

## LOCAL CONVERGENCE FOR SOME THIRD-ORDER ITERATIVE METHODS UNDER WEAK CONDITIONS

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ABSTRACT. The solutions of equations are usually found using iterative methods whose convergence order is determined by Taylor expansions. In particular, the local convergence of the method we study in this paper is shown under hypotheses reaching the third derivative of the operator involved. These hypotheses limit the applicability of the method. In our study we show convergence of the method using only the first derivative. This way we expand the applicability of the method. Numerical examples show the applicability of our results in cases earlier results cannot.

### 1. Introduction

Let  $S = \mathbb{R}$  or  $\mathbb{C}$ ,  $D$  be a convex subset of  $S$  and  $F : D \subseteq S \rightarrow S$  be a nonlinear differentiable function. Most solution methods for computing a solution  $x^*$  of the equation

$$(1.1) \quad F(x) = 0,$$

are Newton-like methods. Classical third order methods are very expensive to implement, since there appears  $F''$  at every step. That is why numerous researchers [3, 4, 5, 6, 8, 18, 20, 22, 27, 28, 29] have used instead multi-point methods defined by:

$$(1.2) \quad x_{n+1} = x_n - \frac{F(x_n)}{F'(z_n)}, \quad z_n = x_n - \frac{F(x_n)}{2F'(x_n)},$$

$$(1.3) \quad x_{n+1} = x_n - \frac{F(w_n)}{F'(x_n)}, \quad w_n = x_n - \frac{F(x_n)}{F'(x_n)},$$

$$(1.4) \quad x_{n+1} = x_n - \frac{2F(x_n)}{F'(x_n) + F'(y_n)}, \quad y_n = x_n - \frac{F(x_n)}{F'(x_n)},$$

where  $x_0$  is an initial point. The methods (1.2)-(1.4) can be found in Chen [12], Frontini and Sormani [18], Kanwar et al. [22] and others. These methods diverge from  $x^*$  if the derivative of the function is either zero or very small

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in the vicinity of the solution. That is why numerous third order iterative methods have been used. For example in [9, 18, 29], they studied the local convergence of the two-point method defined by

$$(1.5) \quad x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n + \alpha y_n)},$$

where

$$y_n = \frac{F(x_n)}{F'(x_n) \pm pF(x_n)},$$

where  $\alpha, p \in S$  are constants. Numerous single and multi-point methods have been given in [2, 7, 22, 27]. The local convergence is based on first, second and third derivative in the preceding works but only the first derivative appears in these methods. For example, define a function  $f$  on  $D = [-\frac{1}{2}, \frac{3}{2}]$  by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Let  $x^* = 1$ . Then we have

$$f'(x) = 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \quad f'(1) = 3,$$

$$f''(x) = 6x \ln x^2 + 20x^3 - 12x^2 + 10x,$$

$$f'''(x) = 6 \ln x^2 + 60x^2 - 24x + 22.$$

Then, clearly, the function  $f'''(x)$  is not bounded on  $D$ . We only suppose Lipschitz conditions on the first derivative to overcome the usage of higher order derivatives.

The rest of the paper is organized as follows: In Sections 2 and 3, we study the local convergence analysis of the methods (1.4) and (1.5), respectively. The numerical examples are presented in the concluding Section 4.

## 2. Local convergence for the method (1.4)

We present the local convergence analysis of the method (1.4) in this section. Let  $U(v, \rho), \bar{U}(v, \rho)$  stand for the open and closed balls in  $S$ , respectively, with center  $v \in S$  and radius  $\rho > 0$ .

It is convenient for the local convergence analysis that follows to define some functions and parameters. Let  $L_0 > 0$  and  $L > 0$  be given parameters with  $L_0 \leq L$ . Define the functions  $g_1, g_0$  and  $h_0$  on the interval  $[0, \frac{1}{L_0})$  by

$$g_1(t) = \frac{Lt}{2(1 - L_0t)}, \quad g_0(t) = \frac{L_0}{2}(1 + g_1(t))t, \quad h_0(t) = g_0(t) - 1$$

and the parameter  $r_1$  by

$$r_1 = \frac{2}{2L_0 + L} < \frac{1}{L_0}.$$

We have that  $h_0(0) = -1 < 0$  and  $h_0(t) \rightarrow +\infty$  as  $t \rightarrow \frac{1}{L_0}^-$ . It follows from the intermediate value theorem that function  $h_0$  has zeros in the interval  $(0, \frac{1}{L_0})$ .

