ_H^1 of a time-dependent Hamiltonian $H = H(t,x)$ supported in the interior. Then based on (3) and some standard calculations in Hamiltonian geometry using the integration by parts, one proves that this integral does not depend on the choice of Hamiltonian $H \mapsto \phi$. Therefore it descends to $\mathrm{Ham}(D^2, \partial D^2) = \mathrm{Diff}^\Omega(D^2, \partial D^2)$. Then another application of Stokes' formula, one can prove that this latter definition indeed coincides with that of Definition 1.1. (See [1] for its proof.)

It is via this second definition how the author attempts to extend the classical Calabi homomorphism $\mathrm{Cal} : \mathrm{Ham}(D^2, \partial D^2) \to \mathbb{R}$ to its topological analog $\overline{\mathrm{Cal}} : \mathrm{Hameo}(D^2, \partial D^2) \to \mathbb{R}$. In [13], the definition (1.2) is extended to a homomorphism

$$\overline{\mathrm{Cal}}^{path} : \mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id) \to \mathbb{R}$$

on the set $\mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ of topological Hamiltonian paths. (See Section 2 for the precise definition.) Here following the notation from [18], we denote by $Sympeo(D^2, \partial D^2)$ the C^0-closure of $Symp(D^2, \partial D^2)$. Gromov-Eliashberg's C^0 symplectic rigidity theorem [5] states

$$Diff(D^2, \partial D^2) \cap Sympeo(D^2, \partial D^2) = Symp(D^2, \partial D^2).$$

In [13, 14], it is shown that a proof of descent of $\overline{\mathrm{Cal}}^{path}$ to the group

$$\mathrm{Hameo}(D^2, \partial D^2) := ev_1(\mathcal{P}^{ham}(Sympeo(M, \omega), id))$$

of Hamiltonian homeomorphisms (or more succinctly *hameomorphisms*) is reduced to the following extension result of Calabi homomorphism.

One important ingredient in our scheme towards the proof of Conjecture 1.2, which itself has its own interest, is the existence of the Alexander isotopy in the topological Hamiltonian category. Recall that the well-known Alexander isotopy on the disc D^2 exists in the homeomorphism category but not in the differentiable category. We will establish that such an Alexander isotopy defines contractions of topological Hamiltonian loops to the identity constant loop in the topological Hamiltonian category.

Theorem 1.3 (Alexander isotopy; Theorem 3.3)**.** *Any topological Hamiltonian loop in* $\mathrm{Hameo}(D^2, \partial D^2)$ *is contractible to the identity loop via topological Hamiltonian homotopy of loops.*

1.2. Basic phase function and Calabi invariant

The scheme of the proof of Conjecture 1.2 we propose is based on the following conjectural result of the *basic phase function* introduced in [9]. This conjecture is also a crucial ingredient needed in the proof of homotopy invariance of the spectral invariance of topological Hamiltonian paths laid out in [15]. Explanation of this conjecture is now in order.

Recall the classical action functional on T^*N for an arbitrary compact manifold N is defined as

$$\mathcal{A}^{cl}_H(\gamma) = \int \gamma^*\theta - \int_0^1 H(t,\gamma(t))\,dt$$

on the space $\mathcal{P}(T^*N)$ of paths $\gamma : [0,1] \to T^*N$, and its first variation formula is given by

(1.3)

$$d\mathcal{A}^{cl}_H(\gamma)(\xi) = \int_0^1 \omega(\dot\gamma - X_H(t,\gamma(t)),\xi(t))\,dt - \langle\theta(\gamma(0)),\xi(0)\rangle + \langle\theta(\gamma(1)),\xi(1)\rangle.$$

The basic phase function graph selector is canonical in that the assignment

$$H \mapsto f_H; \quad C^\infty([0,1]\times T^*N;\mathbb{R}) \to C^0(N)$$

varies continuously in (weak) Hamiltonian topology of $C^\infty([0,1]\times T^*N;\mathbb{R})$ [17, 18]. The construction f_H in [9] is given by considering the Lagrangian pair

$$(o_N, T^*_qN), \quad q \in N$$

and its associated Floer complex $CF(H; o_N, T^*_qN)$ generated by the Hamiltonian trajectory $z : [0,1] \to T^*N$ satisfying

$$\dot z = X_H(t,z(t)), \quad z(0) \in o_N,\ z(1) \in T^*_qN. \tag{1.4}$$

Denote by $\mathcal{C}hord(H; o_N, T^*_qN)$ the set of solutions of (1.4). The differential $\partial_{(H,J)}$ on $CF(H; o_N, T^*_qN)$ is provided by the moduli space of solutions of the perturbed Cauchy-Riemann equation

$$\begin{cases} \frac{\partial u}{\partial\tau} + J\left(\frac{\partial u}{\partial t} - X_H(u)\right) = 0 \\ u(\tau,0) \in o_N,\ u(\tau,1) \in T^*_qN. \end{cases} \tag{1.5}$$

The resulting spectral invariant $\rho^{lag}(H;[q])$ is to be defined by the mini-max value

$$\rho^{lag}(H;[q]) = \inf_{\alpha\in[q]} \lambda_H(\alpha),$$

where $[q]$ is a generator of the homology group $HF(o_N, T^*_qN) \cong \mathbb{Z}$. The basic phase function $f_H : N \to \mathbb{R}$ is then defined by $f_H(q) = \rho^{lag}(H;[q])$ first for generic $q \in N$ and then extending to the rest of M by continuity. (See [9] for the detailed construction and Section 5 of the present paper for a summary.)

Next we relate the basic phase function to the Calabi invariant on the two-disc as follows. Let F be a topological Hamiltonian generating a topological Hamiltonian path ϕ_F on the 2-disc D^2 with $\operatorname{supp} F \subset \operatorname{Int} D^2$. We consider an approximating sequence F_i with $\operatorname{supp} F_i \subset \operatorname{Int} D^2$. We embed D^2 into S^2 as the upper hemisphere and then extend F_i canonically to whole S^2 by zero.

We now specialize the above discussion on the basic phase function to the cases of the Lagrangianization of symplectic diffeomorphisms, i.e., consider their graphs

$$\operatorname{Graph} \phi = \{(\phi(x), x) \mid x \in S^2\} \subset S^2 \times S^2.$$

Applying this to $\phi^t_{\mathbb{F}_i}$ and noting $\operatorname{supp} \phi^t_{\mathbb{F}_i} \subset D^2_+ \times D^2_+$, we obtain

$$\operatorname{Graph} \phi^t_{F_i} \bigcap \Delta \supset \Delta_{D^2_-} \bigcup \Delta_{D^2_+ \setminus D^2_+(1-\delta)}$$

for some $\delta > 0$ for all $t \in [0,1]$, independently of sufficiently large i's but depending only on F. (See [18] or Definition 2.7 of the present paper for the precise definition of approximating sequence on open manifolds.) Then we consider the normalization $\underline{F_i}$ of F_i on S^2 and define Hamiltonian

$$\underline{\mathbb{F}_i}(t, \mathbf{x}) := \chi(\mathbf{x})\, \underline{F_i}(t, x), \quad \mathbf{x} = (x, y)$$

on $T^*\Delta$ with a slight abuse of notation for $\underline{\mathbb{F}_i}$, where χ is a cut-off function such that $\chi \equiv 1$ on a neighborhood V_Δ of Δ with

$$\operatorname{supp} \phi_F \subset V_\Delta \subset \overline{V}_\Delta \subset S^2 \times S^2 \setminus \Delta.$$

Two kinds of the associated generating functions, denoted by $\widetilde{h}_{\mathbb{F}_i}$ and $h_{\mathbb{F}_i}$ respectively, are given by

$$\widetilde{h}_{\mathbb{F}_i}(\mathbf{q}) = \mathcal{A}^{cl}_{\mathbb{F}}\left(z^{\mathbf{q}}_{\mathbb{F}_i}\right), \quad h_{\mathbb{F}_i}(\mathbf{x}) = \mathcal{A}^{cl}_{\mathbb{F}_i}\left(z^{\mathbb{F}_i}_{\mathbf{x}}\right), \tag{1.6}$$

where the Hamiltonian trajectories $z^{\mathbf{q}}_{\mathbb{F}_i}$ and $z^{\mathbb{F}_i}_{\mathbf{x}}$ are defined by

$$z^{\mathbf{q}}_{\mathbb{F}_i}(t) = \phi^t_{\mathbb{F}_i}(\mathbf{q}),\ \mathbf{q} \in o_\Delta,$$
$$z^{\mathbb{F}_i}_{\mathbf{x}}(t) = \phi^t_{\mathbb{F}_i}((\phi^1_{\mathbb{F}_i})^{-1}(\mathbf{x})),\ \mathbf{x} \in \phi^1_{\mathbb{F}_i}(o_\Delta).$$

We note that $z^{\mathbf{q}}_{\mathbb{F}_i}(0) = \mathbf{q}$ and $z^{\mathbb{F}_i}_{\mathbf{x}}(1) = \mathbf{x}$. Later we will review the definition from [9, 17] of the basic phase function $f_{\mathbb{F}_i}$ and the Lagrangian selector $\sigma_{\mathbb{F}_i}$. These maps have the properties that

$$f_{\mathbb{F}_i} = h_{\mathbb{F}_i} \circ \sigma_{\mathbb{F}_i} \tag{1.7}$$

and $\sigma_{\mathbb{F}_i}(\mathbf{q}) = (\mathbf{q}, df_{\mathbb{F}_i}(\mathbf{q})) \in T^*\Delta$ whenever $df_{\mathbb{F}_i}(\mathbf{q})$ exists. This ends the review of construction of basic phase function.

The following theorem exhibits the relationship between the limit of Calabi invariants and that of the basic phase function.

Theorem 1.4 (Theorem 6.1). *Let (M, ω) be an arbitrary closed symplectic manifold. Let $U = M \setminus B$ where B is a closed subset of nonempty interior. Let $\lambda = \phi_F$ be any engulfed topological Hamiltonian loop in $\mathcal{P}^{ham}(Sympeo_U(M, \omega), id)$ with $\phi^t_{F_i} \equiv id$ on B. Then*

$$\lim_{i \to \infty} f_{\underline{\mathbb{F}_i}}(x) = \frac{\overline{\mathrm{Cal}}_U(F)}{\mathrm{vol}_\omega(M)} \tag{1.8}$$

uniformly over $x \in M$, for any approximating sequence F_i of F. In particular, the limit function $f_{\underline{\mathbb{F}}}$ defined by $f_{\underline{\mathbb{F}}}(x) := \lim_{i\to\infty} f_{\underline{\mathbb{F}_i}}(x)$ is constant.

It is crucial for the equality (1.8) to hold in the general case that we are considering topological Hamiltonian *loop*, not just a path. (We refer readers to the proof of Theorem 6.1 to see how the loop property is used therein. We also refer to the proof of Lemma 7.5 [17] for a similar argument used for a similar purpose.)

The following is the main conjecture to beat which was previously proposed by the present author in [15].

Conjecture 1.5 (Main Conjecture)**.** *Let $M = S^2$ be the 2 sphere with standard symplectic structure. Let $\Lambda = \left\{\phi^t_{H(s)}\right\}_{(s,t)\in[0,1]^2}$ be a hameotopy contracting a topological Hamiltonian loop ϕ_F with $F = H(1)$ such that $H(s) \equiv id$ on D^2_- where D^2_- is the lower hemisphere of S^2. Then $f_{\underline{\mathbb{F}}} = 0$.*

It turns out that this conjecture itself is strong enough to directly give rise to a proof of Conjecture 1.2 in a rather straightforward manner with little usage of Floer homology argument in its outset except a few functorial properties of the basic phase function that are automatically carried by the Floer theoretic construction given in [9].

We indicate validity of this conjecture by proving the conjecture for the following special class consisting of weakly graphical topological Hamiltonian loops.

1.3. Graphical Hamiltonian diffeomorphism on D^2 and its Calabi invariant

We start with the following definition. We refer readers to Definition 4.2 for the definition of engulfed diffeomorphisms.

Definition 1.6. Let $\Psi : U_\Delta \to \mathcal{V}$ be a Darboux-Weinstein chart of the diagonal $\Delta \subset M \times M$ and denote $\pi_\Delta = \pi^\Psi_\Delta : U_\Delta \to \Delta$ to be the composition of Ψ followed by the canonical projection $T^*\Delta \to \Delta$.

1. We call an engulfed symplectic diffeomorphism $\phi : M \to M$ Ψ*-graphical* if the projection $\pi_\Delta|_{\mathrm{Graph}\,\phi} \to \Delta$ is one-to-one, and an engulfed symplectic isotopy is $\{\phi^t\}$ Ψ*-graphical* if each element ϕ^t Ψ-graphical. We call a Hamiltonian $F = F(t,x)$ Ψ-graphical if its associated Hamiltonian isotopy ϕ^t_F is Ψ-graphical.
2. We call a topological Hamiltonian loop F is strongly (resp. weakly) Ψ-graphical, if it admits an approximating sequence F_i each element of which is Ψ-graphical (resp. whose time-one map $\phi^1_{F_i}$ is Ψ-graphical).

Denote by F^a the time-dependent Hamiltonian generating the path $t \mapsto \phi^{at}_F$. The statement (2) of this definition is equivalent to saying that each F^a is Ψ-graphical for $a \in [0,1]$.

We remark that any symplectic diffeomorphisms sufficiently C^1-close to the identity is graphical, but not every C^0-close one. We also remark that $\pi_\Delta|_{\text{Graph}\,\phi}$ is surjective and hence a diffeomorphism if ϕ is a Ψ-graphical symplectic diffeomorphism isotopic to the identity via a Ψ-engulfed isotopy.

In 2 dimension, we prove the following interesting phenomenon. We doubt that similar phenomenon occurs in high dimension. This theorem will not be used in the proofs of main results of the present paper but has its own interest.

Theorem 1.7. *Let M be a closed 2 dimensional surface. Suppose $\phi : M \to M$ is a Ψ-graphical symplectic diffeomorphism isotopic to the identity via Ψ-graphical isotopy. and let* $\text{Graph}\,\phi = \text{Image}\,\alpha_\phi$ *for a closed one-form α_ϕ. Then for any $0 \leq r \leq 1$, the projection $\pi_2 : M \times M \to M$ restricts to a one-one map to* $\text{Image}\,r\,\alpha_\phi \subset M \times M$. *In particular*

$$\text{Image}\,r\,\alpha_\phi = \text{Graph}\,\phi_r \tag{1.9}$$

for some symplectic diffeomorphism $\phi_r : M \to M$ for each $0 \leq r \leq 1$.

Finally we prove Conjecture 1.5 for the weakly graphical topological Hamiltonian loop on S^2 that arises as follows.

Theorem 1.8. *Conjecture 1.5 holds for any weakly graphical topological Hamiltonian loop on S^2 arising from one on D^2 as in subsection 1.2.*

The proof of this theorem strongly relies on Theorem 1.3.

An immediate corollary of Theorems 1.4 and 1.8 is the following vanishing result of Calabi invariant.

