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DEFORMING PINCHED HYPERSURFACES OF THE HYPERBOLIC SPACE BY POWERS OF THE MEAN CURVATURE INTO SPHERES

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Abstract. This paper concerns closed hypersurfaces of dimension $n \geq 2$ in the hyperbolic space $\mathbb{H}^{n+1}_{\kappa}$ of constant sectional curvature κ evolving in direction of its normal vector, where the speed equals a power $\beta \geq 1$ of the mean curvature. The main result is that if the initial closed, weakly h-convex hypersurface satisfies that the ratio of the biggest and smallest principal curvature at everywhere is close enough to 1, depending only on n and β, then under the flow this is maintained, there exists a unique, smooth solution of the flow which converges to a single point in $\mathbb{H}^{n+1}_{\kappa}$ in a maximal finite time, and when rescaling appropriately, the evolving hypersurfaces exponential convergence to a unit geodesic sphere of $\mathbb{H}^{n+1}_{\kappa}$.

1. Introduction

This paper consider the following problem. Let M^n be a smooth, compact oriented manifold of dimension $n \geq 2$ without boundary, $(N^{n+1}, \bar{g})$ be an $(n+1)$-dimensional completed Riemannian manifold, and $\mathrm{X}_0 : M^n \to N^{n+1}$ a smooth immersion. Consider a one-parameter family of smooth immersions: $\mathrm{X}_t : M^n \to N^{n+1}$. The hypersurfaces $M_t = \mathrm{X}_t(M^n)$ are said to move by powers of the mean curvature, if $\mathrm{X}_t = \mathrm{X}(\cdot, t)$ satisfies the evolution equation

$$\begin{cases} \frac{\partial}{\partial t}\mathrm{X}(p,t) = -H^{\beta}(p,t) \cdot \nu(p,t), & p \in M^n, \\ \mathrm{X}(\cdot, 0) = \mathrm{X}_0(\cdot), \end{cases} \tag{1.1}$$

where $\beta > 0$, $\nu(p,t)$ is the outer unit normal to M_t at $\mathrm{X}(p,t)$ in the tangent space TN^{n+1}, and $H(p,t)$ the trace of the Weingarten map $\mathscr{W}_{-\nu}(p,t) = -\mathscr{W}_{\nu}(p,t)$ on the tangent space TM^n induced by X_t.

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The flow (1.1) has been considered by Schulze in [21], [22] when N^{n+1} is the Euclidean space $\mathbb{R}^{n+1}$ for M_0 of strictly positive mean curvature hypersuface. Schulze called such a flow an H^β-flow. In [21] he proved the following theorem.

Theorem 1.1 (see [21]). *Let $\mathrm{X}_0 : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion, where $H(M_0) > 0$. Then there exists a unique, smooth solution to the flow (1.1) on a finite maximal time interval $[0, T)$. In the case that*

i) *M_0 is strictly convex for $0 < \beta < 1$,*
ii) *M_0 is weakly convex for $\beta \geq 1$,*

then M_t converges to a point as $t \to T$.

Here "weakly convex" and "strictly convex", resp., are defined as all the eigenvalues $\lambda_i(p)$ of the Weingarten map $\mathscr{W} = \{g^{ik}h_{kj}\} = \{h^i_j\}$ being positive and nonnegative, resp., where $\{g^{ij}\}$ is the inverse of the induced metric $\{g_{ij}\}$ and $\{h_{ij}\}$ the second fundamental form for $1 \leq i, j, k \leq n$ in a local coordinate. Some counterexamples show that under the assumptions of Theorem 1.1 in general the evolving hypersurfaces along the flow (1.1) may not become spherical in shape as the limit is approached. Furthermore, if the initial hypersurface satisfies a stronger assumption on principal curvatures, Schulze in [22] has shown that for $\beta > 1$ the evolving hypersurfaces M_t contract to a point in finite time, becoming spherical in shape as $t \to T$. Precisely, denote the Gauß curvature by K, he prove the following:

Theorem 1.2 (see [22]). *For $\beta \geq 1$ there exists a nonnegative constant $C(n, \beta) < 1/n^n$ such that the following holds: If the initial hypersurface of $\mathbb{R}^{n+1}$ is pinched in the sense that*

$$\frac{K(p)}{H^n(p)} > C(n, \beta) \quad \text{for all} \quad p \in M^n, \tag{1.2}$$

then this remains so under the H^β-flow. The constant $C(n, \beta)$ is increasing in β, $\lim_{\beta\to 1} C(n, \beta) = 0$ and $\lim_{\beta\to+\infty} C(n, \beta) = 1/n^n$. Furthermore the normalized embedding

$$\hat{\mathrm{X}}\left(p, \hat{t}\right) := \left((\beta + 1)n^k(T - t)^{-1/(\beta+1)}\right)\left(\mathrm{X}\left(p, \hat{t}\right) - x_0\right) \tag{1.3}$$

converges for $\hat{t} \to +\infty$ exponentially in the C^∞-topology to the unit sphere of $\mathbb{R}^{n+1}$. Here $\hat{t} := -(\beta + 1)^{-1}n^{-\beta}\ln(1 - t/T)$, where T is the maximal time of existence of the un-normalized flow and x_0 is the point in $\mathbb{R}^{n+1}_\kappa$ where the evolving hypersurfaces shrink down to.

However, the results of [21] and [22] do not closely relate to the ambient space, we face the challenges of extending the above results to hypersurface to more general ambient spaces. But not every Riemannian manifold is well suited to deal with the situation analogous to the setting in the Euclidean space. We want to consider the case that the ambient space is a simply connected Riemannian manifold of constant sectional curvature $\kappa(< 0)$ whose flow behaves quite differently compared with the Euclidean space to a certain extent.

Set $a = \sqrt{|\kappa|}$ and N^{n+1}_κ be isometric to the hyperbolic space $\mathbb{H}^{n+1}_\kappa$ of radius $1/a$:

$$\mathbb{H}^{n+1}_\kappa := \{p \in L^{n+2} : \langle p, p\rangle = -\frac{1}{a^2}\}.$$

Here $(L^{n+2}, \langle\cdot,\cdot\rangle)$ denotes the $(n+2)$-dimensional Lorentz-Minkowski space. To consider the flow (1.1) in N^{n+1}_κ is then equivalent to consider the flow (1.1) in $\mathbb{H}^{n+1}_\kappa$. Indeed, in order to formulate the main result of this work, it is necessary to provide some definitions as in [6, 7] as follows.

Definition 1.3. A horosphere $\mathcal{H}$ of $\mathbb{H}^{n+1}_\kappa$ is the limit of a geodesic sphere of $\mathbb{H}^{n+1}_\kappa$ as its center goes to the infinity along a fixed geodesic ray.

Definition 1.4. An horoball $\mathscr{H}$ is the convex domain whose boundary is a horosphere.

Definition 1.5. A hypersurface M of $\mathbb{H}^{n+1}_\kappa$ is said to be convex by horospheres (h-convex for short) if it bounds a domain Ω satisfying that for every $p \in M = \partial\Omega$, there is a horosphere $\mathcal{H}$ of $\mathbb{H}^{n+1}_\kappa$ through p such that Ω is contained in $\mathscr{H}$ of $\mathbb{H}^{n+1}_\kappa$ bounded by $\mathcal{H}$.

Remark 1.6. In fact, Borisenko-Miquel in [6] have shown that horosphere $\mathcal{H}$ of $\mathbb{H}^{n+1}_\kappa$ is weakly (strictly) h-convex if and only if all its principal curvatures are (strictly) bounded from below by a at each point.

An earlier paper by the authors [13] has obtained the following result which is an analogue of the above Theorem 1.1 of Schulze [21] on the flow (1.1) of convex hypersurface of $\mathbb{R}^{n+1}$ in the context of $\mathbb{H}^{n+1}_\kappa$.

Theorem 1.7. *Let* $\mathrm{X}_0 : M^n \to \mathbb{H}^{n+1}_\kappa$ *be a smooth immersion with the mean curvature strictly bounded from below by* na, *that is* $H(M_0) > na$. *Then there exists a unique, smooth solution to the flow* (1.1) *on a finite maximal time interval* $[0, T)$ *and* T *is between* $\frac{1}{\beta+1}(H_{\max}(M_0))^{-(\beta+1)}$ *and* $\frac{n}{\beta+1}(H_{\min}(M_0) - na)^{-(\beta+1)}$. *In the case that*

i) M_0 *is strictly* h*-convex for* $0 < \beta < 1$,
ii) M_0 *is weakly* h*-convex for* $\beta \geq 1$,

then the hypersurfaces M_t *are strictly* h*-convex for all* $t > 0$ *and they contract to a point in* $\mathbb{H}^{n+1}_\kappa$ *as* t *approaches* T.