Denote by  $r_0$  the smallest such zero. Then, we have that  $g_1(r_1) = 1$ ,  $g_0(r_0) = 1$ ,  $0 \leq g_1(t) < 1$  for each  $t \in [0, r_1)$  and  $0 \leq g_0(t) < 1$  for each  $t \in [0, r_0)$ . Moreover, define the functions  $g_2$  and  $h_2$  on the interval  $[0, r_0)$  by

$$g_2(t) = \frac{2L(1 + g_1(t))t}{1 - g_0(t)}$$

and

$$h_2(t) = 2L(1 + g_1(t))t + g_0(t) - 1.$$

Then we have  $h_2(0) = -1 < 0$  and

$$\begin{aligned} h_2(r_0) &= 2L(1 + g_1(r_0))r_0 + g_0(r_0) - 1 \\ &= 2L(1 + g_1(r_0))r_0 \\ &= 2L\left(1 + \frac{Lr_0}{1 - L_0r_0}\right)r_0 > 0. \end{aligned}$$

That is, the function  $h_2$  has zeros in the interval  $(0, r_0)$ . Denote by  $r_2$  the smallest such zero. Define the parameter

$$(2.1) \quad r = \min\{r_1, r_2\}.$$

Hence we conclude that

$$(2.2) \quad 0 \leq g_0(t) < 1,$$

$$(2.3) \quad 0 \leq g_1(t) < 1$$

and

$$(2.4) \quad 0 \leq g_2(t) < 1$$

for each  $t \in [0, r)$ .

Next, we present the local convergence analysis of the method (1.4).

**Theorem 2.1.** *Let  $F : D \subseteq S \rightarrow S$  be a differentiable function. Suppose that there exist  $x^* \in D$ , parameters  $L_0 > 0$ ,  $L > 0$  such that, for each  $x, y \in D$ ,*

$$(2.5) \quad F(x^*) = 0, \quad F'(x^*) \neq 0,$$

$$(2.6) \quad |F'(x^*)^{-1}(F'(x) - F'(x^*))| \leq L_0|x - x^*|,$$

$$(2.7) \quad |F'(x^*)^{-1}(F'(x) - F'(y))| \leq L|x - y|$$

and

$$(2.8) \quad \bar{U}(x^*, r) \subseteq D,$$

where  $r$  is as defined in (2.1). Then the sequence  $\{x_n\}$  generated for  $x_0 \in U(x^*, r) - \{x^*\}$  by the method (1.4) is well defined, remains in  $U(x^*, r)$  for each  $n \geq 0$  and converges to  $x^*$ . Moreover, the following estimates hold for each  $n \geq 0$ ,

$$(2.9) \quad |y_n - x^*| \leq g_1(|x_n - x^*|)|x_n - x^*| < |x_n - x^*| < r,$$

$$(2.10) \quad |(F'(x_n) + F'(y_n))^{-1}F'(x^*)| \leq \frac{2}{1 - g_0(|x_n - x^*|)}$$

and

$$(2.11) \quad |x_{n+1} - x^*| \leq g_2(|x_n - x^*|)|x_n - x^*| < |x_n - x^*|,$$

where the “ $g$ ” functions are defined previously. Furthermore, for  $T \in [r, \frac{2}{L_0})$  the limit point  $x^*$  is the only solution of the equation  $F(x) = 0$  in  $\bar{U}(x^*, T) \cap D$ .

*Proof.* By hypothesis  $x_0 \in U(x^*, r) - \{x^*\}$ , the definition of  $r$  and (2.6), we get

$$(2.12) \quad |F'(x^*)^{-1}(F'(x_0) - F'(x^*))| \leq L_0|x_0 - x^*| < L_0r < 1.$$

It follows from (2.12) and the Banach Lemma on invertible functions [2, 7, 23, 24, 26, 27] that  $F'(x_0)$  is invertible and

$$(2.13) \quad |F'(x_0)^{-1}F'(x^*)| \leq \frac{1}{1 - L_0|x_0 - x^*|} < \frac{1}{1 - L_0r}.$$