Corollary 1.9. *Suppose $\lambda = \phi_F$ is a weakly graphical topological Hamiltonian loop on D^2. Then* $\overline{\text{Cal}}^{path}(\lambda) = 0$.

Unraveling the definitions, this corollary establishes the main conjecture with the additional graphicality hypothesis on ϕ_i.

Theorem 1.10. *Consider a sequence $\phi_i \in \text{Ham}(D^2, \partial D^2)$ that satisfies the following conditions*:

(1) *Each ϕ_i is graphical, and $\phi_i \to id$ in C^0-topology,*
(2) *$\phi_i = \phi^1_{H_i}$ with convergent H_i in $L^{(1,\infty)}$-topology.*

Then $\lim_{i\to\infty} \text{Cal}(\phi_i) = 0$.

We hope to study elsewhere general engulfed topological Hamiltonian loop dropping the graphicality condition.

Remark 1.11. Previously the author announced a 'proof' of the nonsimpleness result in [14] *modulo the proof of Conjecture 1.5* in which nonsimpleness is derived out of the homotopy invariance of spectral invariants whose proof also strongly relied on this vanishing result. Unlike the previously proposed scheme of the proof, the current scheme does not rely on the homotopy invariance of spectral invariants of topological Hamiltonian paths but more directly follows from the above mentioned vanishing result.

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Part 1. Calabi invariant and basic phase function

2. Calabi homomorphism $\overline{\mathrm{Cal}}^{path}$ on the path space

2.1. Hamiltonian topology and Hamiltonian homotopy

In [18], Müller and the author introduced the notion of Hamiltonian topology on the space

$$\mathcal{P}^{ham}(Symp(M, \omega), id)$$

of Hamiltonian paths $\lambda : [0,1] \to Symp(M, \omega)$ with $\lambda(t) = \phi_H^t$ for some time-dependent Hamiltonian H. We would like to emphasize that we do *not* assume that H is normalized *unless otherwise said explicitly*. This is because we need to consider both compactly supported and mean-normalized Hamiltonians and suitably transform one to the other in the course of the proof of the main theorem of this paper.

We first recall the definition of this Hamiltonian topology.

We start with the case of closed (M, ω). For a given continuous function $h : M \to \mathbb{R}$, we denote

$$\mathrm{osc}(h) = \max h - \min h.$$

We define the C^0-distance $\overline{d}$ on $Homeo(M)$ by the symmetrized C^0-distance

$$\overline{d}(\phi, \psi) = \max\left\{ d_{C^0}(\phi, \psi), d_{C^0}(\phi^{-1}, \psi^{-1}) \right\}$$

and the C^0-distance, again denoted by $\overline{d}$, on

$$\mathcal{P}^{ham}(Symp(M, \omega), id) \subset \mathcal{P}(Homeo(M), id)$$

by

$$\overline{d}(\lambda, \mu) = \max_{t \in [0,1]} \overline{d}(\lambda(t), \mu(t)).$$

The Hofer length of Hamiltonian path $\lambda = \phi_H$ is defined by

$$\mathrm{leng}(\lambda) = \int_0^1 \mathrm{osc}(H_t)\, dt = \|H\|.$$

Following the notations of [18], we denote by ϕ_H the Hamiltonian path

$$\phi_H : t \mapsto \phi_H^t;\ [0,1] \to \mathrm{Ham}(M,\omega)$$

and by $\mathrm{Dev}(\lambda)$ the associated normalized Hamiltonian

$$\mathrm{Dev}(\lambda) := \underline{H}, \quad \lambda = \phi_H, \tag{2.1}$$

where $\underline{H}$ is defined by

$$\underline{H}(t,x) = H(t,x) - \frac{1}{\mathrm{vol}_\omega(M)} \int_M H(t,x)\,\omega^n. \tag{2.2}$$

Definition 2.1. Let (M,ω) be a closed symplectic manifold. Let λ, μ be smooth Hamiltonian paths. The *Hamiltonian topology* is the metric topology induced by the metric

$$d_{ham}(\lambda,\mu) := \overline{d}(\lambda,\mu) + \mathrm{leng}(\lambda^{-1}\mu). \tag{2.3}$$

Now we recall the notion of topological Hamiltonian flows and Hamiltonian homeomorphisms introduced in [18].

Definition 2.2 ($L^{(1,\infty)}$ topological Hamiltonian flow)**.** A continuous map $\lambda : \mathbb{R} \to Homeo(M)$ is called a topological Hamiltonian flow if there exists a sequence of smooth Hamiltonians $H_i : \mathbb{R} \times M \to \mathbb{R}$ satisfying the following:

(1) $\phi_{H_i} \to \lambda$ locally uniformly on $\mathbb{R} \times M$.
(2) the sequence H_i is Cauchy in the $L^{(1,\infty)}$-topology locally in time and so has a limit H_∞ lying in $L^{(1,\infty)}$ on any compact interval $[a,b]$.

We call any such ϕ_{H_i} or H_i an *approximating sequence* of λ. We call a continuous path $\lambda : [a,b] \to Homeo(M)$ a *topological Hamiltonian path* if it satisfies the same conditions with $\mathbb{R}$ replaced by $[a,b]$, and the limit $L^{(1,\infty)}$-function H_∞ called a $L^{(1,\infty)}$ *topological Hamiltonian* or just a *topological Hamiltonian.*

Following the notations from [18], we denote by $Sympeo(M,\omega)$ the closure of $Symp(M,\omega)$ in $Homeo(M)$ with respect to the C^0-metric $\overline{d}$, and by $\mathcal{H}_m([0,1]\times M,\mathbb{R})$ the set of mean-normalized topological Hamiltonians, and by

$$ev_1 : \mathcal{P}^{ham}_{[0,1]}(Sympeo(M,\omega), id) \to Sympeo(M,\omega), id) \tag{2.4}$$

the evaluation map defined by $ev_1(\lambda) = \lambda(1)$. By the uniqueness theorem of Buhovsky-Seyfaddini [2], we can extend the map Dev given in (2.1) to

$$\overline{\mathrm{Dev}} : \mathcal{P}^{ham}_{[0,1]}(Sympeo(M,\omega), id) \to \mathcal{H}_m([0,1]\times M,\mathbb{R})$$

in an obvious way. Following the notation of [13, 18], we denote the topological Hamiltonian path $\lambda = \phi_H$ when $\overline{\mathrm{Dev}}(\lambda) = \underline{H}$ in this general context.

Definition 2.3 (Hamiltonian homeomorphism group)**.** We define

$$\mathrm{Hameo}(M,\omega) = ev_1\left(\mathcal{P}^{ham}_{[0,1]}(Sympeo(M,\omega), id)\right)$$

and call any element therein a *Hamiltonian homeomorphism.*

The group property and its normality in $Sympeo(M, \omega)$ are proved in [18].

Theorem 2.4 ([18])**.** *Let* (M, ω) *be a closed symplectic manifold. Then*

$$\mathrm{Hameo}(M, \omega)$$

is a normal subgroup of $Sympeo(M, \omega)$*.*

Especially when $\dim \Sigma = 2$, we have a smoothing result

$$Sympeo(\Sigma, \omega) = \mathrm{Homeo}^{\Omega}(\Sigma) \tag{2.5}$$

of area-preserving homeomorphisms by area-preserving diffeomorphisms (see [11], [21] for a proof). Therefore combining this with the above theorem, we obtain the following corollary, which is the starting point of our research to apply continuous Hamiltonian dynamics to the study of the simpleness question of the area-preserving homeomorphism group of D^2 (or S^2).

Corollary 2.5. *Let* Σ *be a compact surface with or without boundary and let* Ω *be an area form of* Σ*, which we also regard as a symplectic form* $\omega = \Omega$*. Then* $\mathrm{Hameo}(M, \omega)$ *is a normal subgroup of* $\mathrm{Homeo}^{\Omega}(\Sigma)$*.*

Both results have their counterparts even when $\partial M \neq \emptyset$. We refer to the discussion below at the end of this subsection.

Next we consider the notion of homotopy in this topological Hamiltonian category. The following notion of Hamiltonian homotopy, which we abbreviate as *hameotopy*, of topological Hamiltonian paths is introduced in [14, 16]. The guiding principle for the choice of this as the definition of homotopy in this topological Hamiltonian category is to include the Alexander isotopy we define in Section 3 as a special case.

Definition 2.6 (Hameotopy)**.** Let λ_0, $\lambda_1 \in \mathcal{P}^{ham}(Sympeo(M, \omega), id)$. A hameotopy $\Lambda : [0,1]^2 \to Sympeo(M, \omega)$ between λ_0 and λ_1 based at the identity is a map such that

$$\Lambda(0, t) = \lambda_0(t),\ \Lambda(1, t) = \lambda_1(t), \tag{2.6}$$

and $\Lambda(s, 0) \equiv id$ for all $s \in [0,1]$, and which arises as follows: there is a sequence of smooth maps $\Lambda_j : [0,1]^2 \to \mathrm{Ham}(M, \omega)$ that satisfy

(1) $\Lambda_j(s, 0) = id$,
(2) $\Lambda_j \to \Lambda$ in C^0-topology,
(3) Any s-section $\Lambda_{j,s} : \{s\} \times [0,1] \to \mathrm{Ham}(M, \omega)$ converges in Hamiltonian topology in the following sense: If we write

$$\mathrm{Dev}\left(\Lambda_{j,s} \Lambda_{j,0}^{-1}\right) =: H_j(s),$$

then $H_j(s)$ converges in Hamiltonian topology uniformly over $s \in [0,1]$. We call any such Λ_j an *approximating sequence* of Λ.

When $\lambda_0(1) = \lambda_1(1) = \psi$, a *hameotopy relative to the ends* is one that satisfies $\Lambda(s, 0) = id$, $\Lambda(s, 1) = \psi$ for all $s \in [0,1]$ in addition.

We say that λ_0, $\lambda_1 \in \mathcal{P}^{ham}(Sympeo(M,\omega), id)$ are *hameotopic* (resp. relative to the ends), if there exists a hameotopy (resp. a hameotopy relative to the ends).

We emphasize that by the requirement (3),

$$H_j(0) \equiv 0 \tag{2.7}$$

in this definition.

All the above definitions can be modified to handle the case of open manifolds, either noncompact or compact with boundary, by considering H's compactly supported in the interior as done in Section 6 [18]. We recall the definitions of topological Hamiltonian paths and Hamiltonian homeomorphisms supported in an open subset $U \subset M$ from [18].

We define $\mathcal{P}^{ham}(Symp_U(M,\omega), id)$ to be the set of smooth Hamiltonian paths supported in U. The following definition is taken from Definition 6.2 [18] to which we refer readers for more detailed discussions. First for any open subset $V \subset U$ with compact closure $\overline{V} \subset U$, we can define a completion of $\mathcal{P}^{ham}(Symp_{\overline{V}}(M,\omega), id)$ using the same metric given above.

Definition 2.7. Let $U \subset M$ be an open subset. Define $\mathcal{P}^{ham}(Sympeo_U(M,\omega), id)$ to be the union

$$\mathcal{P}^{ham}(Sympeo_U(M,\omega), id) := \bigcup_{K\subset U} \mathcal{P}^{ham}(Sympeo_K(M,\omega), id)$$

with the direct limit topology, where $K \subset U$ is a compact subset. We define $Hameo_c(U,\omega)$ to be the image

$$Hameo_c(U,\omega) := ev_1(\mathcal{P}^{ham}(Sympeo_U(M,\omega), id)).$$

We would like to emphasize that this set is not necessarily the same as the set of $\lambda \in \mathcal{P}^{ham}(Sympeo(M,\omega), id)$ with compact $\operatorname{supp}\lambda \subset U$. The same definition can be applied to general open manifolds or manifolds with boundary.

2.2. Calabi invariants of topological Hamiltonian paths in D^2

Denote by $\mathcal{P}^{ham}(Symp(D^2, \partial D^2); id)$ the group of Hamiltonian paths supported in $\operatorname{Int}(D^2)$, i.e.,

$$\bigcup_{t\in[0,1]} \operatorname{supp} H_t \subset \operatorname{Int}(D^2).$$

We denote by $\mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ the $L^{(1,\infty)}$ Hamiltonian completion of $\mathcal{P}^{ham}(Symp(D^2, \partial D^2); id)$.

We recall the extended Calabi homomorphism defined in [13] whose well-definedness follows from the uniqueness theorem from [2].

Definition 2.8. Let $\lambda \in \mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ and H be its Hamiltonian supported in $\operatorname{Int} D^2$. We define

$$\overline{\mathrm{Cal}}^{path}(\lambda) = \overline{\mathrm{Cal}}^{path}_{D^2}(\lambda) := \overline{\mathrm{Cal}}(H),$$

where we define $\overline{\mathrm{Cal}}(H) = \lim_{i\to\infty} \mathrm{Cal}(H_i)$ for an (and so any) approximating sequence H_i of H.

It is immediate to check that this defines a homomorphism. The main question to be answered is whether this homomorphism descends to the group $\mathrm{Hameo}(D^2, \partial D^2)$. We recall that two crucial ingredients needed in the proof of well-definedness of this form of the Calabi invariant defined on $\mathrm{Diff}^\Omega(D^2, \partial D^2)$ of area-preserving *diffeomorphisms* is the fact that $\mathrm{Diff}^\Omega(D^2, \partial D^2) = \mathrm{Ham}(D^2, \partial D^2)$ and that it is contractible. In this regard, we would like to prove the following conjecture.

Conjecture 2.9. *Let λ be a contractible topological Hamiltonian loop based at the identity. Then*

$$\overline{\mathrm{Cal}}^{path}(\lambda) = 0.$$

In the next section, we will establish the existence of Alexander isotopy in the topological Hamiltonian category and prove that any topological Hamiltonian loop (based at the identity) on D^2 is indeed contractible and so the contractibility hypothesis in this conjecture automatically holds.

By the homomorphism property of $\overline{\mathrm{Cal}}^{path}$, an immediate corollary of this conjecture would be the following: Suppose that Conjecture 2.9 holds. Let

$$\overline{\mathrm{Cal}}^{path} : \mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id) \to \mathbb{R}$$

be the above extension of the Calabi homomorphism Cal^{path} such that $\lambda_0(1) = \lambda_1(1)$. Then we have

$$\overline{\mathrm{Cal}}^{path}(\lambda_0) = \overline{\mathrm{Cal}}^{path}(\lambda_1).$$

In the next section, we will elaborate this point further.

3. Alexander isotopy of loops in $\mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$

For the description of Alexander isotopy, we need to consider the conjugate action of rescaling maps of D^2

$$R_a : D^2(1) \to D^2(a) \subset D^2(1)$$

for $0 < a < 1$ on $\mathrm{Hameo}(D^2, \partial D^2)$, where $D^2(a)$ is the disc of radius a with its center at the origin. We note that R_a is a conformally symplectic map and so its conjugate action maps a symplectic map to a symplectic map whenever it is defined.

