

## SOME RIGIDITY THEOREMS FOR SELF-SHRINKERS OF THE MEAN CURVATURE FLOW

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ABSTRACT. In this paper, we first prove some Liouville type theorems for elliptic inequalities on weighted manifolds which support a weighted Sobolev-type inequality. Secondly, applying the Liouville type theorems to self-shrinkers, we obtain some global rigidity theorems.

### 1. Introduction

A one-parameter family  $M_t$  ( $t \leq 0$ ) of hypersurfaces in  $\mathbb{R}^{n+1}$  flows by mean curvature if

$$(1.1) \quad (\partial_t x)^\perp = -H\vec{n},$$

where  $\perp$  denotes the projection onto the normal bundle  $NM$ ,  $\vec{n}$  is the outward unit normal and the mean curvature  $H$  is given by  $H = \operatorname{div}\vec{n}$ .

Let  $M^n$  be an  $n$ -dimensional Riemannian manifold, and  $x : M^n \rightarrow \mathbb{R}^{n+1}$  be an isometric immersion. A self-shrinker  $M^n$  as a hypersurface satisfies the following equation for the mean curvature and the normal

$$(1.2) \quad H = \frac{\langle x, \vec{n} \rangle}{2}.$$

Such hypersurface corresponds to expanding homothetic solution of mean curvature flow. The simplest examples are  $\mathbb{R}^n$ ,  $\mathbb{S}^n(\sqrt{2n})$ , and  $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ ,  $0 < k < n$ . Self-shrinkers play an important role in the study of mean curvature flow, not only are they the simplest examples (those where later time slices are rescalings of earlier, that is, if a hypersurface  $\Sigma$  satisfies (1.2), then  $\sqrt{-t}\Sigma$  flows by mean curvature flow (1.1)), but they also describe all possible blow ups at a given singularity of a mean curvature flow. Thus, the classification of type-1 blow ups of a mean curvature flow is equivalent to the classification of self-shrinkers.

For the rigidity property of self-shrinker, there are some results in recent years. In the graphical case, K. Ecker and G. Huisken [7] showed that an

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entire graphical self-shrinker with polynomial volume growth is a hyperplane. L. Wang [13] removed the condition of polynomial volume growth in Ecker-Huisken's theorem. This was later generalized to higher codimension by Q. Ding and Z. Z. Wang [6] under the condition of flat normal bundle.

In the general submanifold case, Colding and Minicozzi [4] gave the classification of self-shrinkers with  $H \geq 0$ , they proved that  $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ ,  $0 \leq k \leq n$ , are the only smooth complete embedded self-shrinkers without boundary, with polynomial volume growth, and  $H \geq 0$  in  $\mathbb{R}^{n+1}$ . Based on an identity of [4], N. Q. Le and N. Sesum [9] proved that if a hypersurface  $M^n \subset \mathbb{R}^{n+1}$  is a smooth complete embedded self-shrinker without boundary and with polynomial volume growth, and satisfies  $|A|^2 < \frac{1}{2}$ , where  $|A|$  denotes the norm of the second fundamental of  $M^n$ , then  $M^n$  is a hyperplane. H. D. Cao and H. Z. Li [1] improved their results to higher codimensional self-shrinkers with  $|A|^2 \leq \frac{1}{2}$  and proved that they must be  $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+1}$  with  $0 \leq k \leq n$ . (Note that, our definition of self-shrinker is slightly different from their's. In [1] and [9], the assumption is  $|A|^2 \leq 1$ , which in our notation becomes  $|A|^2 \leq \frac{1}{2}$ .) For the second gap of  $|A|^2$ , Q. Ding and Y. L. Xin [5] proved that if the self-shrinker  $M^n \subset \mathbb{R}^{n+1}$  has polynomial volume growth and  $|A|^2 \leq \frac{1}{2} + 0.011$ , then  $|A|^2 = \frac{1}{2}$ . Q. M. Cheng and G. X. Wei [3] improved their results to  $|A|^2 \leq \frac{1}{2} + \frac{3}{14}$  and proved that  $M$  is isometric to  $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ ,  $0 \leq k \leq n$ .

Recently, M. Rimoldi [11] proved Colding and Minicozzi's theorem under the conditions that  $H \geq 0$  and  $|A| \in L^2(e^{-\frac{|x|^2}{4}} dvol)$ , without the condition "polynomial volume growth". For the global rigidity property of self-shrinkers, Q. Ding and Y. L. Xin [5] proved that if  $\|A\|_{L^n(M)} < \sqrt{\frac{4}{3n\kappa}}$ , where  $\kappa$  is a constant in Sobolev inequality, then the self-shrinker  $M$  must be a linear subspace.

In this paper, we are interested in the rigidity problem of self-shrinkers in Euclidean space. To achieve this, we first get some more general  $L^p$  vanishing results for elliptic inequalities which come from various geometric situation, under the existence of a weighted Sobolev-type inequality. Our results are proved in a more general setting of weighted manifolds and, therefore, applies to different geometric situations; see Theorem 4.1 and Theorem 4.2.