Denote the turbulent second fundamental form given by $\tilde{h}_{ij} := h_{ij} - ag_{ij}$. Then the turbulent mean curvature $\tilde{H} = H - na$ and the turbulent Gauß curvature $\tilde{K} = \det\{\tilde{h}^j_i\}$. In this paper the turbulent geometric quantities are distinguished by a tilde. The purpose of this paper is to present following extension of the above Theorem 1.2 of Schulze [22] on the flow (1.1) of convex hypersurface of $\mathbb{R}^{n+1}$ to h-convex hypersurface of $\mathbb{H}^{n+1}_\kappa$.

Theorem 1.8. *For* $\beta \geq 1$ *there exists a nonnegative constant* $C(n, \beta) < 1/n^n$ *such that the following holds*: *If the initial closed, weakly* h*-convex hypersurface*

of $\mathbb{H}^{n+1}_\kappa$ *is pinched in the sense that*

$$\frac{\tilde{K}(p)}{\tilde{H}^n(p)} > C(n,\beta) \quad \textit{for all} \quad p \in M^n, \tag{1.4}$$

then this remains under the H^β*-flow and the constant* $C(n,\beta)$ *is increasing in* β*,* $\lim_{\beta\to 1} C(n,\beta) = 0$ *and* $\lim_{\beta\to+\infty} C(n,\beta) = 1/n^n$*. Furthermore the normalized embedding*

$$\hat{\mathrm{X}}\left(p,\hat{t}\right) := \left((\beta+1)n^k(T-t)^{-1/(\beta+1)}\right)\left(\mathrm{X}\left(p,\hat{t}\right) - q_0\right) \tag{1.5}$$

converges for $\hat{t} \to +\infty$ *exponentially in the* C^∞*-topology to the unit geodesic sphere. Here* $\hat{t} := -(\beta+1)^{-1}n^{-\beta}\ln(1-t/T)$*, where* T *is the maximal time of existence of the un-normalized flow and* q_0 *is the point in* $\mathbb{H}^{n+1}_\kappa$ *where the evolving hypersurfaces shrink down to.*

Remark 1.9. In fact, it is well-known that if the condition (1.4) with non-negative constant $C(n,\beta)$ holds on a closed hypersurface, then the principal curvatures are larger than a everywhere and satisfy $\tilde{h}_{ij} > \varepsilon(C)\tilde{H}g_{ij}$ for a suitable $\varepsilon(C) > 0$ which is increasing with C. In other words, (1.4) implies in particular the h-convexity of M_t for $t \in [0,T)$, see Lemma 2.7 for details, exactly as the case in $\mathbb{R}^{n+1}$ of [22]. So it can be viewed as a stronger pinching condition on the principal curvature. A similar pinching condition has also been considered by Chow [9] for Gauß curvature flow, Cabezas-Rivas and Sinestrari [8] for volume-preserving flow by powers of the m-th mean curvature.

There exists a wide literature about the behavior of evolving hypersurfaces in the Euclidean space (or some Riemannian manifolds) in the direction of its inner normal with speed given by some curvature function. For $\beta = 1$, this flow in (1.1) is the well-known mean curvature flow, Huisken [14] showed that, when N^{n+1} is the Euclidean space $\mathbb{R}^{n+1}$, any closed convex hypersurface M_0 evolving by the mean curvature flow contracts to a point in finite time, becoming spherical in shape as the limit is approached. In [15], he extended this result to compact hypersurfaces in general Riemannian manifolds with suitable bounds on curvature. In fact, the speed of the mean curvature flow can be viewed as a symmetric function of the principal curvature with homogeneous degree one, the results of [14] and [15] have been generalized to a class of fully nonlinear parabolic equations of degree one in the Euclidean space (or some Riemannian manifolds), see [1], [2], [9], [10], [16] and [18]. If one considers the flows for which the speed has other positive degrees of homogeneity in the principal curvature it is more difficult to prove corresponding results for the flows. In some case it is known that if the initial hypersurface has an appropriate pinching condition on the principal curvature (unless the case the dimension of the hypersurface is two, see [3], [5] and [19]), then the evolving hypersurfaces converge to a single point (see [4], [9] and [24]).