Furthermore the right composition by R_a defines a map

$$\phi \mapsto \phi \circ R_a^{-1} ;\ \mathrm{Hameo}(D^2(a), \partial D^2(a)) \subset \mathrm{Hameo}(D^2, \partial D^2) \to Homeo(D^2, \partial D^2)$$

and then the left composition by R_a followed by extension to the identity on $D^2 \setminus D^2(a)$ defines a map

$$\mathrm{Hameo}(D^2, \partial D^2) \to \mathrm{Hameo}(D^2, \partial D^2).$$

We have the following important formula for the behavior of Calabi invariants under the Alexander isotopy.

Lemma 3.1. *Let* $\lambda \in \mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ *be a given continuous Hamiltonian path on* D^2*. Suppose* $\operatorname{supp} \lambda \subset D^2(1-\eta)$ *for a sufficiently small* $\eta > 0$*. Consider the one-parameter family of maps* λ_a *defined by*

$$\lambda_a(t,x) = \begin{cases} a\lambda(t, \frac{x}{a}) & \text{for } |x| \le a(1-\eta) \\ x & \text{otherwise} \end{cases}$$

for $0 < a \le 1$*. Then* λ_a *is also a topological Hamiltonian path on* D^2 *and satisfies*

$$\overline{\mathrm{Cal}}^{path}(\lambda_a) = a^4 \overline{\mathrm{Cal}}^{path}(\lambda). \tag{3.1}$$

Proof. A straightforward calculation proves that λ_a is generated by the (unique) continuous Hamiltonian, which we denote by $\mathrm{Dev}(\lambda_a)$ following the notation of [18, 13], which is defined by

$$\mathrm{Dev}(\lambda_a)(t,x) = \begin{cases} a^2 H\left(t, \frac{x}{a}\right) & \text{for } |x| \le a(1-\eta) \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

where $H = \mathrm{Dev}(\lambda)$: Obviously the right hand side function is the Hamiltonian-limit of $\mathrm{Dev}(\lambda_{i,a})$ for a sequence λ_i of smooth Hamiltonian approximation of λ where $\lambda_{i,a}$ is defined by the same formula for λ_i.

From these, we derive the formula

$$\begin{aligned} \overline{\mathrm{Cal}}^{path}(\lambda_a) &= \lim_{i\to\infty} \mathrm{Cal}^{path}(\lambda_{i,a}) \\ &= a^4 \lim_{i\to\infty} \int_0^1 \int_{D^2} H_i(t,y)\Omega \wedge dt \\ &= a^4 \lim_{i\to\infty} \mathrm{Cal}^{path}(\lambda_i) = a^4 \overline{\mathrm{Cal}}^{path}(\lambda). \end{aligned}$$

This proves (3.1). □

We would like to emphasize that the s-Hamiltonian F_Λ of $\Lambda(s,t) = \lambda_s^t$ does not converge in $L^{(1,\infty)}$-topology and so we cannot define its Hamiltonian limit.

Explanation of this relationship is now in order in the following remark.

Remark 3.2. Let $D^{2n} \subset \mathbb{R}^{2n}$ be the unit ball. Consider a smooth Hamiltonian H with $\operatorname{supp} \phi_H \subset \operatorname{Int} D^{2n} \subset \mathbb{R}^{2n}$ and its Alexander isotopy

$$\Lambda(s,t) = \phi_{H^s}^t = \lambda_s(t), \quad \lambda = \phi_H.$$

Denote by H_Λ and K_Λ the t-Hamiltonian and the s-Hamiltonian respectively. Then we have Banyaga's formula $\frac{\partial H}{\partial s} = \frac{\partial K}{\partial t} - \{H, K\}$ which is equivalent to the formula

$$\frac{\partial K}{\partial t} = \frac{\partial}{\partial s}(H \circ \phi_{H^s}^t) \circ (\phi_{H^s}^t)^{-1}. \tag{3.3}$$

(See p. 78 [10] for its derivation, for example.) But we compute

$$H_t \circ \phi_{H^s}^t(x) = s^2 H_t\left(\frac{\phi_{H^s}^t(x)}{s}\right) = s^2 H\left(t, \frac{\phi_{H^s}^t(x)}{s}\right).$$

Therefore we derive

(3.4)

$$\begin{aligned} &K(s,t,x) \\ &= 2s\int_0^t H\left(u,\frac{x}{s}\right)du + s\int_0^t \left\langle \left(d\overline{H}\left(u,\frac{(\phi_{H^s}^u)^{-1}(x)}{s}\right)\right),(\phi_{H^s}^u)^{-1}(x)\right\rangle du.\end{aligned}$$

For the second summand, we use the identity $\overline{H}(t,x) = -H(t,\phi_H^t(x))$. From this expression, we note that K involves differentiating the Hamiltonian H and hence goes out of the $L^{(1,\infty)}$ Hamiltonian category.

Recall that the well-known Alexander isotopy on the disc D^2 exists in the homeomorphism category but not in the differentiable category. We will establish that such an Alexander isotopy defines contractions of topological Hamiltonian loops to the identity constant loop in the topological Hamiltonian category.

Theorem 3.3. *Let λ be a loop in $\mathcal{P}^{ham}(Sympeo(D^2,\partial D^2),id)$. Define $\Lambda : [0,1]^2 \to Sympeo(D^2,\partial D^2)$ by*

$$\Lambda(s,t) = \lambda_s(t).$$

Then Λ is a hameotopy between λ and the constant path id.

Proof. We have $\lambda \in \mathcal{P}^{ham}(Sympeo(D^2,\partial D^2),id)$ with $\lambda(0)=\lambda(1)$. Then λ_s defines a loop contained in $\mathcal{P}^{ham}(Sympeo(D^2,\partial D^2),id)$ for each $0 \le s \le 1$. Let H_i be an approximating sequence of the topological Hamiltonian loop λ.

We fix a sequence $\varepsilon_i \searrow 0$ and define a 2-parameter Hamiltonian family $\Lambda_{i,\varepsilon_i}$ defined by

$$\Lambda_{i,\varepsilon_i}(s,t) := \lambda_{i,\chi_i(s)}(t,\cdot)\circ\lambda_{i,\varepsilon_i}^{-1}(t,\cdot), \tag{3.5}$$

where $\chi_i : [0,1] \to [\varepsilon_i,1]$ is a monotonically increasing surjective function with $\chi_i(t) = \varepsilon_i$ near $t=0$, $\chi_i(1) = 1$ near $t=1$, and $\chi_i \to id_{[0,1]}$ in the Hamiltonian norm (see Definition 3.19 and Lemma 3.20 [18] for this fact). It follows that the sequence $\Lambda_{i,\varepsilon_i}$ is smooth and uniformly converges in Hamiltonian topology as $i \to \infty$ over $s \in [0,1]$ and $\Lambda_{i,\varepsilon_i}^t(1) \to \lambda(t)$ since the Alexander isotopy is smooth as long as $s > 0$ and by definition $\Lambda_{i,\varepsilon_i}$ involves the Alexander isotopy for $s \ge \varepsilon_i > 0$. The convergence immediately follows from the explicit expression of λ_a in Lemma 3.1.

Finally we need to check

$$\|\operatorname{Dev}(\Lambda_{i,\varepsilon_i}(s,\cdot)) - \operatorname{Dev}(\Lambda_{j,\varepsilon_j}(s,\cdot)\| \to 0 \tag{3.6}$$

uniformly over $s \in [0,1]$ as $i,\, j \to \infty$. For this, we apply the standard formula of Dev for the composed flow,

$$\mathrm{Dev}(\lambda\mu^{-1})(t,x) = \mathrm{Dev}(\lambda)(t,x) - \mathrm{Dev}(\mu)(t, \mu_t\lambda_t^{-1}(x))$$

to $\Lambda_{i,\varepsilon_i} := \lambda_{i,\chi_i(s}(t,\cdot) \circ \lambda_{i,\varepsilon_i}^{-1}(t,\cdot)$, which amounts to the more familiar formula $(H\#\overline{G})_t = H_t - G_t \circ \phi_G^t(\phi_H^t)^{-1}$ in the literature. Then we get

(3.7)

$$\mathrm{Dev}(\Lambda_{i,\varepsilon_i}(s,\cdot))(t,x) = \mathrm{Dev}(\lambda_{i,\chi_i(s)})(t,x) - \mathrm{Dev}(\lambda_{i,\varepsilon_i})(t, \lambda_{i,\varepsilon_i}^t \circ (\lambda_{i,\chi_i(s)}^t)^{-1}(x)),$$

where

$$\mathrm{Dev}(\lambda_{i,\chi_i(s)})(t,x) = \begin{cases} \chi_i(s)^2 H_i(t, \frac{x}{\chi_i(s)}) & \text{for } |x| \leq \chi_i(s)(1-\eta) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{Dev}(\lambda_{i,\varepsilon_i})(t,x) = \begin{cases} \varepsilon_i^2 H_i(t, \frac{x}{\varepsilon_i}) & \text{for } |x| \leq \varepsilon_i(1-\eta) \\ 0 & \text{otherwise.} \end{cases}$$

From these expressions, (3.6) immediately follows. This finishes the proof. □

Corollary 3.4. *If $\lambda_0,\, \lambda_1 \in \mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ and $\lambda_0(1) = \lambda_1(1)$, then they are hameotopic relative to the end.*

Proof. Theorem 3.3 implies that the standard Alexander isotopy given in Lemma 3.1 is a hameotopy contracting any topological Hamiltonian loop to the identity in $\mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ with ends points fixed. This proves that the product loop $\lambda_0\lambda_1^{-1}$, which is based at the identity, is contractible via a hameotopy relative to the ends. Then this implies that λ_0 and λ_1 are hameotopic to each other relative to the ends. □

An immediate consequence of Corollary 3.4 is the following.

Proposition 3.5. *Suppose Conjecture 2.9 holds. Then we have*

$$\overline{\mathrm{Cal}}^{path}(\lambda_0) = \overline{\mathrm{Cal}}^{path}(\lambda_1)$$

if $\lambda_0,\, \lambda_1 \in \mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ and $\lambda_0(1) = \lambda_1(1)$.

This theorem implies that $\overline{\mathrm{Cal}}^{path}$ restricted to $\mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ depends only on the final point and so gives rise to the following main theorem on the extension of Calabi homomorphism.

Theorem 3.6. *Suppose Conjecture 2.9 holds. Define a map*

$$\overline{\mathrm{Cal}} : \mathrm{Hameo}(D^2, \partial D^2) \to \mathbb{R}$$

by

$$\overline{\mathrm{Cal}}(g) := \overline{\mathrm{Cal}}^{path}(\lambda)$$

for a (and so any) $\lambda \in \mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ *with* $g = \lambda(1)$. *Then this is well-defined and extends the Calabi homomorphism* $\mathrm{Cal} : \mathrm{Diff}^{\Omega}(D^2, \partial D^2) \to \mathbb{R}$ *to*

$$\overline{\mathrm{Cal}} : \mathrm{Hameo}(D^2, \partial D^2) \to \mathbb{R}.$$

Once this theorem is established, nonsimpleness of $\mathrm{Hameo}(D^2, \partial D^2)$ immediately follows from Conjecture 2.9. (See [13] for the needed argument.)

4. Reduction to the engulfed case and its Lagrangianization

In this section, we reduce the proof of Conjecture 2.9 to the engulfed topological Hamiltonian loops on S^2. Using the given identification of D^2 as the upper hemi-sphere denoted by D^2_+, we can embed

$$\iota^+ : \mathcal{P}^{ham}(Symp(D^2, \partial D^2); id) \hookrightarrow \mathcal{P}^{ham}(Symp(S^2); id)$$

by extending any element $\phi_H \in \mathcal{P}^{ham}(Symp(D^2, \partial D^2); id)$ to the one that is identity on the lower hemisphere D^2_- by setting $H \equiv 0$ thereon.

We first recall the definition of engulfed Hamiltonians from [16].

Definition 4.1. Let (M, ω) be a symplectic manifold. Let a Darboux-Weinstein chart

$$\Phi : \mathcal{V} \subset T^*\Delta \to U_\Delta \subset (M \times M, \omega \oplus -\omega)$$

be given. We call $\mathcal{U}$ a Darboux-Weinstein neighborhood of the diagonal with respect to Φ. In general we call a neighborhood U_Δ of the diagonal a *Darboux-Weinstein neighborhood* if it is the image of a Darboux-Weinstein chart.

With this preparation, we are ready to recall the following definition from [16].

Definition 4.2.
(1) An isotopy of Lagrangian submanifold $\{L_t\}_{0 \leq s \leq 1}$ of L is called V*-engulfed* if there exists a Darboux neighborhood V of L such that $L_s \subset V$ for all s. When we do not specify V, we just call the isotopy engulfed.
(2) We call a (topological) Hamiltonian path ϕ_H $\mathcal{U}$-engulfed if its graph $\mathrm{Graph}\,\phi_H^t$ is engulfed in a Darboux-Weinstein neighborhood $\mathcal{U}$ of the diagonal Δ of $(M \times M, \omega \oplus -\omega)$.

Now let $\lambda = \phi_F$ be a contractible topological Hamiltonian loop contained in $\mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ and $\Lambda = \{\lambda(s)\}_{s \in [0,1]}$ a given hameotopy contracting the loop.

Let $\lambda \in \mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ and consider its extension $\iota^+(\lambda)$ as an element in $\mathcal{P}^{ham}(Sympeo_{D^+}(S^2), id)$ obtained via the embedding ι^+. Denote by $D^1(T^*S^2)$ the unit cotangent bundle and by $\overline{\Delta}$ the anti-diagonal

$$\overline{\Delta} = \{(x, \overline{x}) \in S^2 \times S^2 \mid x \in S^2\}.$$

Then it is well-known that the geodesic flow of the standard metric on S^2 induces a symplectic diffeomorphism

$$\Phi : D^1(T^*S^2) \to S^2 \times S^2 \setminus \overline{\Delta}, \tag{4.1}$$

where $\overline{x}$ is the involution along a (fixed) equator. We regard the image $\mathcal{U} = S^2 \times S^2 \setminus \overline{\Delta}$ as a Darboux-Weinstein neighborhood of the diagonal $\Delta \subset S^2 \times S^2$.

It is then easy to see the following:

Lemma 4.3. *Let $\lambda \in \mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ and denote by $\lambda^+ = \iota^+(\lambda)$ $\in \mathcal{P}^{ham}(Sympeo_{D^+}(S^2), id)$ constructed as above. Then*

$$(\lambda_t^+ \times id)(\Delta) \cap \overline{\Delta} = \emptyset.$$

In particular, the path λ^+ is $\mathcal{U}$-engulfed.

Motivated by the above discussion, we will always consider only the engulfed case in the rest of the paper, unless otherwise said.

Now let $F : [0,1] \times M \to \mathbb{R}$ be a mean normalized engulfed Hamiltonian on a closed symplectic manifold (M, ω). The manifold M carries a natural Liouville measure induced by ω^n. Consider the diagonal Lagrangian $\Delta \subset (M \times M, \omega \oplus -\omega)$ identified with the zero section $o_\Delta \subset T^*\Delta$ in a Darboux chart $(V_\Delta, -d\Theta)$ of Δ in $M \times M$. Put a density ρ_Δ on $\Delta \subset M \times M$ induced by ω^n by the diffeomorphism of the second projection $\pi_2 : \Delta \to M$.