## 2. Preliminaries

A weighted manifold, also known in the literature as a smooth metric measure space, is a triple  $(M, g; e^{-f} dv_g)$ , where  $(M, g)$  is an  $n$ -dimensional Riemannian manifold with metric  $g$ ,  $dv_g$  is the canonical Riemannian volume element and  $f$  is a smooth function on  $M$ . Associated to a weighted manifold  $(M, g; e^{-f} dv_g)$ , we define a weighted Laplacian by

$$\triangle_f u = \triangle u - \langle \nabla f, \nabla u \rangle_g = e^f \operatorname{div}(e^{-f} \nabla u)$$

for  $u \in C^2(M)$ . Then  $\Delta_f$  is self-adjoint in a weighted  $L^2$  space, that is,

$$\int_M u \Delta_f v e^{-f} dv = - \int_M \langle \nabla u, \nabla v \rangle e^{-f} dv$$

for any  $u \in C_c^1(M)$  and  $v \in C^2(M)$ .

Self-shrinkers in  $\mathbb{R}^{n+1}$  can be characterized as minimal hypersurfaces for the conformally changed metric  $g_{ij} = e^{-\frac{|x|^2}{2n}} \delta_{ij}$ , thus we are naturally led to think of a self-shrinker  $x : M^n \rightarrow \mathbb{R}^{n+1}$  as a weighted manifold  $(M^n, e^{-\frac{|x|^2}{4}} dv)$ , whose geometry is governed by analytic properties of the following linear operator  $\mathcal{L}$  which was first introduced by Colding and Minicozzi ([4])

$$\mathcal{L} = \Delta_{\frac{|x|^2}{4}} = \Delta - \frac{1}{2} \langle x, \nabla(\cdot) \rangle = e^{\frac{|x|^2}{4}} \operatorname{div}(e^{-\frac{|x|^2}{4}} \nabla(\cdot)),$$

where  $\operatorname{div}$  is the divergence on  $M^n$ . The operator  $\mathcal{L}$  plays an important role in the study of self-shrinkers.

For our later use, we collect the following two Simons' type identities (cf. [4])

**Lemma 2.1.** *Let  $M^n \rightarrow \mathbb{R}^{n+1}$  be a complete immersed self-shrinker. Then we have*

$$(2.1) \quad \mathcal{L}|A|^2 = 2|\nabla A|^2 + 2|A|^2\left(\frac{1}{2} - |A|^2\right),$$

$$(2.2) \quad \mathcal{L}|H|^2 = 2|\nabla H|^2 + |H|^2 - 2|H|^2|A|^2.$$

For our purposes, it will be more appropriate to deal with the traceless part of  $A$ , which is given by  $\Psi = A - HI$ , with  $I$  the identity operator on  $TM$ . Then,  $\operatorname{tr}(\Psi) = 0$  and

$$|\Psi|^2 = \operatorname{tr}(\Psi^2) = |A|^2 - \frac{H^2}{n} \geq 0$$

with equality at  $p \in M$  if and only if  $p$  is an umbilical point. Thus,  $\Psi \equiv 0$  is equivalent to the fact that the immersion is totally umbilical. For that reason,  $\Psi$  is also called the total umbilicity tensor of  $M$ . Combining (2.1) and (2.2), we get

$$\begin{aligned} \mathcal{L}|\Psi|^2 &= 2\left(|\nabla A|^2 - \frac{|\nabla H|^2}{n}\right) + \left(|A|^2 - \frac{H^2}{n}\right) - 2|A|^4 + 2\frac{|H|^2}{n}|A|^2 \\ &= 2\left|\nabla\left(A - \frac{H}{n}I\right)\right|^2 + \left(|A|^2 - \frac{H^2}{n}\right)\left[1 - \frac{2H^2}{n} - 2\left(|A|^2 - \frac{H^2}{n}\right)\right] \\ &\geq 2\left|\nabla\left(A - \frac{H}{n}I\right)\right|^2 + \left(|A|^2 - \frac{H^2}{n}\right)\left[1 - \frac{2H^2}{n} - 2\left(|A|^2 - \frac{H^2}{n}\right)\right], \end{aligned}$$

where we have used the Kato inequality in the last inequality. Therefore,

$$(2.3) \quad \mathcal{L}|\Psi|^2 \geq 2|\nabla|\Psi||^2 + |\Psi|^2\left(1 - \frac{2H^2}{n} - 2|\Psi|^2\right).$$

### 3. Liouville type theorems on weighted manifolds

In this section, we study the  $L^p$  vanishing property of a special type PDEs on weighted manifolds which support a weighted Sobolev type inequality. Some results of the next section rely on the following weighted version of Liouville-type theorems which might be interesting by itself.

**Theorem 3.1.** *Let  $(M^n, g; dv_f)$  be a complete weighted Riemannian manifold with  $dv_f = e^{-f} dv_g$  and let  $\Phi \in Lip_{loc}(M)$  be a distributional solution of*

$$(3.1) \quad \Delta_f \Phi^2 \geq a|\nabla \Phi|^2 + \Phi^2(\psi - u)$$

*for some functions  $\psi, u \geq 0$  and constant  $a \geq 0$ . Suppose for  $0 \leq \beta < 1$ , the Sobolev-type inequality of the form*

$$(3.2) \quad \left( \int_M \phi^{\frac{2}{1-\beta}} dv_f \right)^{1-\beta} \leq C \int_M |\nabla \phi|^2 dv_f + D \int_M \phi^2 dv_f, \quad \forall \phi \in C_0^\infty(M).$$