Hence  $y_0$  is well defined by the first sub-step of the method (1.4) for  $n = 0$ . We also have

$$(2.14) \quad \begin{aligned} & y_0 - x^* \\ &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ &= -[F'(x_0)^{-1}F'(x^*)] \\ & \quad \times \left[ \int_0^1 F'(x^*)^{-1}[F'(x^* + \theta(x_0 - x^*)) - F'(x_0)](x_0 - x^*)d\theta \right]. \end{aligned}$$

Using (2.3), (2.7), (2.13), (2.14) and the definition of  $r$ , we get

$$\begin{aligned} & |y_0 - x^*| \\ & \leq |F'(x_0)^{-1}F'(x^*)| \left| \int_0^1 F'(x^*)^{-1}[F'(x^* + \theta(x_0 - x^*)) - F'(x_0)](x_0 - x^*)d\theta \right| \\ & \leq \frac{L|x_0 - x^*|^2}{2(1 - L_0|x_0 - x^*|)} \\ & = g_1(|x_0 - x^*|)|x_0 - x^*| < |x_0 - x^*| < r, \end{aligned}$$

which shows (2.9) for  $n = 0$ . Then we show that  $F'(x_0) + F'(y_0)$  is invertible. Indeed, using (2.2), (2.6) and (2.9) (for  $n = 0$ ), we obtain

$$(2.15) \quad \begin{aligned} & |(2F'(x^*))^{-1}(F'(x_0) + F'(y_0) - 2F'(x^*))| \\ & \leq \frac{1}{2}(|F'(x^*)^{-1}(F'(x_0) - F'(x^*))| + |F'(x^*)^{-1}(F'(y_0) - F'(x^*))|) \\ & \leq \frac{L_0}{2}(|x_0 - x^*| + |y_0 - x^*|) \\ & \leq \frac{L_0}{2}(|x_0 - x^*| + g_1(|x_0 - x^*|)|x_0 - x^*|) \\ & = g_0(|x_0 - x^*|) < g_0(r) < 1. \end{aligned}$$

It follows that  $(F'(x_0) + F'(y_0))$  is invertible and (2.10) is satisfied for  $n = 0$ . Moreover,  $x_1$  is well defined by the last sub-step of the method (1.4) for  $n = 0$ . Then, using the last sub-step of the method (1.4) for  $n = 0$ , (2.4), (2.9) and (2.10), we get in turn

$$\begin{aligned}
 & |x_1 - x^*| \\
 & \leq |(F'(y_0) + F'(x_0))^{-1}F'(x^*)| |F'(x^*)^{-1}[(F'(x_0)(x_0 - x^*) - F(x_0) + F(x^*) \\
 & \quad + F'(y_0)(y_0 - x^*) - F(x_0) + F'(x^*))]| \\
 & \leq \frac{\frac{L}{2}|x_0 - x^*|^2 + \left| \int_0^1 F'(x^*)^{-1}[F'(y_0) - F'(x^* + \theta(x_0 - x^*))]d\theta(x_0 - x^*) \right|}{\frac{1}{2}(1 - g_0(|x_0 - x^*|))} \\
 & \leq \frac{\frac{L}{2}|x_0 - x^*|^2 + L \int_0^1 |y_0 - x^* - \theta(x_0 - x^*)|d\theta|x_0 - x^*|}{\frac{1}{2}(1 - g_0(|x_0 - x^*|))} \\
 & \leq \frac{L|x_0 - x^*|^2 + 2L(|y_0 - x^*| + \frac{1}{2}|x_0 - x^*|)|x_0 - x^*|}{1 - g_0(|x_0 - x^*|)} \\
 & \leq \frac{2L|x_0 - x^*|^2 + 2Lg_1(|x_0 - x^*|)|x_0 - x^*|^2}{1 - g_0(|x_0 - x^*|)} \\
 & = g_2(|x_0 - x^*|)|x_0 - x^*| < |x_0 - x^*|,
 \end{aligned}$$

which shows (2.11) for  $n = 0$ . By simply replacing  $x_0, y_0, x_1$  by  $x_k, y_k, x_{k+1}$  in the preceding estimates, respectively, we arrive at estimate (2.9)–(2.11). Using the estimate  $|x_{k+1} - x^*| \leq c|x_k - x^*| < r$ ,  $c = g_2(r) \in [0, 1)$ , we deduce that  $x_{k+1} \in U(x^*, r)$  and  $\lim_{k \rightarrow \infty} x_k = x^*$ .