The rest of the paper is organized as follows: Section 2 first gives some useful preliminary results employed in the rest of the paper, compute the evolution

equation of the turbulent quantity $\tilde{K}/\tilde{H}^n$, and applying the maximum principle to this equation gives that if the initial hypersurface is pinched good enough, then this is preserved for $t > 0$ as long as the flow (1.1) exists, this is a fundamental step in our procedure as in most of the literature quoted above. Furthermore, using this result Section 3 shows that the principal curvature comes close together, at least at those points where the mean curvature tends to infinity. Using these, Section 4 consider a natural normalized equation of the evolution equation (1.1) by keeping some the total area of the normalized hypersurfaces fixed and compute the evolution equations of the various normalized geometric quantities. Using these normalized evolution equations Section 5 shows that the hypersurfaces become spherical in shape as the limit is approached. Since the coefficients of the second order operator of the normalized evolution equations depend on the mean curvature and the normalized hypersurfaces have not uniform bound from below on the normalized mean curvature in time, the evolution equations may become priori degenerate when time goes to infinity, similar to the case of the Euclidean space which was pointed out in [22]. To deal with this problem, following the idea in [22], a regularity result on degenerate parabolic equations, due to DiBenedetto and Friedman [11], will be applied, since the normalized equation can be rewritten as a suitable porous medium equation. Section 6 proves that these normalized hypersurfaces converge to a geodesic sphere of $\mathbb{H}^{n+1}_\kappa$ smoothly and at exponential rate.

2. Preserving pinching of curvature

From now on, we use the same notation as in [13] (or [7, 14, 21]) in local coordinates. As we know that in the case $\beta = 1$ our H^β-flow (1.1) is the mean curvature flow, so we just only consider the flow (1.1) for $\beta > 1$. Theorem 1.7 shows that the evolving hypersurfaces are always strictly h-convex for positive times. So the following always assume that our evolving hypersurfaces are strictly h-convex, such that the mean curvature $H > na$, i.e., the corresponding turbulent mean curvature $\tilde{H} > 0$, the turbulent Gauß curvature $\tilde{K} = \det\{\tilde{h}^j_i\} > 0$, and the inverse of the turbulent Weingarten map $\{\tilde{b}^i_j\} = \{\tilde{h}^i_j\}^{-1}$ is well-defined. To control the pinching of the principal curvature along the flow (1.1) of the Euclidean space, Schulze, in [22], following an idea of Tso [24], looked at a test function $Q = K/H^n$, which was also considered in [8]. An analogous quantity which is the quotient $\tilde{Q} = \tilde{K}/\tilde{H}^n$ is more natural for our flow. By the arithmetic-geometric mean inequality, $\tilde{Q} \leq 1/n^n$ on M_t and equality holds at a point in M_t if and only if $\tilde{\lambda}_1 = \cdots = \tilde{\lambda}_n$, i.e., $\lambda_1 = \cdots = \lambda_n$ at the point. Thus, the only hypersurfaces such that $\tilde{Q} = 1/n^n$ are the geodesic spheres. The rest of this section consists of showing the inequality $\tilde{Q} \geq C > 0$ remains under the evolution.

First recall the evolution equations for geometric quantities and corresponding geometric quantities (see Theorem 3.2 and Theorem 3.3 in [13] for details).

Lemma 2.1. *On any solution M_t of (1.1) the following hold:*

$$\partial_t g_{ij} = -2H^\beta h_{ij}, \tag{2.1}$$

$$\partial_t \nu = \beta H^{\beta-1}\nabla H, \tag{2.2}$$

$$\partial_t(\mathrm{d}\mu_t) = -H^{\beta+1}\mathrm{d}\mu_t, \tag{2.3}$$

$$\begin{aligned}\partial_t h_{ij} = {}& \beta H^{\beta-1}\Delta h_{ij} + \beta(\beta-1)H^{\beta-2}\nabla_i H\nabla_j H - (\beta+1)H^\beta h_i^k h_{kj}\\ & + \beta\left(\left|A\right|^2 + na^2\right)H^{\beta-1}h_{ij} - a^2(\beta+1)H^\beta g_{ij},\end{aligned} \tag{2.4}$$