*holds for  $C > 0$  and  $D \in \mathbb{R}$ . Assume that  $\psi \geq \frac{D}{C}(8\alpha - 16\alpha^2 + 4a\alpha^2 - \epsilon)$ ,  $\|\Phi\|_{L^{\frac{1}{\alpha}}(dv_f)} < \infty$  and  $\|u\|_{L^{\frac{1}{\beta}}(dv_f)} < \frac{1}{C}(8\alpha - 16\alpha^2 + 4a\alpha^2 - \epsilon)$  with  $\alpha > 0$  and  $0 < \epsilon \leq 8\alpha - 16\alpha^2 + 4a\alpha^2$ . Then  $\Phi \equiv 0$ .*

*Proof.* For  $R > 0$ , let us choose a cut-off function  $\varphi_R \in C_c^\infty(M)$  satisfying the properties that

$$\varphi_R(x) = \begin{cases} 1 & \text{on } B_R(q); \\ 0 & \text{on } M \setminus B_{2R}(q) \end{cases}$$

and

$$|\nabla \varphi_R| \leq \frac{2}{R} \quad \text{on } B_{2R}(q) \setminus B_R(q),$$

where  $B_R(q)$  is the geodesic ball in  $M$  with radius  $R$  centered at  $q \in M$ . Multiplying  $\varphi_R^2 \Phi^{\frac{1}{\alpha}-2} e^{-f}$  on both sides of inequality (3.1) and integrating by parts, we get

$$\begin{aligned} 0 &\geq 4 \int_M \varphi_R \Phi^{\frac{1}{\alpha}-1} \langle \nabla \varphi_R, \nabla \Phi \rangle dv_f + (8\alpha - 16\alpha^2) \int_M \varphi_R^2 |\nabla \Phi^{\frac{1}{2\alpha}}|^2 dv_f \\ &\quad + 4a\alpha^2 \int_M \varphi_R^2 |\nabla \Phi^{\frac{1}{2\alpha}}|^2 dv_f + \int_M \varphi_R^2 \Phi^{\frac{1}{\alpha}} (\psi - u) dv_f \\ &= (8\alpha - 16\alpha^2 + 4a\alpha^2) \int_M \varphi_R^2 |\nabla \Phi^{\frac{1}{2\alpha}}|^2 dv_f + \int_M \varphi_R^2 \Phi^{\frac{1}{\alpha}} (\psi - u) dv_f \\ &\quad + 4 \int_M \varphi_R \Phi^{\frac{1}{\alpha}-1} \langle \nabla \varphi_R, \nabla \Phi \rangle dv_f \\ &= (8\alpha - 16\alpha^2 + 4a\alpha^2 - \epsilon) \int_M \varphi_R^2 |\nabla \Phi^{\frac{1}{2\alpha}}|^2 dv_f \\ &\quad + \left( 8 - 16\alpha + 4a\alpha - \frac{\epsilon}{\alpha} \right) \int_M \varphi_R \Phi^{\frac{1}{\alpha}-1} \langle \nabla \varphi_R, \nabla \Phi \rangle dv_f \end{aligned}$$

$$\begin{aligned}
& + (8\alpha - 16\alpha^2 + 4a\alpha^2 - \epsilon) \int_M \Phi^{\frac{1}{\alpha}} |\nabla \varphi_R|^2 dv_f \\
& + \left( -4 + 16\alpha - 4a\alpha + \frac{\epsilon}{\alpha} \right) \int_M \varphi_R \Phi^{\frac{1}{\alpha}-1} \langle \nabla \varphi_R, \nabla_b \Phi \rangle dv_f \\
& + \epsilon \int_M \varphi_R^2 |\nabla \Phi^{\frac{1}{2\alpha}}|^2 dv_f + \tau \int_M \Phi^{\frac{1}{\alpha}} |\nabla \varphi_R|^2 dv_f + \int_M \varphi_R^2 \Phi^{\frac{1}{\alpha}} (\psi - u) dv_f \\
(3.3) \quad & - (8\alpha - 16\alpha^2 + 4a\alpha^2 - \epsilon + \tau) \int_M \Phi^{\frac{1}{\alpha}} |\nabla \varphi_R|^2 dv_f,
\end{aligned}$$

where  $0 < \epsilon \leq 8\alpha - 16\alpha^2 + 4a\alpha^2$ . Choosing  $\tau$  such that  $\tau \geq \frac{1}{\epsilon}(-4\alpha + 16\alpha^2 - 4a\alpha^2 + \epsilon)^2$ , and applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
& \left( -4 + 16\alpha - 4a\alpha + \frac{\epsilon}{\alpha} \right) \int_M \varphi_R \Phi^{\frac{1}{\alpha}-1} \langle \nabla \varphi_R, \nabla_b \Phi \rangle dv_f \\
& + \epsilon \int_M \varphi_R^2 |\nabla \Phi^{\frac{1}{2\alpha}}|^2 dv_f + \tau \int_M \Phi^{\frac{1}{\alpha}} |\nabla \varphi_R|^2 dv_f \geq 0.
\end{aligned}$$

Hence (3.3) reduces to

$$\begin{aligned}
0 & \geq (8\alpha - 16\alpha^2 + 4a\alpha^2 - \epsilon) \int_M |\nabla(\varphi_R \Phi^{\frac{1}{2\alpha}})|^2 dv_f + \int_M \varphi_R^2 \Phi^{\frac{1}{\alpha}} (\psi - u) dv_f \\
(3.4) \quad & - (8\alpha - 16\alpha^2 + 4a\alpha^2 - \epsilon + \tau) \int_M \Phi^{\frac{1}{\alpha}} |\nabla \varphi_R|^2 dv_f.
\end{aligned}$$