To show the uniqueness part, let  $Q = \int_0^1 F'(y^* + \theta(x^* - y^*))d\theta$  for some  $y^* \in \bar{U}(x^*, T)$  with  $F(y^*) = 0$ . Using (2.6), we have

$$\begin{aligned}
 (2.16) \quad |F'(x^*)^{-1}(Q - F'(x^*))| & \leq \int_0^1 L_0|y^* + \theta(x^* - y^*) - x^*|d\theta \\
 & \leq \int_0^1 (1 - \theta)|x^* - y^*|d\theta \\
 & \leq \frac{L_0}{2}T < 1.
 \end{aligned}$$

It follows from (2.16) and the Banach Lemma on invertible functions that  $Q$  is invertible. Finally, from the identity  $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$ , we conclude that  $x^* = y^*$ . This completes the proof.  $\square$

### 3. Local convergence of the method (1.5)

As in Section 2, we need to introduce some functions and parameters. Let  $L_0 > 0$ ,  $L > 0$ ,  $M > 0$ ,  $p \in S$  and  $\alpha \in S$  with  $L_0 \leq L$ . Suppose that

$$(3.1) \quad \max\{|1 + \alpha|, L|\alpha|\} < \frac{1}{M}.$$

Define the functions  $G_1, G_2, G_3$  and  $H_1, H_2, H_3$  on the interval  $[0, \frac{1}{L_0})$  by

$$\begin{aligned} G_1(t) &= \frac{(\frac{L}{2} + |p|M)t + |1 + \alpha|M}{1 - (L_0 + |p|M)t}, \\ H_1(t) &= \left(\frac{L}{2} + L_0 + 2|p|M\right)t + |1 + \alpha|M - 1, \\ G_2(t) &= L_0 G_1(t)t, \\ H_2(t) &= G_2(t) - 1, \\ G_3(t) &= \frac{L}{2(1 - L_0 t)} \left[ t + \frac{2M|\alpha|}{(1 - (L_0 + |p|M)t)(1 - G_2(t))} \right], \\ H_3(t) &= G_3(t) - 1, \end{aligned}$$

respectively, and the parameter

$$R_0 = \frac{1}{L_0 + |p|M}.$$

It follows from (3.1) that

$$H_1(0) = |1 + \alpha|M - 1 < 0$$

and

$$\begin{aligned} H_1(R_0) &= \left(\frac{L}{2} + L_0 + 2|p|M\right)R_0 + |1 + \alpha|M - 1 \\ &= \left(\frac{L}{2} + |p|M\right)R_0 + |1 + \alpha|M > 0 \end{aligned}$$

since  $(L_0 + |p|M)R_0 - 1 = 0$  by the definition of  $R_0$ . It follows from the Intermediate Value Theorem that the function  $H_1$  has zeros in the interval  $(0, R_0)$ . Denote by  $R_1$  the smallest such zero. We also get

$$\begin{aligned} H_1(r_1) &= \left(\frac{L}{2} + L_0\right)r_1 - 1 + |1 + \alpha|M + 2|p|Mr_1 \\ &= 2|p|Mr_1 + |1 + \alpha|M > 0 \end{aligned}$$

since  $(\frac{L}{2} + L_0)r_1 - 1 = 0$  by the definition of  $r_1$ , so

$$R_1 < r_1.$$

Moreover, using the definition of the functions  $G_2$  and  $H_2$ , we have  $H_2(0) = -1 < 0$  and  $H_2(t) \rightarrow \infty$  as  $t \rightarrow R_0^-$ . Hence the function  $H_2$  has zeros in the interval  $(0, R_0)$ . Denote by  $R_2$  the smallest such zero. We also get

$$H_2(R_1) = L_0 G_1(R_1)R_1 - 1 = L_0 R_1 - 1 < 0$$

since  $L_0 R_1 < L_0 R_0 = \frac{L_0}{L_0 + |p|M} \leq 1$  by the definition of  $R_1, R_0$  and, since  $G_1(R_1) = 1$ , we have

$$R_1 < R_2.$$

Furthermore, it follows from (3.1) that

$$H_3(0) = ML|\alpha| - 1 < 0$$

and  $H_3(t) \rightarrow +\infty$  as  $t \rightarrow R_0^-$ . It follows that the function  $H_3$  has zeros in the interval  $(0, R_0)$ . Denote by  $R_3$  the smallest such zero. Set

$$(3.2) \quad R = \min\{R_1, R_3\}.$$

Notice that

$$(3.3) \quad 0 < R < R_0 < \frac{1}{L_0}.$$

We also have

$$(3.4) \quad 0 \leq G_1(t) < 1,$$

$$(3.5) \quad 0 \leq G_2(t) < 1$$

and

$$(3.6) \quad 0 \leq G_3(t) < 1$$

for each  $t \in [0, R)$ .