We fix Darboux neighborhoods

$$V_\Delta \subset \overline{V}_\Delta \subset U_\Delta$$

and let $\omega \oplus -\omega = -d\Theta$ on U_Δ regarded as a neighborhood of the zero section of $T^*\Delta$ once and for all. Then

$$\operatorname{Graph} \phi_F^t \subset V_\Delta \quad \text{for all } t \in [0,1].$$

Here we define

$$\operatorname{Graph} \phi_F^t := \{(\phi_F^t(y), y) \mid y \in M\}.$$

We consider the Hamiltonian π_1^*F, i.e., the one defined by

$$\pi_1^*F(t,(x,y)) = F(t,x)$$

on $T^*\Delta$. This itself is not supported in U_Δ but we can multiply a cut-off function χ of U_Δ so that

$$\chi \equiv 1 \quad \text{on } V_\Delta, \quad \operatorname{supp} \chi \subset U_\Delta$$

and consider the function $\mathbb{F}$ defined by

$$\mathbb{F}(t,(x,y)) = \chi(x,y)\pi_1^*F(t,(x,y)) = \chi(x,y)F(t,x)$$

so that the associated Hamiltonian deformations of $\psi^t(o_N)$ are unchanged. We note that $\mathbb{F}$ *is compactly supported in* $T^*\Delta$ and automatically satisfies the normalization condition

$$\int_\Delta \mathbb{F}(t, \phi_{\mathbb{F}}^t(q))\, \rho_\Delta = 0 \tag{4.2}$$

for all $t \in [0,1]$ where ρ_Δ is the measure on Δ induced by the Liouville measure on M under the projection $\pi_2 : \Delta \subset M \times M \to M$.

Now we denote by $f_{\mathbb{F}}$ the basic phase function of $\operatorname{Graph} \phi_F^1 = \phi_{\mathbb{F}}^1(o_\Delta)$. In the next section, we will examine the relationship between this function and the Calabi invariant of F.

5. Basic phase function f_H and its axioms

In this section, we first recall the definition of *basic phase function* constructed in [9] and summarize its axiomatic properties. Following the terminology of [19], we first introduce the following definition.

Definition 5.1. Let $L \subset T^*N$ be a Hamiltonian deformation of the zero section o_N. We call any continuous function $f : N \to \mathbb{R}$ a *graph selector* such that

$$(q, df(q)) \in L,$$

where $df(q)$ exists.

Existence of such a single-valued continuous function was proved by Sikorav, Chaperon [4] by the generating function method and by the author [9] using the Lagrangian Floer theory. Lipschitz continuity of this particular graph selector follows from the continuity result established in Section 6 [9] specialized to the submanifold S to be a point. The detail of another proof of this Lipschitz continuity is also given in [19] using the generating function techniques.

We denote by $\operatorname{Sing} f$ the set of non-differentiable points of f. Then by definition

$$N_0 = \operatorname{Reg} f := N \setminus \operatorname{Sing} f$$

is a subset of full measure and f is differentiable thereon. In fact, for a generic choice of $L = \phi_H^1(o_N)$, N_0 is open and dense and $\operatorname{Sing} f$ is a stratified submanifold of N of codimension at lease 1. (See [17] for its proof.)

By definition,

$$|df(q)| \leq \max_{x \in L} |p(x)| \tag{5.1}$$

for any $q \in N_0$, where $x = (q(x), p(x))$ and the norm $|p(x)|$ is measured by any given Riemannian metric on N.

The following is an immediate corollary of the definition. We denote by d_{H} the Hausdorff distance.

Corollary 5.2. *As* $d_{\mathrm{H}}(\phi_H^1(o_N), o_N) \to 0$, $|df(q)| \to 0$ *uniformly over* $q \in N_0$.

However this result itself does not tell us much about the convergence of the values of the function f itself because a priori the value of f might not be bounded for a sequence H_i such that $d_{\mathrm{H}}(\phi_H^1(o_N), o_N) \to 0$.

In [9], a canonical choice of f is constructed via the chain level Floer theory, *provided* the generating Hamiltonian H of $L = \phi_H^1(o_N)$ is given. The author called the corresponding graph selector f the *basic phase function* of $L =$

$\phi_H^1(o_N)$ and denoted it by f_H. We give a quick outline of the construction referring the readers to [9] for the full details of the construction.

Consider the Lagrangian pair

$$(o_N, T_q^*N), \quad q \in N$$

and its associated Floer complex $CF(H; o_N, T_q^*N)$ generated by the Hamiltonian trajectory $z : [0,1] \to T^*N$ satisfying

$$\dot{z} = X_H(t, z(t)), \quad z(0) \in o_N,\, z(1) \in T_q^*N. \tag{5.2}$$

Denote by $\mathcal{C}hord(H; o_N, T_q^*N)$ the set of solutions. The differential $\partial_{(H,J)}$ on $CF(H; o_N, T_q^*N)$ is provided by the moduli space of solutions of the perturbed Cauchy-Riemann equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + J\left(\frac{\partial u}{\partial t} - X_H(u)\right) = 0 \\ u(\tau, 0) \in o_N,\, u(\tau, 1) \in T_q^*N. \end{cases} \tag{5.3}$$

An element $\alpha \in CF(H; o_N, T_q^*N)$ is expressed as a finite sum

$$\alpha = \sum_{z \in \mathcal{C}hord(H; o_N, T_q^*N)} a_z[z], \quad a_z \in \mathbb{Z}.$$

We denote the level of the chain α by

$$\lambda_H(\alpha) := \max_{z \in \operatorname{supp} \alpha} \{\mathcal{A}_H^{cl}(z)\}.$$

The resulting invariant $\rho^{lag}(H; [q])$ is to be defined by the mini-max value

$$\rho^{lag}(H; [q]) = \inf_{\alpha \in [q]} \lambda_H(\alpha),$$

where $[q]$ is a generator of the homology group $HF(o_N, T_q^*N) \cong \mathbb{Z}$.

A priori, $\rho^{lag}(H; [q])$ is defined when $\phi_H^1(o_N)$ intersects T_q^*N transversely but can be extended to non-transversal q's by continuity. By varying $q \in N$, this defines a function $f_H : N \to \mathbb{R}$ which is precisely the one called the basic phase function in [9].

Proposition 5.3 (Section 7 [9])**.** *There exists a solution $z : [0,1] \to T^*N$ of $\dot{z} = X(t, z)$ such that $z(0) = q$, $z(1) \in o_N$ and $\mathcal{A}_H^{cl}(z) = \rho^{lag}(H; \{q\})$ whether or not $\phi_H^1(o_N)$ intersects T_q^*N transversely.*

We summarize the main properties of f_H established in [9].

Proposition 5.4 (Theorem 9.1 [9])**.** *When the Hamiltonian $H = H(t,x)$ such that $L = \phi_H^1(o_N)$ is given, there is a canonical lift f_H defined by $f_H(q) := \rho^{lag}(H; \{pt\})$ that satisfies*

$$f_H \circ \pi(x) = h_H(x) = \mathcal{A}_H^{cl}(z_x^H) \tag{5.4}$$

for some Hamiltonian chord z_x^H ending at $x \in T_q^*N$. This f_H satisfies the following property in addition

$$\|f_H - f_{H'}\|_\infty \le \|H - H'\|. \tag{5.5}$$

An immediate corollary of this proposition is the following proved in [9, 15].

Corollary 5.5. *If H_i converges in $L^{(1,\infty)}$, then f_{H_i} converges uniformly.*

Remark 5.6. We would like to emphasize that there is no such C^0-control of the basic generating function h_H even when $H \to 0$ in Hamiltonian topology.

Based on the above proposition, we define:

Definition 5.7. Denote by H^a the Hamiltonian generating the rescaled isotopy $t \mapsto \phi_H^{at}$ for $a > 0$. For any given topological Hamiltonian $H = H(t,x)$, we define its timewise basic phase function by

$$\mathbf{f}_H(t,x) := \lim_{i\to\infty} f_{H_i^t}(x) \tag{5.6}$$

for any approximation sequence H_i of H.

We will always denote a parametric version in bold-faced letters.

We note that the basic generating function h_{H_i} could behave wildly as a whole. But Proposition 5.4 shows that h_{H_i} restricted to the basic Lagrangian selector converges nicely. Note that $\pi_H = \pi|_{L_H} : L_H = \phi_H^1(o_N) \to N$ is surjective for all H and so $\pi_H^{-1}(q) \subset o_N$ is a non-empty compact subset of $o_N \cong N$. Therefore we can regard the 'inverse' $\pi_H^{-1} : N \to L_H \subset T^*N$ as an everywhere defined multivalued section of $\pi : T^*N \to N$.

We introduce the following general definition.

Definition 5.8. Let $L \subset T^*N$ be a Lagrangian submanifold projecting surjectively to N. We call a single-valued section σ of T^*N with values lying in L a *Lagrangian selector* of L.

Once the graph selector f_H of L_H is picked out, it provides a natural Lagrangian selector defined by

$$\sigma_H(q) := \mathrm{Choice}\{x \in L_H \mid \pi(x) = q,\, \mathcal{A}_H^{cl}(z_x^H) = f_H(q)\}$$

via the axiom of choice where Choice is a choice function. It satisfies

$$\sigma_H(q) = df_H(q) \tag{5.7}$$

whenever $df_H(q)$ is defined. We call this particular Lagrangian selector of L_H the *basic Lagrangian selector.* The general structure theorem of the wave front (see [5], [19] for example) proves that the section σ_H is a differentiable map on a set of full measure for a generic choice of H which is, however, *not necessarily continuous*: This is because as long as $q \in N \setminus \mathrm{Sing}\, f_H$, we can choose a small open neighborhood of $U \subset N \setminus \mathrm{Sing}\, f_H$ of q and $V \subset L_H = \phi_H^1(o_N)$ of $x \in V$ with $\pi(x) = q$ so that the projection $\pi|_V : V \to U$ is a diffeomorphism.

6. Calabi homomorphism and basic phase function

Suppose F is a topological Hamiltonian and F_i its approximating sequence and define $\mathbb{F}_i$ and $\mathbb{F}$ as in Section 4.

We first prove the following general theorem in arbitrary dimension. We recall that $f_{\underline{\mathbb{F}_i}}$ converges to $f_{\underline{\mathbb{F}}}$ uniformly.

Theorem 6.1. *Let $\lambda = \phi_F$ be any contractible topological Hamiltonian loop in $\mathcal{P}^{ham}(Sympeo_U(M,\omega), id)$ and with $U = M \setminus B$ where B is a closed subset of nonempty interior. Choose an approximating sequence F_i. Denote by*

$$\overline{\mathrm{Cal}}(F) = \int_0^1 \int_M F \, \mu_\omega \, dt$$

for the Liouville measure associated to ω. Then

$$f_{\underline{\mathbb{F}}}(x) \equiv \frac{\overline{\mathrm{Cal}}(F)}{\mathrm{vol}_\omega(M)} \tag{6.1}$$

for all $x \in M$.

Proof. Let $\underline{F_i} = \mathrm{Dev}(\phi_{F_i})$ which is given by

$$\underline{F_i}(t,x) = F_i(t,x) - c_i(t),$$

where

$$c_i(t) = \frac{1}{\mathrm{vol}_\omega(M)} \int_M F_i(t,x)\, \mu_\omega.$$

Then we have

$$\underline{F_i}(t,x) \equiv -c_i(t) \tag{6.2}$$

and so

$$\int_0^1 \underline{F_i}(t,x)\, dt = -\int_0^1 c_i(t)\, dt = -\frac{\mathrm{Cal}_U(F_i)}{\mathrm{vol}_\omega(M)}$$

for all $x \in B$.

Since F_i is an approximating sequence of topological Hamiltonian F, it follows $F_i \to F$ in $L^{(1,\infty)}$-topology. Therefore applying (5.5) to $H = F_i$ and $H' = 0$ and using the convergence $\|F - F_i\| \to 0$ as $i \to \infty$, we obtain the inequality

$$|f_{\mathbb{F}_i}| \leq \|F\| + \frac{1}{2}$$

for all sufficiently large i's.

Here now enters in a crucial way the fact that ϕ_F generates a topological Hamiltonian loop, not just a path. Together with the Lipschitz property of $f_{\mathbb{F}_i}$ and the inequality (see (5.1))

$$|df_{\underline{\mathbb{F}_i}}| \leq \overline{d}(\phi^1_{\underline{F_i}}, id) \to 0,$$

it immediately follows from the co-area formula (see Theorem 1 of Section 3.4.2 [6], for example) that we can choose a subsequence, again denoted by F_i, so that $f_{\underline{\mathbb{F}_i}} \to c$ uniformly for some constant c.

Therefore it remains to show that this constant is indeed the value $\frac{\overline{\text{Cal}}(F)}{\text{vol}_\omega(M)}$. Denote $K = \operatorname{supp} F$ which is a compact subset of $U = M \setminus B$. We now recall the definition of Hamiltonian topology on noncompact manifolds, Definition 2.7. By definition, there exists $\delta > 0$ such that

$$\operatorname{supp} F_i \subset \operatorname{Int} K(1+\delta/2) \subset K(1+\delta) \subset U,$$

where $K(1+\delta)$ is the (closed) δ-neighborhood of K for all sufficiently large i's. In particular,

$$B(1+\delta/2) \subset M \setminus K(1+\delta/2). \tag{6.3}$$

For any such i's, we also have

$$F_i \equiv 0, \quad \phi^t_{F_i} \equiv id$$

on $B(1+\frac{\delta}{2})$. In particular,

$$\operatorname{Graph} F_i \cap \Delta \supset \Delta_{B(1+\frac{\delta}{2})}.$$

Therefore the same properties stated above as for F_i still hold for $\underline{F_i}$ except the values thereof on B are changed to $-c_i(t)$.

Let $\mathbf{q} \in \Delta_B$ be any point in its interior. By the spectrality of the values of $f_{\underline{\mathbb{F}_i}}(\mathbf{q})$ (Theorem 5.3 [17]), there is a point $\mathbf{x} \in T^*_{\mathbf{q}}\Delta \cap \operatorname{Graph} \phi^1_{F_i}$ such that $(\phi^1_{\mathbb{F}_i})^{-1}(\mathbf{x}) \in o_\Delta$ and

$$f_{\underline{\mathbb{F}_i}}(\mathbf{q}) = \mathcal{A}^{cl}(z^{\underline{\mathbb{F}_i}}_{\mathbf{x}}).$$

We denote $(\phi^1_{\mathbb{F}_i})^{-1}(\mathbf{x}) = (q', q')$.