From (3.2), we have

$$(3.5) \quad \int_M |\nabla(\varphi_R \Phi^{\frac{1}{2\alpha}})|^2 dv_f \geq \frac{1}{C} \left( \int_M (\varphi_R^2 \Phi^{\frac{1}{\alpha}})^{\frac{1}{1-\beta}} dv_f \right)^{1-\beta} - \frac{D}{C} \int_M \varphi_R^2 \Phi^{\frac{1}{\alpha}} dv_f.$$

Substituting (3.5) into (3.4), and using the Hölder inequality, we obtain

$$\begin{aligned}
0 & \geq \frac{1}{C} (8\alpha - 16\alpha^2 + 4a\alpha^2 - \epsilon) \left( \int_M (\varphi_R^2 \Phi^{\frac{1}{\alpha}})^{\frac{1}{1-\beta}} dv_f \right)^{1-\beta} \\
& - \frac{D}{C} (8\alpha - 16\alpha^2 + 4a\alpha^2 - \epsilon) \int_M \varphi_R^2 \Phi^{\frac{1}{\alpha}} dv_f + \int_M \varphi_R^2 \Phi^{\frac{1}{\alpha}} (\psi - u) dv_f \\
& - (8\alpha - 16\alpha^2 + 4a\alpha^2 - \epsilon + \tau) \int_M \Phi^{\frac{1}{\alpha}} |\nabla \varphi_R|^2 dv_f \\
& \geq \left\{ \frac{1}{C} (8\alpha - 16\alpha^2 + 4a\alpha^2 - \epsilon) - \|u\|_{L^{\frac{1}{\beta}}} \right\} \|\varphi_R^2 \Phi^{\frac{1}{\alpha}}\|_{L^{\frac{1}{1-\beta}}(dv_f)} \\
& + \int_M \varphi_R^2 \Phi^{\frac{1}{\alpha}} \left\{ \psi - \frac{D}{C} (8\alpha - 16\alpha^2 + 4a\alpha^2 - \epsilon) \right\} dv_f \\
(3.6) \quad & - (8\alpha - 16\alpha^2 + 4a\alpha^2 - \epsilon + \tau) \int_M \Phi^{\frac{1}{\alpha}} |\nabla \varphi_R|^2 dv_f.
\end{aligned}$$

Since  $\|\Phi\|_{L^{\frac{1}{\alpha}}(dv_f)} < \infty$ , we get

$$\begin{aligned} 0 &\leq \lim_{R \rightarrow \infty} (8\alpha - 16\alpha^2 + 4a\alpha^2 - \epsilon + \tau) \int_M \Phi^{\frac{1}{\alpha}} |\nabla \varphi_R|^2 dv_f \\ (3.7) \quad &\leq \lim_{R \rightarrow \infty} (8\alpha - 16\alpha^2 + 4a\alpha^2 - \epsilon + \tau) \frac{4}{R^2} \int_M \Phi^{\frac{1}{\alpha}} dv_f = 0. \end{aligned}$$

Combining (3.6) and (3.7), letting  $R \rightarrow \infty$ , using monotone and dominated convergence theorems, we obtain

$$\begin{aligned} 0 &\geq \left\{ \frac{1}{C} (8\alpha - 16\alpha^2 + 4a\alpha^2 - \epsilon) - \|u\|_{L^{\frac{1}{\beta}}(dv_f)} \right\} \|\Phi^{\frac{1}{\alpha}}\|_{L^{\frac{1}{1-\beta}}(dv_f)} \\ &\quad + \int_M \Phi^{\frac{1}{\alpha}} \left\{ \psi - \frac{D}{C} (8\alpha - 16\alpha^2 + 4a\alpha^2 - \epsilon) \right\} dv_f. \end{aligned}$$

Since  $\psi \geq \frac{D}{C} (8\alpha - 16\alpha^2 + 4a\alpha^2 - \epsilon)$  and  $\|u\|_{L^{\frac{1}{\beta}}(dv_f)} < \frac{1}{C} (8\alpha - 16\alpha^2 + 4a\alpha^2 - \epsilon)$ , we conclude that  $\Phi = 0$ .  $\square$

**Theorem 3.2.** Let  $(M^n, g; e^{-f} dv_g)$ ,  $n \geq 2$ , be a complete weighted Riemannian manifold with  $\inf f > -\infty$  and let  $\Phi \in Lip_{loc}(M)$  be a distributional solution of

$$\Delta_f \Phi^2 \geq a |\nabla \Phi|^2 + \Phi^2 (\psi - b \Phi^2)$$

for some function  $\psi \geq 0$  and constants  $a, b \geq 0$ . Assume that

$$(3.8) \quad \left( \int_M (\phi^2 e^{-f})^{\frac{n-2}{n-2}} dv \right)^{\frac{n-2}{n}} \leq C \int_M |\nabla \phi|^2 e^{-f} dv + D \int_M \phi^2 e^{-f} dv, \quad \forall \phi \in C_0^\infty(M)$$

for some constants  $C > 0$  and  $D \geq 0$ . Assume further that  $\psi \geq \frac{D(4a+8n-16)}{n^2 C}$  and  $\|\Phi^2\|_{L^{\frac{n}{2}}(dv)} < \frac{4a+8n-16}{n^2 b C}$ . Then  $\Phi \equiv 0$ .