Next, using the preceding notation, we present the local convergence analysis of the method (1.5).

**Theorem 3.1.** *Let  $F : D \subseteq S \rightarrow S$  be a differentiable function. Suppose that there exist  $x^* \in D$ , parameters  $L_0 > 0, L > 0, M > 0, p \in S, \alpha \in S$  such that, for each  $x, y \in D$ , the conditions (2.5)–(2.7) hold,*

$$\max\{|1 + \alpha|, L|\alpha|\} < \frac{1}{M},$$

$$(3.7) \quad |F'(x^*)^{-1}F'(x)| \leq M$$

and

$$(3.8) \quad \bar{U}(x^*, R) \subseteq D,$$

where  $R$  is defined by (3.2). Then the sequence  $\{x_n\}$  generated for  $x_0 \in U(x^*, R) - \{x^*\}$  by the method (1.5) is well defined, remains in  $U(x^*, R)$  for each  $n \geq 0$  and converges to  $x^*$ . Moreover, the following estimates hold, for each  $n \geq 0$ ,

$$(3.9) \quad |(F'(x_n) + pF'(x_n))^{-1}F'(x^*)| \leq \frac{1}{1 - (L_0 + |p|M)|x_n - x^*|},$$

$$(3.10) \quad |\alpha y_n + x_n - x^*| \leq G_1(|x_n - x^*|)|x_n - x^*| < |x_n - x^*| < R,$$

$$(3.11) \quad |F'(x_n + \alpha y_n)^{-1}F'(x^*)| \leq \frac{1}{1 - G_2(|x_n - x^*|)}$$

and

$$(3.12) \quad |x_{n+1} - x^*| \leq G_3(|x_n - x^*|)|x_n - x^*| < |x_n - x^*|,$$

where the “ $G$ ” functions are defined above Theorem 3.1. Furthermore, for  $T \in [R, \frac{2}{L_0})$  the limit point  $x^*$  is the only solution of the equation  $F(x) = 0$  in  $\bar{U}(x^*, T) \cap D$ .

*Proof.* As in Theorem 2.1, it follows that  $F'(x_0)^{-1}$  exists and

$$(3.13) \quad |F'(x_0)^{-1}F'(x^*)| \leq \frac{1}{1 - L_0|x_0 - x^*|} < \frac{1}{1 - L_0R}.$$

We shall show that  $F'(x_0) \pm pF'(x_0)$  is invertible. Using the definition of  $R_0, R$  (2.6) and (3.7), the identity

$$F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta$$

and the estimate

$$(3.14) \quad |F'(x^*)^{-1}F(x_0)| \leq M|x_0 - x^*|,$$

we get in turn

$$(3.15) \quad \begin{aligned} & |(F'(x^* \pm pF(x^*))^{-1}(F'(x_0) \pm pF(x_0) - (F'(x^*) \pm pF'(x^*))))| \\ &= |F'(x^*)^{-1}((F'(x_0) - F'(x^*)) \pm p(F(x_0) - F(x^*)))| \\ &\leq |F'(x^*)^{-1}(F'(x_0) - F'(x^*))| \\ &\quad + |p| \left| F'(x^*)^{-1} \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta \right| \\ &\leq (L_0 + |p|M)|x_0 - x^*| < (L_0 + |p|M)R_0 < 1. \end{aligned}$$

It follows from (3.15) and the Banach Lemma on invertible functions that  $F'(x_0) \pm pF'(x_0)$  is invertible and (3.9) holds for  $n = 0$ . Hence  $y_0$  is well defined by the first sub-step of the method (1.5) for  $n = 0$ .