Because of this, $\phi^1_{F_i} \to id$ as $i \to \infty$ by definition of the approximating sequence F_i of F. Combining this with $\mathbf{q} = (q,q) \in \operatorname{Int} \Delta_B$, $\pi_\Delta(\mathbf{x}) = \mathbf{q}$, we derive

$$d((\phi^1_{\mathbb{F}_i})^{-1}(\mathbf{x}), \mathbf{x}),\ d(\mathbf{x}, \pi_\Delta(\mathbf{x})) < \frac{\delta}{4}$$

for all sufficiently large i's. Then $d((\phi^1_{\mathbb{F}_i})^{-1}(\mathbf{x}), (q,q)) < \frac{\delta}{2}$. Since $\underline{\mathbb{F}_i}(t,\mathbf{x}) = \underline{F_i}(t,x)$ for $\mathbf{x} = (x,y)$, the associated Hamiltonian trajectory $z^{\mathbb{F}}_{\mathbf{x}}$ has the form $(\phi^t_{F_i}(q'), q')$ where $(\phi^1_{\mathbb{F}_i})^{-1}(\mathbf{x}) = (q', q')$. But $d(q,q') < \frac{\delta}{2}$ and hence $q' \in M \setminus B(1+\delta) \subset K(1+\frac{\delta}{2})$. (We refer to the proof of Lemma 7.5 [17] for a similar argument used for a similar purpose.)

Therefore $\phi^t_{F_i}(q') \equiv q'$ for all $t \in [0,1]$. This proves that $z^{\mathbb{F}}_{\mathbf{x}}$ must be the constant trajectory $z^{\mathbb{F}}_{\mathbf{x}}(t) \equiv \mathbf{q}$. Then we compute its action value

$$\begin{aligned} f_{\underline{\mathbb{F}_i}}(\mathbf{q}) &= \mathcal{A}^{cl}(z^{\underline{\mathbb{F}_i}}_{\mathbf{x}}) \\ &= -\int_0^1 \underline{\mathbb{F}_i}(t,\mathbf{q})\,dt = -\int_0^1 \underline{F_i}(t,q)\,dt = \int_0^1 c_i(t)\,dt = \frac{\operatorname{Cal}_U(F_i)}{\operatorname{vol}_\omega(M)}. \end{aligned}$$

Since $F_i \to F$ in $L^{(1,\infty)}$-topology and $\operatorname{supp}\phi_{F_i}$, $\operatorname{supp}\phi_F \subset U$, it also follows $\operatorname{Cal}_U(F_i) \to \overline{\operatorname{Cal}_U}(F)$ as $i \to \infty$. This proves indeed $f_{\underline{\mathbb{F}_i}} \to \frac{\overline{\operatorname{Cal}_U}(F)}{\operatorname{vol}_\omega(M)}$. $\square$

An examination of the argument at the end of the proof leading to the identification of the constant with the $\frac{\overline{\mathrm{Cal}}_U(F)}{\mathrm{vol}_\omega(M)}$ shows that the reason why the convergence $\phi^1_{F_i} \to id$ enters is because we need for the projection $\pi_\Delta(\mathbf{x})$ to lie outside $\operatorname{supp} F_i$ to get the required identification. This needed property automatically holds for the projection $U_\Delta \to \Delta$ with $B = D^2_-$ of the canonical Darboux-Weinstein neighborhood obtained through the embedding (4.1) in Section 4. This is because under this embedding the projection $\pi_\Delta(x, y)$ is nothing but the mid-point projection of (x, y) along the geodesic connecting the points $x, y \in S^2$. Since the upper hemisphere $D^2_+ \subset S^2$ is geodesically convex, $\pi_\Delta(x, y)$ is always contained in $\operatorname{Int} D^2_+$ whenever $x, y \in \operatorname{Int} D^2_+$. In particular $\pi_\Delta(\phi^t_{F_i}(q'), q') \in \operatorname{Int} \Delta_{D^2_+}$ for all $t \in [0, 1]$ if $q' \in \operatorname{Int} D^2_+$ and hence the point $\pi_\Delta(\mathbf{x}) = \pi_\Delta(\phi^1_{F_i}(q'), q')$ cannot be projected to a point (q, q) with $q \in B = D^2_-$ *irrespective of the convergence* $\phi^1_{F_i} \to id$. This eliminates the above somewhat subtle argument for the case of our main interest. An implication of this consideration leads to the following stronger result for this case in that it applies to an arbitrary *path* not just to loops.

Theorem 6.2. *Let $\lambda = \phi_F$ be any topological Hamiltonian path supported in* $\operatorname{Int} D^2$. *Denote by $\mathbb{F}$ the associated Hamiltonian on $D^1(T^*\Delta_{S^2}) \cong S^2 \times S^2 \setminus \overline{\Delta}_{S^2}$ constructed as before* (*via the embedding* (4.1)). *Then*

$$f_{\underline{\mathbb{F}}}(\mathbf{q}) = \frac{\overline{\mathrm{Cal}}(F)}{\mathrm{vol}(S^2)}$$

for all $\mathbf{q} \in \Delta_{D^2_-}$.

Of course, in this case, $f_{\underline{\mathbb{F}}}$ will not be constant on $\Delta_{D^2_+}$ in general.

7. Extension of Calabi homomorphism

We recall from the definition of $\mathcal{P}^{ham}(Symp_U(M, \omega), id)$ with $U = M \setminus B$ that if $\phi_F \in \mathcal{P}^{ham}(Symp_U(M, \omega), id)$, then there exists a 2-parameter Hamiltonian $H = H(s, t, x)$ such that $\phi^t_{H(s)} \equiv id$ and $H \equiv 0$ on $B = M \setminus U$ for a nonempty open subset of M. In particular, we have $\underline{H}(s) \equiv c(s)$ on B with

$$c(s) = \frac{1}{\mathrm{vol}_\omega(M)} \int_M H(s)\,\omega.$$

Engulfedness of H enables us to do computations on a Darboux-Weinstein neighborhood V_Δ of the diagonal $\Delta \subset M \times M$, which we regard either as a subset of $M \times M$ or that of $T^*\Delta$ depending on the given circumstances. At the end, we will apply the computations to the given approximating sequence of hameotopy of contractible topological Hamiltonian loop.

Now we further specialize to the case of our main interest D^2. We embed D^2 into S^2 as the upper hemisphere D^2_+ and denote $B = D^2_-$, the lower hemisphere.

The following is the main conjecture to beat which was originally proposed in [15]. This is the only place where the restriction to the two-disc D^2 is needed, but we expect the same vanishing result hold for higher dimensional disc D^{2n} or even for general pair (M, B), which is a subject of future study.

Conjecture 7.1. *Assume $M = S^2$ and $B = D^2_-$ be the lower hemisphere as above. Let $\Lambda = \left\{\phi^t_{H(s)}\right\}_{(s,t)\in[0,1]^2}$ be a hameotopy contracting a topological Hamiltonian loop ϕ_F with $F = H(1)$ such that $\phi^1_{H(s)} \equiv \phi^t_{H(0)} \equiv id$ for all t, $s \in [0,1]$.*

Let $f_{\underline{\mathbb{F}}}$ be the limit basic phase function defined by $f_{\underline{\mathbb{F}}} = \lim_{i\to\infty} f_{\underline{\mathbb{F}_i}}$. Then

$$f_{\underline{\mathbb{F}}} = 0.$$

Combining Theorem 6.1 and Conjecture 7.1, we now prove the following.

Theorem 7.2. *Suppose Conjecture 7.1 holds. Then the homomorphism*

$$\overline{\mathrm{Cal}}^{path} : \mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id) \to \mathbb{R}$$

descends to a homomorphism

$$\overline{\mathrm{Cal}} : \mathrm{Hameo}(D^2, \partial D^2) \to \mathbb{R}$$

which restricts to $\mathrm{Cal} : \mathrm{Ham}(D^2, \partial D^2) \to \mathbb{R}$.

Proof. Let $\phi \in \mathrm{Hameo}(D^2, \partial D^2)$. We will show that for any topological Hamiltonian paths λ, λ' with $\lambda(1) = \lambda'(1) = \phi$, $\overline{\mathrm{Cal}}^{path}(\lambda) = \overline{\mathrm{Cal}}^{path}(\lambda')$. By the homomorphism property, it is enough to prove $\overline{\mathrm{Cal}}^{path}(\lambda^{-1}\lambda') = 0$. But we have $\lambda^{-1}(0)\lambda'(0) = \lambda^{-1}(1)\lambda'(1)) = id$, i.e., the path $\lambda^{-1}\lambda'$ defines a topological Hamiltonian loop based at the identity. Therefore Conjecture 7.1 and Theorem 6.2 imply $\overline{\mathrm{Cal}}^{path}(\lambda^{-1}\lambda') = 0$.

Then the theorem follows by defining $\overline{\mathrm{Cal}} : \mathrm{Hameo}(D^2, \partial D^2) \to \mathbb{R}$ to be

$$\overline{\mathrm{Cal}}(\phi) = \overline{\mathrm{Cal}}^{path}(\lambda)$$

for a (and so any) topological Hamiltonian path λ with $\lambda(1) = \phi$. □

Therefore we have proved:

So the main remaining task is to prove Conjecture 7.1 which will prove all the conjectures stated in the present paper. In the next section, we will prove the conjecture for the *weakly graphical* topological Hamiltonian loop on the disc.

Part 2. Weakly graphical topological Hamiltonian loops on D^2

8. Geometry of graphical symplectic diffeomorphisms in 2-dimension

We start with the following definition in general dimension.

Definition 8.1. Let $\Psi : U_\Delta \to \mathcal{V}$ be a Darboux-Weinstein chart of the diagonal $\Delta \subset M \times M$ and $\pi_\Delta : U_\Delta \to \Delta$ the associated projection.

(1) We call an engulfed symplectic diffeomorphism $\phi : M \to M$ Ψ-*graphical* if the projection π_Δ is one-one, and an engulfed symplectic isotopy $\{\phi^t\}$ Ψ-*graphical* if each element ϕ^t Ψ-graphical. We call a Hamiltonian $F = F(t,x)$ Ψ-graphical if its associated Hamiltonian isotopy ϕ_F^t Ψ-graphical.
(2) We call a topological Hamiltonian loop F is strongly (resp. weakly) Ψ-graphical, if it allows an approximating sequence F_i each element of which is Ψ-graphical (resp. whose time-one map $\phi_{F_i}^1$ is Ψ-graphical).

Denote by F^a the time-dependent Hamiltonian generating the path $t \mapsto \phi_F^{at}$. The statement (2) of this definition is equivalent to saying that each F^a is Ψ-graphical for $a \in [0,1]$. We remark that any symplectic diffeomorphisms sufficiently C^1-close to the identity is graphical, but not every C^0-close one.

In the rest of this section, we restrict ourselves to the two dimensional case.

We identify $U_y \times U_y \hookrightarrow T^*\Delta$ by the explicit linear coordinate changes

$$\mathbf{q}_1 = \frac{q+Q}{2},\ \mathbf{q}_2 = \frac{p+P}{2},\ \mathbf{p}_2 = q - Q,\ \mathbf{p}_1 = P - p, \tag{8.1}$$

where $(Q,P) = (Q,P)\circ\pi_1$ and $(q,p) = (Q,P)\circ\pi_2$ in this Darboux-Weinstein chart. (We note that this chart can be chosen globally on D^2.) Then we have

$$\begin{aligned} Q &= \mathbf{q}_1 - \frac{\mathbf{p}_2}{2}, \quad q = \mathbf{q}_1 + \frac{\mathbf{p}_2}{2} \\ P &= \mathbf{q}_2 + \frac{\mathbf{p}_1}{2}, \quad p = \mathbf{q}_2 - \frac{\mathbf{p}_1}{2}. \end{aligned} \tag{8.2}$$

In short, we write

$$x = (Q,P) = \mathbf{q} + \frac{1}{2}j\mathbf{p}, \quad y = (q,p) = \mathbf{q} - \frac{1}{2}j\mathbf{p},$$

where $j : \mathbb{R}^2_{\mathbf{p}} \times \mathbb{R}^2_{\mathbf{p}}$ is the linear map given by $j(\mathbf{p}_1,\mathbf{p}_2) = (-\mathbf{p}_2,\mathbf{p}_1)$.

In dimension 2, we prove the following interesting phenomenon. Although we have not checked it, it is unlikely that similar phenomenon occurs in higher dimensions. This theorem has its own interest. The theorem itself will not be used in the proofs of main results of the present paper except that the same kind of the proof will be used later in the proof of Proposition 9.1.

Theorem 8.2. *Suppose $\phi : M \to M$ is a Ψ-graphical symplectic diffeomorphism and let* $\mathrm{Graph}\,\phi = \mathrm{Image}\,\alpha_\phi$ *for a closed one-form α_ϕ on Δ. Then for any $0 \le r \le 1$, the projection $\pi_2 : M \times M \to M$ restricts to a one-one map to* $\mathrm{Image}\, r\,\alpha_\phi \subset M \times M$. *In particular*

$$\mathrm{Image}\, r\,\alpha_\phi = \mathrm{Graph}\,\phi_r \tag{8.3}$$

for some symplectic diffeomorphism $\phi_r : M \to M$ for each $0 \le r \le 1$.

Proof. We have only to prove the map

$$\mathbf{q} \mapsto \mathbf{q} - \frac{r}{2} j\alpha_\phi(\mathbf{q}) \tag{8.4}$$

is one-one. This is because it is the composition of the maps

$$\Delta \to \mathrm{Image}\, \alpha_\phi; \quad \mathbf{q} \mapsto (\mathbf{q}, r\, \alpha_\phi(\mathbf{q}))$$

and the projection $\pi_2 : \mathrm{Image}\, r\, \alpha_\phi \to M$ where the first map is a bijective map. Denote this map by ψ_r.

Since the map ψ_r has degree 1, it will be enough to prove that it is an immersion since the latter will imply that the map must be a covering projection. Therefore we need to prove that the derivative

$$d\psi(\mathbf{q}) = I - \frac{r}{2} j\, \nabla \alpha_\phi(\mathbf{q})$$

is invertible for all $\mathbf{q}$ and $0 \le r \le 1$. Here $\nabla \alpha_\phi$ is the covariant derivative of the one-form α_ϕ with respect to the flat affine connection ∇. We regard it as a section of $Hom(T\Delta, T^*\Delta)$, i.e., a bundle map

$$\nabla \alpha_\phi : T\Delta \to T^*\Delta.$$

Lemma 8.3. *At each point $\mathbf{q} \in \Delta$, the linear map*

$$\nabla : v \mapsto \nabla_v \alpha_\phi$$

is a symmetric operator, i.e., it satisfies

$$\langle \nabla_v \alpha_\phi, w\rangle = \langle \nabla_w \alpha_\phi, v\rangle \tag{8.5}$$

for all v, $w \in T_{\mathbf{q}}\Delta$ at any $\mathbf{q} \in \Delta$.

Proof. This immediately follows from the fact that any closed one-form can be locally written as $\alpha_\alpha = df_\phi$ for some function on Δ. Then $\nabla \alpha_\phi = D^2 f_\phi$ which is the Hessian of the function f_ϕ which is obviously symmetric. □

We first prove the following general result on the set of 2×2 symmetric matrices.

Lemma 8.4. *Let A be a 2×2 symmetric matrix. Then*

$$\det(I - rj\, A) > 0 \tag{8.6}$$

for all $r \in [0, 1]$, provided it holds at $r = 1$, i.e., provided

$$\det(I - j\, A) > 0.$$

The same holds for the opposite inequality.