*Proof.* Letting  $u = b\Phi^2$ ,  $\alpha = \frac{1}{n}$  and  $\epsilon = \frac{1}{R}$  with  $R$  large enough in (3.4), we have

$$\begin{aligned} 0 &\geq \left\{ \frac{4a+8(n-2)}{n^2} - \frac{1}{R} \right\} \int_M |\nabla(\varphi_R \Phi^{\frac{n}{2}})|^2 e^{-f} dv + \int_M \varphi_R^2 \Phi^n (\psi - b\Phi^2) e^{-f} dv \\ (3.9) \quad &- \left\{ \frac{4a+8(n-2)}{n^2} - \frac{1}{R} + \tau \right\} \int_M \Phi^n |\nabla \varphi_R|^2 e^{-f} dv. \end{aligned}$$

Choosing  $\phi = \varphi_R \Phi^{\frac{n}{2}}$  in (3.8), we obtain

$$(3.10) \quad \int_M |\nabla(\varphi_R \Phi^{\frac{n}{2}})|^2 e^{-f} dv \geq \frac{1}{C} \left( \int_M (\varphi_R^2 \Phi^n e^{-f})^{\frac{n-2}{n-2}} dv \right)^{\frac{n-2}{n}} - \frac{D}{C} \int_M \varphi_R^2 \Phi^n e^{-f} dv.$$

Substituting (3.10) into (3.9), and using the Hölder inequality yields

$$0 \geq \frac{4aR+8(n-2)R-n^2}{n^2 CR} \left( \int_M (\varphi_R^2 \Phi^n e^{-f})^{\frac{n-2}{n-2}} dv \right)^{\frac{n-2}{n}}$$

$$\begin{aligned}
& - \frac{D[4aR+8(n-2)R-n^2]}{n^2CR} \int_M \varphi_R^2 \Phi^n e^{-f} dv + \int_M \varphi_R^2 \Phi^n (\psi - b\Phi^2) e^{-f} dv \\
& - \left\{ \frac{4a+8(n-2)}{n^2} - \frac{1}{R} + \tau \right\} \int_M \Phi^n |\nabla \varphi_R|^2 e^{-f} dv \\
& \geq \left\{ \frac{4aR+8(n-2)R-n^2}{n^2CR} - b\|\Phi^2\|_{\frac{n}{2}} \right\} \|\varphi_R^2 \Phi^n e^{-f}\|_{\frac{n}{n-2}} \\
& + \int_M \varphi_R^2 \Phi^n \left\{ \psi - \frac{D[4aR+8(n-2)R-n^2]}{n^2CR} \right\} e^{-f} dv \\
(3.11) \quad & - \left\{ \frac{4a+8(n-2)}{n^2} - \frac{1}{R} + \tau \right\} \int_M \Phi^n |\nabla \varphi_R|^2 e^{-f} dv.
\end{aligned}$$

Since  $\inf f > -\infty$  and  $\|\Phi^2\|_{L^{\frac{n}{2}}(dv)} < \frac{4a+8(n-2)}{n^2bC}$ , we get

$$\begin{aligned}
0 & \leq \lim_{R \rightarrow \infty} \left\{ \frac{4a+8(n-2)}{n^2} - \frac{1}{R} + \tau \right\} \int_M \Phi^n |\nabla \varphi_R|^2 e^{-f} dv \\
& \leq \lim_{R \rightarrow \infty} \left\{ \frac{4a+8(n-2)}{n^2} - \frac{1}{R} + \tau \right\} \frac{4}{R^2} \int_M \Phi^n e^{-f} dv = 0.
\end{aligned}$$

Therefore, letting  $R \rightarrow \infty$  in (3.11), using monotone and dominated convergence theorems, we have

$$0 \geq \left\{ \frac{4a+8(n-2)}{n^2C} - b\|\Phi^2\|_{\frac{n}{2}} \right\} \|\Phi^n e^{-f}\|_{\frac{n}{n-2}} + \int_M \Phi^n \left\{ \psi - \frac{D(4a+8n-16)}{n^2C} \right\} e^{-f} dv.$$

Since  $\psi \geq \frac{D(4a+8n-16)}{n^2C}$  and  $\|\Phi^2\|_{L^{\frac{n}{2}}} < \frac{4a+8(n-2)}{n^2bC}$ , we conclude that  $\Phi = 0$ .  $\square$

*Remark 3.1.* On a complete Riemannian manifold  $M^n$  with

$$\text{Ric} + \nabla^2 f - \frac{\nabla f \otimes \nabla f}{m-n} \geq -K$$

for some constant  $m > n$ ,  $K \geq 0$  and with the uniformly lower bound condition

$$\inf_{x \in M} \int_{B_{r_0}(x)} e^{-f} dv > 0$$

for some fixed radius  $r_0 > 0$ , the following Sobolev inequality (cf. [10])

$$C \left( \int_M \phi^{\frac{2n}{n-2}} e^{-f} dv \right)^{\frac{n-2}{n}} \leq \int_M |\nabla \phi|^2 e^{-f} dv + \int_M \phi^2 e^{-f} dv, \quad \forall \phi \in C_0^\infty(M)$$

holds for some constant  $C > 0$ . Thus, if additionally  $f \geq 0$ , then

$$C \left( \int_M (\phi^2 e^{-f})^{\frac{n}{n-2}} dv \right)^{\frac{n-2}{n}} \leq \int_M |\nabla \phi|^2 e^{-f} dv + \int_M \phi^2 e^{-f} dv, \quad \forall \phi \in C_0^\infty(M).$$

When  $\psi = \sigma \Phi^{2(k-1)}$  with  $\sigma \geq 0$ ,  $k \geq 1$ , and  $M$  supports an Euclidean type weighted Sobolev inequality, we have the following corollary.