Next, we show that  $\alpha y_0 + x_0 \in U(x^*, R)$ . We have in turn

$$(3.16) \quad \begin{aligned} \alpha y_0 + x_0 - x^* &= \frac{\alpha F(x_0)}{F'(x_0) \pm pF'(x_0)} + x_0 - x^* \\ &= \frac{-[F(x_0) - F(x^*) - F'(x_0)(x_0 - x^*)]}{F'(x_0) \pm pF'(x_0)} \\ &\quad + \frac{(1 + \alpha)F(x_0) \pm pF'(x_0)(x_0 - x^*)}{F'(x_0) \pm pF'(x_0)}. \end{aligned}$$

Then, using (2.6), (3.9) (for  $n = 0$ ), (3.14), (3.2) and (3.4), we have

$$(3.17) \quad \begin{aligned} & |\alpha y_0 + x_0 - x^*| \\ &\leq |(F'(x_0) \pm pF'(x_0))^{-1}F'(x^*)| \\ &\quad \times \left[ \left| \int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x_0))(x_0 - x^*)d\theta \right| \right. \\ &\quad \left. + |1 + \alpha| \left| \int_0^1 F'(x^*)^{-1}F'(x^* + \theta(x_0 - x^*))d\theta \right| \right. \\ &\quad \left. + |p| \left| \int_0^1 F'(x^*)^{-1}F'(x^* + \theta(x_0 - x^*))d\theta \right| \right] \end{aligned}$$



$$\begin{aligned} &\leq \frac{(\frac{L}{2} + |p|M)|x_0 - x^*| + |1 + \alpha|M||x_0 - x^*|}{1 - (L_0 + |p|M)|x_0 - x^*|} \\ &= G_1(|x_0 - x^*|)|x_0 - x^*| < |x_0 - x^*| < R, \end{aligned}$$

which shows (3.10) for  $n = 0$  and  $\alpha y_0 + x_0 \in U(x^*, R)$ . Also, it follows from (3.2), (3.5), (2.6) and (3.17) that

$$\begin{aligned} |F'(x^*)^{-1}[F'(x_0 + \alpha y_0) - F'(x^*)]| &\leq L_0|x_0 + \alpha y_0 - x^*| \\ &\leq L_0G_1(|x_0 - x^*|)|x_0 - x^*| \\ (3.18) \qquad \qquad \qquad &= G_2(|x_0 - x^*|) \\ &< G_2(R) < 1. \end{aligned}$$

It follows from (3.18) and the Banach Lemma on invertible functions that  $F'(x_0 + \alpha y_0)$  is invertible and

$$(3.19) \qquad |F'(x_0 + \alpha y_0)^{-1}F'(x^*)| \leq \frac{1}{1 - G_2(|x_0 - x^*|)},$$

which shows (3.11) for  $n = 0$ . Hence  $x_1$  is well defined by the second sub-step of the method (1.5) for  $n = 0$ . Using the second sub-step of the method (1.5) for  $n = 0$ , we have the approximation

$$\begin{aligned} &x_1 - x^* \\ &= [x_0 - x^* - F'(x_0)^{-1}F(x_0)] + F'(x_0)^{-1}F'(x^*)[F'(x_0 + \alpha y_0) - F'(x_0)] \\ (3.20) \quad &\times [F'(x_0 + \alpha y_0)^{-1}F'(x^*)][F'(x^*)^{-1}F'(x^*)]. \end{aligned}$$

Then, in view of (3.20), (3.2), (3.3), (2.7), we obtain in turn

$$\begin{aligned} &|x_1 - x^*| \\ &\leq \frac{L|x_0 - x^*|^2}{2(1 - L_0|x_0 - x^*|)} \\ &\quad + |F'(x_0)^{-1}F'(x^*)||F'(x^*)^{-1}[F'(x_0 + \alpha y_0) - F'(x_0)]| \\ &\quad \times |F'(x_0 + \alpha y_0)^{-1}F'(x^*)||F'(x^*)^{-1}F'(x^*)| \\ &\leq \frac{L|x_0 - x^*|^2}{2(1 - L_0|x_0 - x^*|)} \\ &\quad + \frac{ML|\alpha||x_0 - x^*|}{(1 - L_0|x_0 - x^*|)(1 - (L_0 + |p|M)|x_0 - x^*|)(1 - G_2(|x_0 - x^*|))} \\ &= G_3(|x_0 - x^*|)|x_0 - x^*| < |x_0 - x^*| < R, \end{aligned}$$

which shows (3.12) for  $n = 0$ . By simply replacing  $x_0, y_0, x_1$  by  $x_k, y_k, x_{k+1}$  in the preceding estimates, respectively, we arrive at estimate (3.9)–(3.12). Using the estimate  $|x_{k+1} - x^*| < |x_k - x^*| < R$ , we deduce that  $x_{k+1} \in U(x^*, R)$  and  $\lim_{k \rightarrow \infty} x_k = x^*$ . The uniqueness part is given in Theorem 2.1 with  $R$  replacing  $r$ . This completes the proof.  $\square$

*Remark 3.2.* (1) In view of (2.6) and the estimate

$$\begin{aligned}\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \\ &\leq 1 + L_0\|x - x^*\|,\end{aligned}$$

the condition (3.7) can be dropped and  $M$  can be replaced by

$$M(t) = 1 + L_0t$$

or simply by  $M = 2$ , since  $t \in [0, \frac{1}{L_0})$ .