$$\begin{aligned}\partial_t h_i^j = {}& \beta H^{\beta-1}\Delta h_i^j + \beta(\beta-1)H^{\beta-2}\nabla_i H\nabla^j H - (\beta-1)H^\beta h_i^k h_k^j\\ & + \beta\left(\left|A\right|^2 + na^2\right)H^{\beta-1}h_i^j - a^2(\beta+1)H^\beta\delta_i^j,\end{aligned} \tag{2.5}$$

$$\partial_t H = \beta H^{\beta-1}\Delta H + \beta(\beta-1)H^{\beta-2}|\nabla H|^2 + \left(\left|A\right|^2 - na^2\right)H^\beta, \tag{2.6}$$

$$\begin{aligned}\partial_t H^l = {}& \beta H^{\beta-1}\Delta H^l + l\beta(\beta-l)H^{\beta+l-3}|\nabla H|^2\\ & + l\left(\left|A\right|^2 - na^2\right)H^{\beta+l-1}, \quad l\in\mathbb{R}.\end{aligned} \tag{2.7}$$

Lemma 2.2. *On any solution M_t of (1.1) the following hold:*

$$\begin{aligned}\partial_t \tilde{h}_{ij} = {}& \beta H^{\beta-1}\Delta\tilde{h}_{ij} + \beta(\beta-1)H^{\beta-2}\nabla_i\tilde{H}\nabla_j\tilde{H} - (\beta+1)H^\beta h_i^k\tilde{h}_{kj}\\ & + \beta H^{\beta-1}\left|\tilde{A}\right|^2\tilde{h}_{ij} + a(\beta+1)H^\beta\tilde{h}_{ij},\end{aligned} \tag{2.8}$$

$$\begin{aligned}\partial_t \tilde{h}_i^j = {}& \beta H^{\beta-1}\Delta\tilde{h}_i^j + \beta(\beta-1)H^{\beta-2}\nabla_i\tilde{H}\nabla^j\tilde{H} - (\beta-1)H^\beta h_i^k\tilde{h}_k^j\\ & + \beta\left|\tilde{A}\right|^2 H^{\beta-1}h_i^j + a(\beta+1)H^\beta\tilde{h}_i^j,\end{aligned} \tag{2.9}$$

$$\partial_t\tilde{H} = \beta H^{\beta-1}\Delta\tilde{H} + \beta(\beta-1)H^{\beta-2}|\nabla\tilde{H}|^2 + H^\beta\left|\tilde{A}\right|^2 + 2aH^\beta\tilde{H}, \tag{2.10}$$

$$\begin{aligned}\partial_t\tilde{H}^l = {}& \beta H^{\beta-1}\Delta\tilde{H}^l + l\beta\left[(\beta-1)\tilde{H} - (l-1)H\right]\tilde{H}^{l-2}H^{\beta-2}|\nabla\tilde{H}|^2\\ & + lH^\beta\tilde{H}^{l-1}\left|\tilde{A}\right|^2 + 2alH^\beta\tilde{H}^l, \quad l\in\mathbb{R}.\end{aligned} \tag{2.11}$$

The following algebraic property proved by Schulze in ([22], Lemma 2.5) will be needed in the later sections.

Lemma 2.3. *For any $\varepsilon > 0$ assume that $\lambda_i \geq \varepsilon H > 0$, $i = 1,\ldots,n$, at some point of an n-dimensional hypersurface. Then at the same point there exists a $\delta = \delta(\varepsilon, n) > 0$ such that*

$$\frac{n\left|A\right|^2 - H^2}{H^2} \geq \delta\left(\frac{1}{n^n} - \frac{K}{H^n}\right).$$

Consider the functions as in [7] and [13]:

$$\mathrm{s}_\kappa(x) = \frac{\sinh(\sqrt{|\kappa|}x)}{\sqrt{|\kappa|}} = \frac{\sinh(ax)}{a}, \qquad \mathrm{c}_\kappa(r) = \mathrm{s}_\kappa'(x),$$

$$\mathrm{ta}_\kappa(x) = \frac{\mathrm{s}_\kappa(x)}{\mathrm{c}_\kappa(x)}, \qquad \mathrm{co}_\kappa(x) = \frac{1}{\mathrm{ta}_\kappa(x)}.$$