**Corollary 3.1.** *Let  $(M^n, g; e^{-f} dv_g)$  be a complete weighted Riemannian manifold with  $\inf f > -\infty$  and let  $\Phi \in Lip_{loc}(M)$  be a distributional solution of*

$$\Delta_f \Phi^2 \geq a|\nabla \Phi|^2 + \sigma \Phi^{2k} - b\Phi^4$$

*for some constants  $k \geq 1$  and  $a, \sigma, b \geq 0$ . Assume that  $M^n$  supports an inequality of the form*

$$\left( \int_M (\phi^2 e^{-f})^{\frac{n-2}{n-2}} dv \right)^{\frac{n-2}{n}} \leq C \int_M |\nabla \phi|^2 e^{-f} dv, \quad \forall \phi \in C_0^\infty(M)$$

*for some constant  $C > 0$ . Assume further that  $\|\Phi^2\|_{L^{\frac{n}{2}}(dv)} < \frac{4a+8n-16}{n^2bC}$ . Then  $\Phi \equiv 0$ .*

#### 4. The classification of self-shrinkers as weighted manifolds

In order to apply the results in the above section to self-shrinkers, we need the following Sobolev type inequality, which can be obtained by a method similar to [14] (or [15]) in the case of submanifolds with parallel mean curvature. In the following argument, we denote by  $f(x) = \frac{|x|^2}{4}$ .

**Lemma 4.1.** *Let  $x : M^n \rightarrow \mathbb{R}^{n+1}$ ,  $n \geq 3$ , be a complete immersed self-shrinker. Then for any  $\varphi \in C_0^\infty(M)$ , we have the following inequality*

$$(4.1) \quad \left( \int_M (\varphi^2 e^{-f})^{\frac{n-2}{n-2}} dv \right)^{\frac{n-2}{n}} \leq \frac{2D^2(n)[(n-2)^2 + 2n^2(n-1)^2]k}{n^2(n-2)^2} \left( \int_M |\nabla \varphi|^2 e^{-f} dv + \frac{n}{4k} \int_M \varphi^2 e^{-f} dv \right)$$

*for any  $k \geq 1$ , where  $D(n) = 2^n(1+n)^{(n+1)/n}(n-1)^{-1}\sigma_n^{-1/n}$  with  $\sigma_n = \text{volume of the unit ball in } \mathbb{R}^n$ .*

*Proof.* From [8], we have the following Sobolev inequality

$$(4.2) \quad \left( \int_M g^{\frac{n-1}{n-2}} dv \right)^{\frac{n-1}{n}} \leq D(n) \int_M \left( |\nabla g| + \frac{|H|}{n} g \right) dv$$

for all  $0 \leq g \in C_c^\infty(M)$ . Substituting  $g = \varphi^{\frac{2(n-1)}{n-2}}$  into (4.2) yields

$$\left( \int_M \varphi^{\frac{2n}{n-2}} dv \right)^{\frac{n-1}{n}} \leq \frac{2(n-1)}{n-2} D(n) \int_M \varphi^{\frac{n}{n-2}} |\nabla \varphi| dv + \frac{D(n)}{n} \int_M |H| \varphi^{\frac{2(n-1)}{n-2}} dv.$$

By the Hölder inequality, we obtain

$$\|\varphi\|_{\frac{2n}{n-2}} \leq D(n) \left\{ \frac{2(n-1)}{n-2} \|\nabla \varphi\|_2 + \frac{1}{n} \|H\varphi\|_2 \right\}.$$

Thus,

$$(4.3) \quad \|\varphi\|_{\frac{2n}{n-2}}^2 \leq D^2(n) \left\{ \frac{4(n-1)^2(1+s)}{(n-2)^2} \|\nabla \varphi\|_2^2 + \left(1 + \frac{1}{s}\right) \frac{1}{n^2} \|H\varphi\|_2^2 \right\}$$



for all  $s > 0$ . Replacing  $\varphi$  by  $\varphi e^{-\frac{f}{2}}$  and using (1.2), we have

$$(4.4) \quad \left( \int_M (\varphi e^{-\frac{f}{2}})^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}} \leq D^2(n) \left\{ \frac{4(n-1)^2(1+s)}{(n-2)^2} \int_M |\nabla(\varphi e^{-\frac{f}{2}})|^2 dv \right. \\ \left. + \frac{1}{4n^2} \left(1 + \frac{1}{s}\right) \int_M \varphi^2 e^{-f} |x^\perp|^2 dv \right\}.$$

Consider the first integration at the right hand side of (4.4). Integrating by parts and noting that  $\nabla|x|^2 = 2x^\top$ , where  $\top$  denotes the projection onto the tangent bundle  $TM$ , we obtain

$$(4.5) \quad \int_M |\nabla(\varphi e^{-\frac{f}{2}})|^2 dv \\ = \int_M |\nabla\varphi|^2 e^{-f} dv + \frac{1}{2} \int_M \langle \nabla\varphi^2, \nabla e^{-f} \rangle dv + \frac{1}{4} \int_M \varphi^2 e^f |\nabla e^{-f}|^2 dv \\ = \int_M |\nabla\varphi|^2 e^{-f} dv - \frac{1}{2} \int_M \varphi^2 \Delta e^{-f} dv + \frac{1}{16} \int_M \varphi^2 |x^\top|^2 e^{-f} dv.$$