Proof. Denote $A = \left(\begin{smallmatrix} a & c \\ c & b \end{smallmatrix}\right)$. Then straightforward computation shows

$$jA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} -c & -b \\ a & c \end{pmatrix}.$$

In particular $\mathbf{tr}(j\, A) = 0$ and hence

$$\det(I - j\, A) = 1 + \det(j\, A) = 1 + (ab - c^2).$$

Therefore $\det(I - j\, A) > 0$ is equivalent to

$$1 + (ab - c^2) > 0.$$

For $r = 0$, $I - rj\,A = I$ and so the inequality obviously holds. On the other hand, if $r \in (0,1]$, we derive

$$1 + r^2(ab - c^2) \geq r^2(1 + (ab - c^2)) > 0$$

which finishes the proof. □

Remark 8.5. Note that if A is symmetric, then $jA \in sp(2)$ the Lie algebra of the symplectic group $Sp(2)$. Then the set $\{B \in sp(2) \mid \det(I - B) = 0\}$ is given by the equation

$$1 + (ab - c^2) = 0; \quad B = \begin{pmatrix} c & b \\ -a & -c \end{pmatrix}$$

which defines a hypersurface in $sp(2)$. If we denote

$$sp_{\pm}(2) = \{B \in sp(2) \mid \pm \det(I - B) > 0\}$$

what this lemma shows that each component thereof is star-shaped centered at I.

By the hypothesis, it follows that $\psi = \psi_1$ is an orientation preserving diffeomorphism and so $\det d\psi(\mathbf{q}) > 0$. We now compute

$$\det d\psi(\mathbf{q}) = \det\left(I - \frac{1}{2}\nabla\alpha_\phi(\mathbf{q})\right)$$

and

$$d\psi_r(\mathbf{q}) = I - \frac{r}{2}\nabla\alpha_\phi(\mathbf{q}).$$

By Lemma 8.4, we derive $d\psi_r(\mathbf{q}) > 0$ and so ψ_r is immersed for all r. This finishes the proof of Theorem 8.2. □

Remark 8.6. A similar proof also gives rise to the following theorem with v and r replaced, whose proof will be omitted since it will not be used in the present paper: Suppose $\phi : M \to M$ is a Ψ-graphical symplectic diffeomorphism. Consider the family of maps $\phi_v : M \to M$ defined by $\phi_v(y) = y + v(\phi(y) - y)$ for $v \in [0,1]$. Then ϕ_v is also Ψ-graphical for all v, i.e., we can express

$$\operatorname{Graph}\phi_v = \operatorname{Image}\alpha_v$$

for some one-form α_v on Δ for all v.

9. Weakly graphical Hamiltonian diffeomorphism and Alexander isotopy

The following proposition reflects some special characteristic of Alexander isotopy relative to the general hameotopy.

Proposition 9.1. *Suppose that $\phi^1_{F_i}$ is Ψ-graphical. Then $\phi^1_{F_i,a}$ defined as in Lemma 3.1 is also Ψ-graphical for all $0 \leq a \leq 1$.*

Proof. The proof of this proposition is similar to that of Theorem 8.2 in its spirit but is much simpler than it. It is enough to prove the map $\kappa_a = \pi_\Delta \circ (\pi_2^\Delta)^{-1} : S^2 \to S^2$ is one-one since the map $(\pi_2^\Delta)^{-1} : S^2 \to \mathrm{Graph}\, \phi^1_{F_i,a}$ is bijective. But the map κ_a is given by

$$\kappa_a(y) = \frac{1}{2}(y + \phi^1_{F_i,a}(y))$$

in the affine chart. A straightforward computation shows

$$d\kappa_a(y) = \begin{cases} \frac{1}{2}(Id + d\phi^1_{F_i}(\frac{y}{a})) & \text{for } y \text{ with } |y| \leq a(1-\eta) \\ Id & \text{otherwise.} \end{cases}$$

Since κ_1 is an orientation-preserving diffeomorphism and S^2 is compact, there exists $\delta > 0$ such that

$$\det(d\kappa_1(y)) > \delta > 0$$

for all $y \in S^2$. From the expression of $d\kappa_a(y)$, it follows $d\kappa_a(y) = d\kappa_1(\frac{y}{a})$ and hence

$$\det(d\kappa_a(y)) = \det\Big(d\kappa_1(\frac{y}{a})\Big) > \delta > 0 \tag{9.1}$$

for all $a \in [0,1]$ and $y \in S^2$. This implies $\kappa_a : S^2 \to S^2$ is an immersion and so a covering map of degree 1. Therefore it must be a one-one map. □

We now consider 3-chain Ξ_i parameterized by the map

$$\Xi_i : [0,1] \times M \to T^*\Delta$$

defined by

$$\Xi_i(a,x) = \begin{cases} a\ (\phi_i(x), x) & x \in D^2_+ \\ (x,x) & x \in S^2 \setminus D^2_+. \end{cases} \tag{9.2}$$

We note that this chain defines the same chain as the trace $\mathbf{Tr}_{\mathcal{G}}(\phi_i)$ of the Alexander isotopy of the 2-disc extended to S^2 by the identity: it is by definition parameterized by

$$\mathbf{Tr}_{\mathcal{G}}(\phi_i)(a,y) = \begin{cases} \big(a\phi_i(\frac{y}{a}), y\big) & |y| \leq a \leq 1 \\ (y,y) & y \in S^2 \setminus D^2_+(a). \end{cases}$$

Remark 9.2. The way (9.2) of parameterizing the trace of the Alexander isotopy as a Lagrangian chain associated, *which is smooth everywhere including* $a = 0$, is crucial for us to establish the C^0-convergence property of a-Hamiltonian of the Alexander isotopy in the context of graphical case. See Corollary 10.4 later in the next section. This aspect of Alexander isotopy seems to be something new which, as far as we know, has not been utilized in the literature before.

We will fix i and just write $\phi = \phi_i$ until we need to vary i. Using the coordinate change (8.1), we have the equalities

$$\mathbf{q}^a = a\,\frac{\phi(x) + x}{2},\ \mathbf{p}^a = -a\, j(\phi(x) - x) \tag{9.3}$$

on D^2_+ which extend to the $S^2 \setminus D^2_+$ by the identity map. If we parameterize Graph ϕ by $\mathbf{q}(1,x) = \frac{\phi(x)+x}{2}$ and $\mathbf{p}(1,x) = -j(\phi(x) - x)$ (using the canonical identification of $S^2 \cong \Delta_{S^2}$), then for each given $0 \leq a \leq 1$, we may use Proposition 9.1 to parameterize the chain $\Xi(a,\cdot)$

$$\mathbf{p}(a,x) = d_{\mathbf{q}}\mathbf{g}(a,\mathbf{q}(a,x)),\ \mathbf{q}(a,x) = a\mathbf{q}(1,x)$$

for $(a,x) \in [0,1] \times S^2$ for some continuous function $\mathbf{g} = \mathbf{g}(a,\mathbf{q})$ which is smooth on $(0,1] \times S^2$. More precisely, if we denote $g_a(\mathbf{q}) = \mathbf{g}(a,\mathbf{q})$, the first equation becomes

$$\mathbf{p}_1(a,x) = \frac{\partial g_a}{\partial \mathbf{q}_1}(\mathbf{q}(a,x)),\ \mathbf{p}_2(a,x) = \frac{\partial g_a}{\partial \mathbf{q}_2}(\mathbf{q}(a,x))$$

on D^2_+. Substituting $\mathbf{q}(a,x) = a\mathbf{q}(1,x)$ thereinto, we obtain

$$\mathbf{p}_1(a,x) = \frac{\partial g_a}{\partial \mathbf{q}_1}(a\mathbf{q}(1,x)),\ \mathbf{p}_2(a,x) = \frac{\partial g_a}{\partial \mathbf{q}_2}(a\mathbf{q}(1,x)).$$

Substituting the last three relations into the second equation of (9.3), we obtain

$$\frac{\partial g_a}{\partial \mathbf{q}_j}(a\mathbf{q}(1,x)) = a\frac{\partial g_1}{\partial \mathbf{q}_j}(\mathbf{q}(1,x)), \quad j = 1,\ 2. \tag{9.4}$$

If we define a function $\widetilde{g}_a$ by $\widetilde{g}_a(\mathbf{q}) = g_a(a\,\mathbf{q})$, then $\widetilde{g}_1 = g_1$ and

$$\frac{\partial \widetilde{g}_a}{\partial \mathbf{q}_j}(\mathbf{q}) = a\frac{\partial g_a}{\partial \mathbf{q}_j}(a\,\mathbf{q}), \quad j = 1,\ 2. \tag{9.5}$$

By the graphicality of ϕ, it follows that the map

$$x \mapsto \mathbf{q}(1,x) = \left(\frac{\phi(x)+x}{2}, \frac{\phi(x)+x}{2}\right)$$

is a bijective map. Therefore by setting $\mathbf{q} = \mathbf{q}(1,x)$ and varying x, we derive, from (9.4) and (9.5),

$$\frac{\partial \widetilde{g}_a}{\partial \mathbf{q}_j}(\mathbf{q}) = a^2\frac{\partial \widetilde{g}_1}{\partial \mathbf{q}_j}(\mathbf{q}), \quad j = 1,\ 2$$

for all $\mathbf{q} \in D^2_+$. This, combined with vanishing of $\widetilde{g}_a$ and $\widetilde{g}_1$ outside D^2_+, implies

$$\widetilde{g}_a = a^2\widetilde{g}_1, \qquad \text{i.e.,} \quad g_a(a\cdot) = a^2 g_1.$$

Proposition 9.3. *Let $g_a : \Delta_{S^2} \to \mathbb{R}$ be the function such that $g_a \equiv 0$ outside $\Delta^{D^2_+}$ and* Image dg_a *represents the 2-chain $\Xi(a,\cdot)$. Denote $\mathbf{g}(a,\mathbf{q}) = g_a(\mathbf{q})$. Then*

$$\frac{\partial \mathbf{g}}{\partial a}(a,\mathbf{q}) = 2ag_1\left(\frac{\mathbf{q}}{a}\right) - \frac{1}{a}dg_a(\mathbf{q})\cdot\mathbf{q} \tag{9.6}$$

for all $\mathbf{q} \in \Delta_{S^2}$ and for $0 < a \leq 1$.

Proof. We rewrite the identity $g_a(a\mathbf{q}) = a^2 g_1(\mathbf{q})$ on $\Delta_{D^2_+}$ as $\mathbf{g}(a, a\mathbf{q}) = a^2 g_1(\mathbf{q})$ on D^2_+. By differentiating the latter identity with respect to a, we obtain

$$\frac{\partial \mathbf{g}}{\partial a}(a, a\mathbf{q}) + dg_a(a\mathbf{q}) \cdot \mathbf{q} = 2ag_1(\mathbf{q}).$$

By setting $\widetilde{\mathbf{q}} = a\mathbf{q}$ for $\mathbf{q} \in \Delta_{D^2_+}$, we can rewrite the equation as

$$\frac{\partial \mathbf{g}}{\partial a}(a, \widetilde{\mathbf{q}}) + \frac{1}{a} dg_a(\widetilde{\mathbf{q}}) \cdot \widetilde{\mathbf{q}} = 2ag_1\left(\frac{\widetilde{\mathbf{q}}}{a}\right).$$

By rewriting the variable $\widetilde{\mathbf{q}}$ by $\mathbf{q}$, this proves the equality (9.6) on $\Delta_{D^2_+}$. It also obviously holds on $\Delta_{S^2} \setminus \Delta_{D^2_+}$ since both sides vanish. This finishes the proof. □

The upshot of this proposition is that the right hand side of (9.6) changes continuously with respect to the C^0-topology for the set of graphical ϕ's. This will play a fundamental role in the proof of Theorem 1.8 later in the next section.

Remark 9.4. Strictly speaking, we should use the modified Alexander isotopy as given in the proof of Theorem 3.3. We fix a sequence of weakly Ψ-graphical approximating sequence F_i and its Alexander isotopy $\Lambda_i = \Lambda_{i,\varepsilon_i}$ defined as in (3.5). We also denote by $K_i = K_i(a, t, x)$ the unique associated a-Hamiltonian supported in $\operatorname{Int} D^2$ chosen as in Lemma A.1, and denote $G_i(a, x) = K_i(a, 1, x)$. Then we obtain $\operatorname{Cal}(G_i^a) = (\chi_i(a)^4 - \varepsilon_i^4) \operatorname{Cal}(G_i)$ from Lemma 3.1. As $i \to \infty$, the right hand side converges to $a^4 \overline{\operatorname{Cal}}(F)$ with $F = \lim_{i \to \infty} F_i$ since $\operatorname{Cal}(G_i) = \operatorname{Cal}(F_i)$ from Lemma 3.1.

Having said this remark, we will just use the standard Alexander isotopy given in Lemma 3.1 ignoring the fact that it is not smooth at $a = 0$. All of our arguments can be justified using the above modified Alexander isotopy and taking the limit.

10. Vanishing of basic phase function for the graphical case on D^2

Now we restrict to the context of Theorem 1.8. Let F be a topological Hamiltonian generating a topological Hamiltonian loop ϕ_F on the 2-disc D^2 with $\operatorname{supp} F \subset \operatorname{Int} D^2$. We consider an approximating sequence H_i and $F_i = H_i(1)$ with $\operatorname{supp} F_i \subset \operatorname{Int} D^2$. We embed D^2 into S^2 as the upper hemisphere and then extend F_i canonically to whole S^2 by zero, and consider the graphs $\operatorname{Graph} \phi^1_{F_i}$ in $S^2 \times S^2$. Note $\operatorname{supp} \phi_{\mathbb{F}_i} \subset D^2_+ \times D^2_+$ and hence

$$\operatorname{Graph} \phi^t_{F_i} \bigcap \Delta \supset \Delta_{D^2_-} \bigcup \Delta_{D^2_+ \setminus D^2_+(1-\delta)}$$

for some $\delta > 0$ for all $t \in [0,1]$ independent of sufficiently large i's depending only on F, *provided* $\overline{d}(\phi^1_F, id) \le \frac{\delta}{2}$. We fix the given topological Hamiltonian loop ϕ_F and fix such $\delta > 0$.

Then we consider the normalization $\underline{F_i}$ of F_i on S^2 and define Hamiltonian

$$\underline{\mathbb{F}_i}(t, \mathbf{x}) := \chi(\mathbf{x})\, \underline{F_i}(t, x), \quad \mathbf{x} = (x, y)$$

on $T^*\Delta$ with a slight abuse of notation for $\underline{\mathbb{F}_i}$.

Theorem 10.1. *Conjecture 1.5 holds for any weakly graphical topological Hamiltonian loop on S^2 arising as above.*

An immediate corollary of Theorems 6.1 and 10.1 is the following vanishing result of Calabi invariant.