(2) The results obtained here can be used for operators  $F$  satisfying the autonomous differential equations [7] of the form

$$F'(x) = P(F(x)),$$

where  $P$  is a continuous operator. Then, since  $F'(x^*) = P(F(x^*)) = P(0)$ , we can apply the results without actually knowing  $x^*$ . For example, let  $F(x) = e^x - 1$ . Then we can choose:  $P(x) = x + 1$ .

(3) The local results obtained here can be used for projection methods such as Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method (GCR) for the combined Newton/finite projection methods and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies (see [2, 7, 24, 27]).

(4) The radius  $r_1$  given by (2.1) was shown by us to be the convergence radius of Newton's method [7, 8]

$$(3.21) \quad x_{n+1} = x_n - F'(x_n)^{-1}F(x_n)$$

for each  $n \geq 0$ , under the conditions (2.6) and (2.7). It follows from (2.1) and  $r < r_1$  that the convergence radius  $r$  of the method (1.4) cannot be larger than the convergence radius  $r_1$  of the second order Newton's method (3.21). As already noted in [2, 7],  $r_1$  is at least as large as the convergence ball given by Rheinboldt [26]

$$(3.22) \quad r_R = \frac{2}{3L}.$$

In particular, for  $L_0 < L$ , we have

$$r_R < r_1$$

and

$$\frac{r_R}{r_1} \rightarrow \frac{1}{3}$$

as  $\frac{L_0}{L} \rightarrow 0$ . That is, our convergence ball  $r_1$  is at most three times larger than Rheinboldt's. The same value for  $r_R$  was given by Traub [27].

(5) It is worth noticing that the method (1.4) is not changing when we use the conditions of Theorem 2.1 instead of the stronger conditions used in

[1], [9]-[25], [27]-[30]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi = \ln \left( \frac{|x_{n+1} - x^*|}{|x_n - x^*|} \right) / \ln \left( \frac{|x_n - x^*|}{|x_{n-1} - x^*|} \right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln \left( \frac{|x_{n+1} - x_n|}{|x_n - x_{n-1}|} \right) / \ln \left( \frac{|x_n - x_{n-1}|}{|x_{n-1} - x_{n-2}|} \right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator  $F$ .

#### 4. Numerical examples

Now, we present two numerical examples in this section.

**Example 4.1.** Returning back to the motivational example at the introduction of this study, we have  $L_0 = L = 146.6629073$ ,  $M = 101.5578008$ ,  $\alpha = -0.9902$ ,  $p = 1$ . The parameters are given in Table 1.

TABLE 1

the method (1.4)	the method (1.5)
$r_0=0.0068$	$R_0=0.0040$
$r_1=0.0045$	$R_1=7.0847e-18$
$r_2=3.6990e-05$	$R_3=0.0030$
$r = r_2$	$R=7.0847e-18$
$\xi_1=2.8072$	$\xi_1=1.9966$
$\xi = 2.9827$	$\xi = 2.0125$

**Example 4.2.** Let  $D = [-1, 1]$ . Define a function  $f$  of  $D$  by

$$(4.1) \quad f(x) = e^x - 1.$$

Using (4.1) and  $x^* = 0$ , we get  $L_0 = e - 1 < L = M = e$ ,  $\alpha = -0.8161$ ,  $p = 1$ .

TABLE 2

the method (1.4)	the method (1.5)
$r_0=0.5820$	$R_0=0.2254$
$r_1=0.3249$	$R_1=0.0587$
$r_2=0.0973$	$R_3=0.0737$
$r = r_2$	$R=0.0587$
$\xi_1=2.4782$	$\xi_1=1.7317$
$\xi = 2.6654$	$\xi = 2.0064$

The parameters are given in Table 2.

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