Since  $\Delta x = -H\vec{n}$  on any hypersurface and  $H = \frac{\langle x, \vec{n} \rangle}{2}$  on a self-shrinker, we have

$$(4.6) \quad \Delta e^{-f} = -\frac{1}{4} (2\langle \nabla x, \nabla x \rangle + 2\langle \Delta x, x \rangle) e^{-f} + \frac{1}{16} |\nabla|x|^2|^2 e^{-f} \\ = -\frac{1}{4} (2n - 2\langle H\vec{n}, x \rangle) e^{-f} + \frac{1}{4} |x^\top|^2 e^{-f} \\ = -\frac{1}{4} (2n - |x^\perp|^2) e^{-f} + \frac{1}{4} |x^\top|^2 e^{-f} \\ = -\frac{n}{2} e^{-f} + \frac{1}{4} |x|^2 e^{-f}.$$

Substituting (4.6) into (4.5), we have

$$(4.7) \quad \int_M |\nabla(\varphi e^{-\frac{f}{2}})|^2 dv = \int_M |\nabla\varphi|^2 e^{-f} dv - \frac{1}{8} \int_M \varphi^2 |x^\perp|^2 e^{-f} + \frac{n}{4} \int_M \varphi^2 e^{-f} dv \\ - \frac{1}{16} \int_M \varphi^2 |x^\top|^2 e^{-f} dv.$$

Substituting (4.7) into (4.4) and choosing  $s > 0$  such that

$$\frac{1}{8} \times \frac{4(n-1)^2(1+s)}{(n-2)^2} = \frac{1}{4n^2} \left(1 + \frac{1}{s}\right),$$

that is,  $s = \frac{(n-2)^2}{2n^2(n-1)^2}$ , we conclude that

$$\left( \int_M (\varphi e^{-\frac{f}{2}})^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}} \leq \frac{2D^2(n)[(n-2)^2 + 2n^2(n-1)^2]}{n^2(n-2)^2} \left( \int_M |\nabla\varphi|^2 e^{-f} dv \right. \\ \left. + \frac{n}{4} \int_M \varphi^2 e^{-f} dv \right),$$

where we discard the negative term at the right hand side. Therefore,

$$\begin{aligned} \left( \int_M (\varphi^2 e^{-f})^{\frac{n-2}{n-2}} dv \right)^{\frac{n-2}{n}} &\leq \frac{2D^2(n)[(n-2)^2 + 2n^2(n-1)^2]k}{n^2(n-2)^2} \left( \int_M |\nabla \varphi|^2 e^{-f} dv \right. \\ &\quad \left. + \frac{n}{4k} \int_M \varphi^2 e^{-f} dv \right) \end{aligned}$$

for any  $k \geq 1$ .  $\square$

*Remark 4.1.* In the inequality (4.1), we introduce the constant  $k$  to suit our purposes. The flexibility of  $k$  enables us to apply Theorem 3.2 directly to prove global rigidity theorem for self-shrinkers.

In the following argument, we will use the above lemma to prove our main theorems. Considering the inequality (2.1), an immediate application of Theorem 3.2 and Lemma 4.1 yields the following result.

**Theorem 4.1.** *Let  $M^n \rightarrow \mathbb{R}^{n+1}$ ,  $n > 2$ , be a complete immersed self-shrinker. Assume that*

$$(4.8) \quad \left( \int_M |A|^n dv \right)^{1/n} < \frac{n-2}{D(n)} \sqrt{\frac{n}{(n-2)^2 + 2n^2(n-1)^2}}.$$

*Then  $M^n$  is a hyperplane.*

*Proof.* It follows from (2.1) that

$$\mathcal{L}|A|^2 \geq 2|\nabla|A||^2 + |A|^2(1 - 2|A|^2).$$

By (4.1), we take  $C = \frac{2D^2(n)[(n-2)^2 + 2n^2(n-1)^2]k}{n^2(n-2)^2}$ ,  $D = \frac{D^2(n)[(n-2)^2 + 2n^2(n-1)^2]}{2n(n-2)^2}$ ,  $a = b = 2$  and  $\psi = 1$  in Theorem 3.2. Let  $k = \frac{2n-2}{n}$ , then  $\psi = 1 = \frac{D(4a+8n-16)}{n^2C}$ , and (4.8) is equivalent to  $\|A\|_{L^n(M)}^2 < \frac{4a+8(n-2)}{n^2bC}$ . According to Theorem 3.2,  $|A| = 0$ , that is,  $M^n$  is totally geodesic. Therefore,  $M^n$  is a hyperplane.  $\square$

*Remark 4.2.* Substituting  $s = \frac{(n-2)^2}{2n^2(n-1)^2}$  into (4.3), we have

$$\kappa^{-1} \|\varphi\|_{\frac{2n}{n-2}}^2 \leq \|\nabla \varphi\|_2^2 + \frac{1}{2} \|H\varphi\|_2^2,$$

where  $\kappa = \frac{2D^2(n)[(n-2)^2 + 2n^2(n-1)^2]}{n^2(n-2)^2}$ . We see that the constant at the right hand side of (4.8) is equal to  $\sqrt{\frac{2}{n\kappa}}$ , which is larger than  $\sqrt{\frac{4}{3n\kappa}}$ . Hence the constant in Theorem 4.1 seems better and more explicit than that of [5] in the case where the self-shrinker  $M^n$  is a hypersurface in  $\mathbb{R}^{n+1}$ .