Corollary 10.2. *Suppose $\lambda = \phi_F$ be an engulfed topological Hamiltonian loop as in Theorem 6.1. Assume λ is weakly graphical. Then $\overline{\mathrm{Cal}}^{path}(\lambda) = 0$.*

The remaining section will be occupied by the proof of Theorem 10.1. Let F be a graphical topological Hamiltonian loop and F_i be an approximating sequence that is Ψ-graphical for a Darboux-Weinstein chart Ψ.

Proof of Theorem 10.1. Let F be the Hamiltonian associated to the topological Hamiltonian loop on S^2 arising from the compactly supported Hamiltonian F on D^2 that is weakly Ψ-graphical. We fix a sequence of weakly Ψ-graphical approximating sequence F_i and its Alexander isotopy $\Lambda_i = \Lambda_{i,\varepsilon_i}$ defined as in (3.5). (As we mentioned at the end of the last section, we will just use the standard Alexander isotopy given in Lemma 3.1 below for the simplicity of exposition.) We also denote $K_i = K_i(a, t, x)$ the unique associated a-Hamiltonian supported in $\mathrm{Int}\, D^2$ chosen as in Lemma A.1, and denote $G_i(a, x) = K_i(a, 1, x)$. In particular, $G_i(0, \cdot) = 0$. Recall $\Lambda_i(0, t) = id$ for all $t \in [0, 1]$.

We denote $H_i(a)$ the t-Hamiltonian defined by $H_i(a)(t, x) = H_i(a, t, x)$. By the weak Ψ-graphicality of $H_i(a)$ from Proposition 9.1 for $a \in [0, 1]$, G_i^a is Ψ-graphical where $G_i^a = G_i^a(s, x) = a\, G_i(as, x)$. We recall from Proposition A.3 $\phi^1_{H_i(a)} = \phi^a_{G_i}$ and

$$f_{\mathbb{H}_i(a)} = f_{\mathbb{G}_i^a}.$$

By the Ψ-graphicality of G_i^a, the basic phase function $f_{\mathbb{G}_i^a}$ is defined everywhere on Δ as a smooth (single-valued) function.

Then the function $\mathbf{f}_{\underline{\mathbb{G}_i}} : [0, 1] \times \Delta \to \mathbb{R}$ defined by $\mathbf{f}_{\underline{\mathbb{G}_i}}(a, \mathbf{q}) = f_{\underline{\mathbb{G}_i^a}}(\mathbf{q})$ satisfies the Hamilton-Jacobi equation

$$\frac{\partial \mathbf{f}_{\underline{\mathbb{G}_i}}}{\partial a}(a, \mathbf{q}) + \underline{\mathbb{G}_i}\Big(a, d_{\mathbf{q}} \mathbf{f}_{\underline{\mathbb{G}_i}}(a, \mathbf{q})\Big) = 0. \tag{10.1}$$

We postpone the derivation of this equation till Appendix. We note $d_{\mathbf{q}} \mathbf{f}_{\underline{\mathbb{G}_i}}(a, \mathbf{q}) = df_{\underline{\mathbb{G}_i^a}}(\mathbf{q})$ by definition of $f_{\underline{\mathbb{G}_i^a}}$.

Now we consider the mean-normalized Hamiltonian $\underline{G_i^a}$ and its associated Hamiltonian $\underline{\mathbb{G}_i^a}$ on $T^*\Delta$.

Lemma 10.3. *Let $\mathbf{g}_i$ be the function associated to $\phi = \phi^1_{F_i}$ as defined in Proposition 9.3. Consider the a-Hamiltonian K_i associated to the Alexander* *isotopy. Then*

$$\mathbf{f}_{\underline{\mathbb{G}_i}}(a,\mathbf{q}) = \mathbf{g}_i(a,\mathbf{q}) + a^4\frac{\mathrm{Cal}(F_i)}{\mathrm{vol}(S^2)}. \tag{10.2}$$

Proof. We first recall $\mathrm{Cal}(F_i) = \mathrm{Cal}(G_i)$ (Lemma A.2).

Since $f_{\mathbb{G}_i^a}$ and $g_{i,a}(:= \mathbf{g}_i(a,\cdot))$ satisfy $\mathrm{Image}\, df_{\mathbb{G}_i^a} = \mathrm{Image}\, dg_{i,a}$ with $f_{\mathbb{G}_i^a} = g_{i,a} \equiv 0$ on $\Delta_{S^2} \setminus \Delta_{D^2_+}$, $f_{\mathbb{G}_i^a} = g_{i,a}$ everywhere on Δ_{S^2}. Then applying Theorem 6.2, we have finished the proof. □

Corollary 10.4. *We have*

$$\underline{\mathbb{G}_i}\Big(a, d_{\mathbf{q}}\mathbf{f}_{\underline{\mathbb{G}_i}}(a,\mathbf{q})\Big) = -2af_{\mathbb{F}_i}\left(\frac{\mathbf{q}}{a}\right) + \frac{1}{a}df_{\mathbb{H}_i(a)}(\mathbf{q})\cdot\mathbf{q} - 4a^3\frac{\mathrm{Cal}(F_i)}{\mathrm{vol}(S^2)}. \tag{10.3}$$

In particular, the function $(a,\mathbf{q}) \mapsto \underline{\mathbb{G}_i}(a, d_{\mathbf{q}}\mathbf{f}_{\underline{\mathbb{G}_i}}(a,\mathbf{q}))$ *uniformly converges to a continuous function*

$$-2af_{\mathbb{F}}\left(\frac{\mathbf{q}}{a}\right) - 4a^3\frac{\mathrm{Cal}(F)}{\mathrm{vol}(S^2)}.$$

Proof. By differentiating (10.2), and comparing (10.1) and (9.6), we immediately obtain the first statement.

For the second, we rewrite

$$\left|\frac{1}{a}df_{\mathbb{H}_i(a)}(\mathbf{q})\cdot\mathbf{q}\right| = \left|df_{\mathbb{H}_i(a)}(\mathbf{q})\cdot\frac{\mathbf{q}}{a}\right| \le |df_{\mathbb{H}_i(a)}(\mathbf{q})|\left|\frac{\mathbf{q}}{a}\right|.$$

Then we recall $\mathrm{supp}\, f_{\mathbb{H}(a)} \subset D^2_+(a)$ (see (3.2)) from which we derive

$$|df_{\mathbb{H}_i(a)}(\mathbf{q})|\left|\frac{\mathbf{q}}{a}\right| \le \max_{\mathbf{q}}|df_{\mathbb{H}_i(a)}(\mathbf{q})| \to 0$$

as $i \to \infty$. Once this is established, we derive the second statement by taking the limit of (10.3). □

The explicit expression of the right hand side of (10.3) will not play any role later in the present paper but only the conclusion that *the function* $(a,\mathbf{q}) \mapsto \underline{\mathbb{G}_i}(a, d_{\mathbf{q}}\mathbf{f}_{\underline{\mathbb{G}_i}}(a,\mathbf{q}))$ *uniformly converges to a continuous function* will do later.

We also derive from this corollary that there exists some constant $C > 0$ independent of i's such that

$$\left|\underline{\mathbb{G}_i}\Big(a, d_{\mathbf{q}}\mathbf{f}_{\underline{\mathbb{G}_i}}(a,\mathbf{q})\Big)\right| \le C \tag{10.4}$$

for all a, $\mathbf{q}$ and for all sufficiently large i's. (In fact, we have

$$\left|\underline{\mathbb{G}_i}\Big(a, d_{\mathbf{q}}\mathbf{f}_{\underline{\mathbb{G}_i}}(a,\mathbf{q})\Big)\right| \le 2a\|f_{\mathbb{F}_i}\|_{C^0} + \|df_{\mathbb{H}_i(a)}\|_{C^0} + 4a^3\frac{|\mathrm{Cal}(F_i)|}{\mathrm{vol}(S^2)} \to 4a^3\frac{|\mathrm{Cal}(F)|}{\mathrm{vol}(S^2)}$$

as $i \to \infty$.)

We now consider the integrals

$$I_{G_i}(a) := \int_\Delta f_{\underline{\mathbb{G}_i^a}}\,\pi_2^*\omega.$$

Then $I_{G_i}(0) = 0$ and

$$I'_{G_i}(a) = \int_\Delta \frac{\partial \mathbf{f}_{\mathbb{G}_i}}{\partial a}\, \pi_2^*\omega = -\int_\Delta \underline{\mathbb{G}_i}\left(a, df_{\underline{\mathbb{G}_i^a}}(\mathbf{q})\right)\, \pi_2^*\omega$$

and so

$$I_{G_i}(1) = -\int_0^1 \int_\Delta \underline{\mathbb{G}_i}\left(a, df_{\underline{\mathbb{G}_i^a}}(\mathbf{q})\right)\, \pi_2^*\omega\, da.$$

By the identity

$$\phi^a_{\underline{\mathbb{G}_i}}(o_\Delta) = \operatorname{Graph} \phi^a_{\underline{G_i}} = \operatorname{Image} df_{\underline{\mathbb{G}_i^a}}$$

and the bijectivity of the projection,

$$\pi_2 : \phi^a_{\underline{\mathbb{G}_i}}(o_\Delta) \to S^2$$

we can write $\pi_1(df_{\underline{\mathbb{G}_i^a}}(\mathbf{q})) = \phi^a_{\underline{G_i}}(y(\mathbf{q}))$ for the unique $y(\mathbf{q})$ satisfying

$$\pi_2(df_{\underline{\mathbb{G}_i^a}}(\mathbf{q})) = y(\mathbf{q})$$

for each given $\mathbf{q}$. We denote by $\pi_1^{\phi^a_{\underline{\mathbb{G}_i}}(o_\Delta);\Delta} : \phi^a_{\underline{\mathbb{G}_i}}(o_\Delta) \to \Delta$ the projection of $\phi^a_{\underline{\mathbb{G}_i}}(o_\Delta)$ to Δ along π_1-direction.

Consider the sequence of maps $\iota_{\phi^a_{\underline{\mathbb{G}_i}}} : \Delta \to \Delta$ defined by

$$\iota_{\phi^a_{\underline{\mathbb{G}_i}}}(\mathbf{q}) = \pi_1^{\phi_{\underline{\mathbb{G}_i^a}}(\Delta);\Delta} \circ \sigma_{\underline{\mathbb{G}_i^a}},$$

where $\pi_1^{\phi_{\underline{\mathbb{G}_i^a}}(\Delta);\Delta}$ is the projection of $\phi^a_{\underline{\mathbb{G}_i}}(\Delta)$ onto Δ along the π_1-direction.

Then we obtain the following from the graphicality of $\phi^a_{\underline{\mathbb{G}_i}}$.

Lemma 10.5. *Each element of the sequence is a diffeomorphism and that the sequence $\iota_{\phi^a_{\underline{\mathbb{G}_i}}}$ uniformly converges to the identity map id_Δ over $a \in [0,1]$ as $i \to \infty$.*

Now we recall the following well-known fact whose proof follows from a straightforward 3 ϵ argument and the weak continuity of the pushforward operation of measures under the C^0-topology of continuous maps.

Lemma 10.6. *Let X be a compact topological space. Denote by μ a finite measure on X, by f a continuous real-valued function. Consider a sequence f_i of continuous function uniformly converging to f and ι_i a sequence of continuous maps uniformly converging to the identity map. Then*

$$\lim_{i\to\infty} \int_X f_i\, \mu = \lim_{i\to\infty} \int_X (f_i \circ \iota_i)\, \mu \left(= \int_X f\, \mu \right).$$

Therefore applying this lemma to the current context of

$$X = \Delta,\ \mu = \mu_\omega,\ \iota = (\iota_{\phi^a_{\underline{\mathbb{G}_i}}})^{-1},\ f_i = \underline{\mathbb{G}_i}\left(a, df_{\underline{\mathbb{G}_i^a}}(\cdot)\right)$$

and combining Corollary 10.4, Lemma 10.5 and (10.4), we derive

$$\lim_{i\to\infty}\int_\Delta \underline{\mathbb{G}_i}(a, df_{\underline{\mathbb{G}_i^a}}(\mathbf{q}))\,\mu_\omega = \lim_{i\to\infty}\int_\Delta \underline{\mathbb{G}_i}\left(a, df_{\underline{\mathbb{G}_i^a}}(\iota^{-1}_{\phi^a_{\underline{\mathbb{G}_i}}}(\mathbf{q}))\right)\,\mu_\omega.$$

But for $\mathbf{q} = (y, y)$, recalling $\mathbb{G}_i = \pi_1^* G_i$ on $\operatorname{Graph}\phi^a_{\mathbb{G}_i} = \operatorname{Image} df_{\underline{\mathbb{G}_i^a}}$, we derive

$$\begin{aligned}\underline{\mathbb{G}_i}\left(a, df_{\underline{\mathbb{G}_i^a}}((\iota_{\phi^a_{\underline{\mathbb{G}_i}}})^{-1}(\mathbf{q})\right) &= \underline{G_i}\left(a, \pi_1(df_{\underline{\mathbb{G}_i^a}}((\iota_{\phi^a_{\underline{\mathbb{G}_i}}})^{-1}\mathbf{q})\right)\\ &= \underline{\mathbb{G}_i}\left(a, \pi_1^{\phi^a_{\underline{\mathbb{G}_i}}(o_\Delta);\Delta}\circ\sigma_{\underline{\mathbb{G}_i}^a}((\iota_{\phi^a_{\underline{\mathbb{G}_i}}})^{-1}\mathbf{q})\right)\\ &= \underline{\mathbb{G}_i}(a, \mathbf{q}) = \pi_1^*\underline{G_i}(a, \mathbf{q}) = \underline{G_i}(a, y).\end{aligned}$$

For the second equality above, we also use the obvious identity

$$\pi_1^{\phi^a_{\underline{\mathbb{G}_i}}(o_\Delta)} = \pi_1^\Delta\circ\pi_1^{\phi^a_{\underline{\mathbb{G}_i}}(o_\Delta);\Delta}.$$

Here for any given subset $L\subset S^2\times S^2$, we denote by $\pi_1^L : L\to S^2$ the restriction of $\pi_1 : S^2\times S^2\to S^2$ to L.

Combining these, we evaluate the integral and derive

$$\lim_{i\to\infty}\int_\Delta \underline{\mathbb{G}_i}\left(a, df_{\underline{\mathbb{G}_i^a}}(\mathbf{q})\right)\,\pi_2^*\omega = \lim_{i\to\infty}\int_\Delta \underline{\mathbb{G}_i}(a,\mathbf{q})\,\pi_2^*\omega = \lim_{i\to\infty}\int_{S^2}\underline{G_i}(a,y)\,\omega = 0$$

where the last vanishing occurs by the mean-normalization condition of $\underline{G_i}$. This proves $\lim_{i\to\infty} I_{G_i}(1) = 0$ in particular.