**Theorem 4.2.** *Let  $M^n \rightarrow \mathbb{R}^{n+1}$ ,  $n > 2$ , be a complete immersed self-shrinker. Assume that  $H^2 \leq \frac{(k-2)n+2}{2k}$  and*

$$\left( \int_M |\Psi|^n dv \right)^{\frac{2}{n}} < \frac{2(n-1)(n-2)^2}{D^2(n)[(n-2)^2 + 2n^2(n-1)^2]k}$$

*for  $k \geq \frac{2n-2}{n}$ . Then  $M^n$  is a hyperplane.*

*Proof.* Applying Theorem 3.2 and Lemma 4.1 to the inequality (2.3), we conclude that  $|\Psi| = 0$ , i.e.,  $|A|^2 = \frac{H^2}{n}$ , that is,  $M^n$  is a totally umbilical hypersurface in  $\mathbb{R}^{n+1}$ . Hence,  $M^n$  is a hyperplane or a hypersphere. If  $M^n$  is a hypersphere, it must be  $\mathbb{S}^n(\sqrt{2n}) \subset \mathbb{R}^{n+1}$  and  $|H|^2 = \frac{n}{2}$ , which contradicts to the assumption that  $H^2 \leq \frac{(k-2)n+2}{2k} < \frac{n}{2}$ . Therefore,  $M^n$  is a hyperplane in  $\mathbb{R}^{n+1}$ .  $\square$

*Remark 4.3.* It is obvious that the constant  $\frac{(k-2)n+2}{2k} < \frac{n}{2}$  and  $\lim_{k \rightarrow \infty} \frac{(k-2)n+2}{2k} = \frac{n}{2}$ . Cao and Li [1] proved that if a complete noncompact self-shrinker  $M^n$  in  $\mathbb{R}^{n+1}$  has polynomial volume growth and satisfies  $H^2 \geq \frac{n}{2}$ , then  $H^2 \equiv \frac{n}{2}$ .

**Corollary 4.1.** *Let  $M^n \rightarrow \mathbb{R}^{n+1}$ ,  $n > 2$ , be a complete immersed self-shrinker. Assume that  $\sup H^2 < \frac{n}{2}$ . Then there exist an explicit positive constant  $C$  depending on  $\sup H^2$  such that if*

$$\left( \int_M |\Psi|^n dv \right)^{\frac{1}{n}} < C,$$

*then  $M^n$  is a hyperplane.*

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## References

- [1] H. D. Cao and H. Z. Li, *A gap theorem for self-shrinkers of the mean curvature flow in arbitrary codimension*, Calc. Var. Partial Differential Equations **46** (2013), no. 3-4, 879–889.
- [2] Q. M. Cheng and Y. Peng, *Complete self-shrinkers of the mean curvature flow*, Calc. Var. Partial Differential Equations **52** (2015), no. 3-4, 497–506.
- [3] Q. M. Cheng and G. X. Wei, *A gap theorem of self-shrinkers*, Trans. Amer. Math. Soc. **367** (2015), no. 7, 4895–4915.
- [4] T. H. Colding and W. P. Minicozzi II, *Generic mean curvature flow I: Generic singularities*, Ann. of Math. **175** (2012), no. 2, 755–833.
- [5] Q. Ding and Y. L. Xin, *The rigidity theorems of self shrinkers*, Trans. Amer. Math. Soc. **366** (2014), no. 10, 5067–5085.
- [6] Q. Ding and Z. Z. Wang, *On the self-shrinking systems in arbitrary codimension spaces*, arXiv:1012.0429v1, 2011.
- [7] K. Ecker and G. Huisken, *Mean curvature evolution of entire graphs*, Ann. of Math. **130** (1989), no. 3, 453–471.
- [8] D. Hoffman and J. Spruck, *Sobolev and isoperimetric inequalities for Riemannian submanifolds*, Comm. Pure Appl. Math. **27** (1974), 715–727.
- [9] N. Q. Le and N. Sesum, *Blow-up rate of the mean curvature during the mean curvature flow and a gap theorem for self-shrinkers*, Comm. Anal. Geom. **19** (2011), no. 4, 633–659.
- [10] X. D. Li, *Sobolev inequalities on forms and  $L^{p,q}$ -cohomology on complete Riemannian manifolds*, J. Geom. Anal. **20** (2010), no. 2, 354–387.
- [11] M. Rimoldi, *On a classification theorem for self-shrinkers*, Proc. Amer. Math. Soc. **142** (2014), no. 10, 3605–3613.
- [12] K. Smoczyk, *Self-shrinkers of the mean curvature flow in arbitrary codimension*, Int. Math. Res. Not. **2015** (2005), no. 48, 2983–3004.

- [13] L. Wang, *A Bernstein type theorem for self-similar shrinkers*, *Geom. Dedicata* **151** (2011), 297–303.
- [14] H. W. Xu,  *$L_{n/2}$ -pinching theorems for submanifolds with parallel mean curvature in a sphere*, *J. Math. Soc. Japan* **46** (1994), no. 3, 503–515.
- [15] H. W. Xu and J. R. Gu, *A general gap theorem for submanifolds with parallel mean curvature in  $R^{n+p}$* , *Comm. Anal. Geom.* **15** (2007), no. 1, 175–194.

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