But we have $f_{\underline{\mathbb{G}_i}} = f_{\underline{\mathbb{H}_i(1)}}(= f_{\underline{\mathbb{F}_i}})$ by Proposition A.3 and in particular

$$f_{\underline{\mathbb{G}_i}} = f_{\underline{\mathbb{F}_i}} \to f_{\underline{\mathbb{F}}}$$

uniformly. Combining the above discussion, we have proved

$$\int_\Delta f_{\underline{\mathbb{F}}}\,\pi_2^*\omega = \lim_{i\to\infty}\int_\Delta f_{\underline{\mathbb{F}_i}}\,\pi_2^*\omega = \lim_{i\to\infty}\int_\Delta f_{\underline{\mathbb{G}_i}}\,\pi_2^*\omega = 0.$$

This finishes the proof. □

Remark 10.7. We would like to point out that while the average of $\underline{G_i}$ vanishes and $\phi^s_{G_i}\to id$ uniformly over $s\in[0,1]$, unlike the t-Hamiltonian F_i which converges in Hamiltonian topology, there is no a priori control of the C^0 behavior of the s-Hamiltonian G_i itself in general according to the definition of approximation sequence H_i of the hameotopy in Definition 2.6. (See (3.4) for the explicit form of $G_i(s,\cdot) = K_i(s,1,\cdot)$ in the case of Alexander isotopy, which evidently involves taking the derivative the t-Hamiltonian.)

In this regard, the above proof strongly relies on the graphicality of the topological Hamiltonian, or more precisely on the graphicality of its approximation sequence. Without this graphicality, one has to deal with emergence of the caustics of the projection $\pi_\Delta : \operatorname{Graph}\phi_{\underline{F_i}}\to\Delta$ or equivalently the non-differentiability locus of the basic phase function $f_{\underline{\mathbb{F}_i}}$. Here seems to enter the piecewise smooth Hamiltonian geometry of Lagrangian chains. We will elaborate this aspect elsewhere.

Appendix A. Homotopy invariance of basic phase function

In this section, we prove some homotopy invariance property of the basic phase function.

Let $\Lambda = \{\phi^t_{H(s)}\}$ be a smooth two-parameter family satisfying $H \equiv 0$ on a neighborhood of B by definition of $\mathcal{P}^{ham}(Symp_U(M,\omega), id)$ with $U = M \setminus B$. We denote by $K = K(s,t,x)$ a s-Hamiltonian of the 2-parameter family $\Lambda = \left\{\phi^t_{H(s)}\right\}$ with $K(s,0,\cdot) \equiv 0$: The latter choice is possible since we have the s-Hamiltonian flow $s \mapsto \phi^0_{H(s)} \equiv id$ and so we can set $K(s,0,\cdot) \equiv 0$. Recall that K is determined uniquely modulo the addition of constants depending on s, t.

We first prove a few lemmata.

Lemma A.1. *Let H and K be as above. Suppose B is connected and has nonempty interior. Then we can choose the s-Hamiltonian K so that $K(s,t,\cdot) \equiv 0$ on a neighborhood of $B \subset M$ for all s, $t \in [0,1]$.*

Proof. Let H be as above and consider its associated s-Hamiltonian vector field Y, i.e.,

$$Y = \frac{\partial \phi^t_{H(s)}}{\partial s} \circ (\phi^t_{H(s)})^{-1}.$$

By definition, we have $Y \rfloor \omega$ is an exact one form that vanishes on B. Considering the exact sequence

$$\to H^0(B) \to H^1(M,B) \to H^1(M) \to$$

and noting $H^0(B) = 0$, the map $H^1(M,B) \to H^1(M)$ has zero kernel. This implies that $Y \rfloor \omega = dK_{s,t}$ for some $K_{s,t} : M \to \mathbb{R}$ with $\operatorname{supp} K \subset M \setminus B$. This finishes the proof. □

This in particular implies $\phi_{K^1} \in \mathcal{P}^{ham}(Symp_U(M,\omega), id)$. Next we have the following coincidence of the Calabi invariant.

Lemma A.2. *Assume $H(0) = 0$ and let K be chosen as in Lemma A.1. Then*

$$\mathrm{Cal}_U(K^1) = \mathrm{Cal}_U(H(1)).$$

Proof. First note $\phi^1_{K^1} = \phi^1_{H(1)}$. Denote by $\Lambda(s,t) = \phi^t_{H(s)}$ the two-parameter family associated to H. Then

$$\Lambda(0,t) \equiv id \equiv \Lambda(s,0)$$

by the requirement $H(0,t,x) \equiv 0$. Therefore the Hamiltonian path $t \mapsto \phi^t_{H(1)} := \Lambda(1,t)$ is smoothly homotopic to the path $s \mapsto \phi^s_{K^1} := \Lambda(s,1)$ relative to the ends and hence we have the lemma by the smooth homotopy invariance of Cal_U: In fact, an explicit homotopy $\Upsilon : [0,1]^2 \to Symp_U(M,\omega)$ between them is given by the formula

$$\Upsilon(s,t) = \begin{cases} \Lambda(t, 1 + 2s(t-1)) & \text{for } 0 \le s \le \frac{1}{2} \\ \Lambda\left(2(s-1/2) + 2t(1-s), t\right) & \text{for } \frac{1}{2} \le s \le 1. \end{cases}$$

The map Υ satisfies

$$\Upsilon(0,t) = \Lambda(t,1) = \phi^t_{K^1}, \quad \Upsilon(1,t) = \phi^t_{H(1)},$$
$$\Upsilon(s,0) = id, \quad \Upsilon(s,1) = \Lambda(1,1) = \phi^1_{H(1)} = \phi^1_F$$

and hence is the required homotopy relative to the ends. □

Now we prove homotopy invariance of the basic generating function and the basic phase functions.

Proposition A.3. $\widetilde{h}_{\mathbb{K}^1} = \widetilde{h}_{\mathbb{H}(1)}$ *and* $f_{\mathbb{K}^1} = f_{\mathbb{H}(1)}$.

Proof. We apply the first variation formula (1.3) to $z^{\mathbf{q}}_{\mathbb{K}^1}(s)$ and $z^{\mathbf{q}}_{\mathbb{H}(1)}(t)$ respectively, and obtain

$$d\widetilde{h}_{\mathbb{K}^1}(v) = \langle \theta(\phi^1_{\mathbb{K}^1}(\mathbf{q})), T\phi^1_{K^1}(v)\rangle,$$
$$d\widetilde{h}_{\mathbb{H}^1}(v) = \langle \theta(\phi^1_{\mathbb{H}(1)}(\mathbf{q})), T\phi^1_{\mathbb{H}(1)}(v)\rangle$$

for any given $v \in T_{\mathbf{q}}\Delta$. Since $\phi^1_{\mathbb{K}^1} = \phi^1_{\mathbb{H}(1)}$, we have proved $d\widetilde{h}_{\mathbb{K}^1} = d\widetilde{h}_{\mathbb{H}(1)}$. On the other hand, for any point $\mathbf{q} \in \Delta_B$, $\mathbb{H} \equiv 0 \equiv \mathbb{K}^1$ on a neighborhood of $\mathbf{q}$ in $T^*\Delta$ and so both $z^{\mathbf{q}}_{\mathbb{K}^1}$ and $z^{\mathbf{q}}_{\mathbb{H}(1)}$ are constant. Therefore the values of both $\widetilde{h}_{\mathbb{K}^1}$ and $\widetilde{h}_{\mathbb{H}(1)}$ are zero at such a point $\mathbf{q} \in \Delta_B$. This finishes the proof of the first equality.

For the proof of $\widetilde{f}_{\mathbb{K}^1} = \widetilde{f}_{\mathbb{H}(1)}$, the first equality in particular implies that the sets of critical values of the action functionals

$$\mathcal{A}^{cl}_{\mathbb{K}^1}, \mathcal{A}^{cl}_{\mathbb{H}(1)} : \Omega(o_N, T^*_q M) \to \mathbb{R}$$

coincide. Then standard homotopy argument used in the homotopy invariance of (in fact any type of) the spectral invariant applies to prove $\rho^{lag}(H,\{q\}) = f_H(q)$ for each $q \in N$ for general H. This finishes the proof. □

Then combining Lemma A.2 and Proposition A.3, we also derive

$$f_{\underline{\mathbb{K}}^1} = f_{\mathbb{K}^1} + \frac{\mathrm{Cal}_U(K^1)}{\mathrm{vol}_\omega(M)} = f_{\mathbb{H}(1)} + \frac{\mathrm{Cal}_U(H(1))}{\mathrm{vol}_\omega(M)} = f_{\underline{\mathbb{H}}(1)}. \tag{A.1}$$

With this preparation, in the proof of Theorem 10.1 later, we used $\mathbb{K}^1$ instead of $\mathbb{H}(1)$ in our proof. This is because we exploited the fact that $\phi^s_{K^1}$ is C^0-small.

Appendix B. Timewise basic phase function as a solution to Hamilton-Jacobi equation

In this section, we show that the space-time basic phase function $\mathbf{f}_H$ defined by

$$\mathbf{f}_H(t,q) = f_{H^t}(q)$$

satisfies the Hamilton-Jacobi equation. More precise description of this statement is now in order.

Let N be an arbitrary compact manifold without boundary and let $H = H(t,x)$ be a time-dependent Hamiltonian defined on the cotangent bundle T^*N and $L = \phi_H^1(o_N)$ be the associated Hamiltonian deformation of o_N. In this case, there is a canonical generating function of L associated to the Hamiltonian H given as follows.

We first start with the discussion on the basic generating function. (We refer the readers to [17] for more detailed exposition on this.) For any given time-dependent Hamiltonian $H = H(t,x)$, the classical action functional on the space

$$\mathcal{P}(T^*N) := C^\infty([0,1], T^*N)$$

is defined by

$$\mathcal{A}_H^{cl}(\gamma) = \int \gamma^*\theta - \int_0^1 H(t,\gamma(t))\,dt. \tag{B.1}$$

We denote $L_H = \phi_H^1(o_N)$ and by $i_H : L_H \hookrightarrow T^*N$ the inclusion map. For given $x \in L_H$, we define the Hamiltonian trajectory

$$z_x^H(t) = \phi_H^t((\phi_H^1)^{-1}(x))$$

which is one satisfying

$$z_x^H(0) \in o_N, \quad z_x^H(1) = x.$$

The function $h_H : L_H \to \mathbb{R}$, called the *basic generating function* in [17], is defined by

$$h_H(x) = \mathcal{A}_H(z_x^H).$$

It satisfies $i_H^*\theta = dh_H$ on L_H, i.e., h_H is a canonical generating function of L_H in that it satisfies

$$i_H^*\theta = dh_H.$$

Then we consider the parametric version of basic generating function (1.6) which is defined by

$$\mathbf{h}_H(t,x) := h_{H^t}(x) \tag{B.2}$$

on $\mathbf{Tr}_{\phi_H}(o_N) := \bigcup_{t\in[0,1]}\{t\} \times \phi_H^t(o_N)$. A straightforward calculation leads to:

Proposition B.1. *Consider the map*

$$\Psi_H : [0,1] \times N \to T^*[0,1] \times T^*N \cong T^*([0,1] \times N)$$

defined by the formula

$$\Psi_H(t,q) = \left(t, -H(t,\phi_H^t(o_q)), \phi_H^t(o_q)\right), \tag{B.3}$$

where $o_q \in o_N$ associated to the point $q \in N$. Then Ψ_H is an exact Lagrangian embedding of $[0,1] \times N$. Denote the associated exact Lagrangian submanifold by

$$\widehat{L} := \operatorname{Image}\Psi_H$$

and by $i_{\widehat{L}} : \widehat{L} \to T^*([0,1] \times N)$ the inclusion map. We also denote by $p : \widehat{L} \to [0,1] \times T^*N$ the restriction to $\widehat{L}$ of the natural projection $T^*([0,1] \times N) \to$

$[0,1]\times T^*N$. *Let* (t,a) *be the canonical coordinate of* $T^*[0,1]$. *Then the timewise basic generating functions* $\widetilde{\mathbf{h}}_H$, $\mathbf{h}_H$ *satisfy*

$$\begin{aligned} d\widetilde{\mathbf{h}}_H &= \Psi_H^*(\theta + a\,dt), \\ p^* d\mathbf{h}_H &= i_{\widehat{L}}^*(\theta + a\,dt), \end{aligned} \tag{B.4}$$

on $[0,1]\times N$ *and on* $\widehat{L}$ *respectively. In particular,* $\mathbf{h}_H\circ p$ *is a generating function of the exact Lagrangian submanifold* $\widehat{L}\subset T^*([0,1]\times N)$.

As a function on the zero section $o_N \cong N$, not on L_H, the basic generating function h_H is a multi-valued function. But the basic phase function $\mathbf{f}_H$, as a timewise graph selector, satisfies the identity

$$\mathbf{h}_H(t,x) = \mathbf{f}_H(t,\pi(x)). \tag{B.5}$$

In particular, substituting $x = d_q\mathbf{f}_H(t,q)$ into (B.5) and noting $\pi(d_q\mathbf{f}_H(t,q)) = q$, we obtain

$$\begin{aligned} \mathbf{f}_H(t,q) &= \mathbf{h}_H(t, d_x\mathbf{f}_H(t,q)) \\ &= \mathbf{h}_H\circ\sigma_H(t,q). \end{aligned}$$

Here σ_H is the map defined by $\sigma_H(t,q) = df_{H^t}(q)$, which is the timewise version of the definition (5.7) whose image is contained in $\widehat{L}$.

Therefore

$$\begin{aligned} d\mathbf{f}_H &= d(\mathbf{h}_H\circ\sigma_H) \\ &= \sigma_H^*(d\mathbf{h}_H) \\ &= \sigma_H^* i_{\widehat{L}}^*(\theta + a\,dt) \\ &= (i_{\widehat{L}}\circ\sigma_H)^*(\theta + a\,dt) \end{aligned}$$

on the smooth locus of $\mathbf{f}_H$ in $[0,1]\times N$. But

$$i_{\widehat{L}}\circ\sigma_H(t,q) = (t, -H(t,\sigma_H(t,q)), \sigma_H(t,q)).$$

Therefore

$$(i_{\widehat{L}}\circ\sigma_H)^*(\theta + a\,dt) = \sigma_H^*\theta - H(t,\sigma_H(t,q))\,dt$$

and hence

$$d\mathbf{f}_H = \sigma_H^*\theta - H(t,\sigma_H(t,q))\,dt.$$

We also have $d\pi d\sigma_H(\frac{\partial}{\partial t}) = 0$ since $\pi\sigma_H(t,q) = q$ for all t. This implies

$$\sigma_H^*\theta(\frac{\partial}{\partial t}) = \sigma_H(t,q)\left(d\pi d\sigma_H(\frac{\partial}{\partial t})\right) = 0.$$

In particular, we have derived

$$\frac{\partial \mathbf{f}_H}{\partial t} = -H(t,\sigma_H(t,q))$$

(on the smooth locus $N \setminus \mathrm{Sing}(d\mathbf{f}_H)$). This is equivalent to the Hamilton-Jacobi equation

$$\frac{\partial \mathbf{f}_H}{\partial t}(t,q) + H(t, d_q \mathbf{f}_H(t,q)) = 0.$$

By repeating the above discussion for the time s instead of t in our present context of $N = \Delta$ we have derived the Hamilton-Jacobi equation (10.1).

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