Denote r_p the function "distance to p" in $\mathbb{H}^{n+1}_\kappa$ and use the notation $\partial_{r_p} = \bar{\nabla} r_p$. And denote the component of ∂_{r_p} by $\partial^\top_{r_p}$ tangent to M_t, which satisfies $\partial_{r_p} = \nabla(r_p|_{M^n})$. Define the inner radius ρ_- and the circumradius radius ρ_+ by

$$\rho_+(t) = \inf\{r : B_r(q) \text{ encloses } M_t \text{ for some } q \in \mathbb{H}^{n+1}_\kappa\},$$
$$\rho_-(t) = \sup\{r : B_r(q) \text{ is enclosed by } M_t \text{ for some } q \in \mathbb{H}^{n+1}_\kappa\},$$

where $B_r(q)$ is the geodesic ball of radius r with centered at q. The following well-known result for h-convex hypersurfaces in $\mathbb{H}^{n+1}_\kappa$ will be applied in later sections.

Lemma 2.4. *Let Ω be a compact h-convex domain, o the center of an inball of Ω, ρ_- its inner radius, and ρ_+ its circumradius radius. Furthermore let $\tau := \mathrm{ta}_\kappa(\frac{a\rho_-}{2})$, then*

i) *The maximal distance $\max d(o, \partial\Omega)$ between o and the points in $\partial\Omega$ satisfies the inequality*

$$\mathrm{maxd}(o, \partial\Omega) \le \rho_- + a\frac{\ln(1+\sqrt{\tau})^2}{1+\tau} < \rho_- + a\ln 2.$$

ii) *For any interior point p of Ω, $\langle \nu, \partial_{r_p}\rangle \ge a\mathrm{ta}_\kappa(\mathrm{dist}((p, \partial\Omega))$, where* dist *denotes the distance in the ambient space $\mathbb{H}^{n+1}_\kappa$.*

iii) *There exists a constant $C = C(a) > 0$ such that*

$$\rho_+ \le C\left(\rho_- + \sqrt{\rho_-}\right).$$

Proof. See ([6], Theorem 3.1) for the proof of i) and ii) in the lemma. As a consequence of i) and ii) in the lemma, iii) in the lemma has been proved by Makowski (see ([17], Theorem 5.2). □

Lemma 2.5. *On any solution M_t of (1.1) the quantity $\tilde{K}$ satisfies the following evolution equation*:

$$\begin{aligned}
\partial_t \tilde{K} &= \beta H^{\beta-1}\Delta\tilde{K} - \beta H^{\beta-1}\frac{\left|\nabla\tilde{K}\right|^2}{\tilde{K}} - \beta H^{\beta-1}\tilde{K}\nabla^m\tilde{h}^j_i\nabla_m\tilde{b}^i_j \\
&\quad + \beta(\beta-1)H^{\beta-2}\tilde{K}\nabla_i\tilde{H}\nabla^j\tilde{H}\tilde{b}^i_j + (1-\beta)H^{\beta+1}\tilde{K} + \beta n H^{\beta-1}\left|\tilde{A}\right|^2\tilde{K} \\
&\quad + na(\beta+1)H^\beta\tilde{K} + a\beta H^{\beta-1}\left|\tilde{A}\right|^2\tilde{K}\sum_{i=1}^n\tilde{b}^i_i \\
&= \beta H^{\beta-1}\Bigg\{\Delta\tilde{K} - \frac{(n-1)}{n}\frac{\left|\nabla\tilde{K}\right|^2}{\tilde{K}} + \frac{\tilde{K}}{\tilde{H}^2}\left|\tilde{H}\nabla_i\tilde{h}^n_m - \tilde{h}^n_m\nabla_i\tilde{H}\right|^2_{g,\tilde{b}} \\
&\quad - \frac{\tilde{H}^{2n}}{n\tilde{K}}\left|\nabla(\tilde{K}\tilde{H}^{-n})\right|^2 + \frac{(\beta-1)}{H}\tilde{K}\nabla_i\tilde{H}\nabla^j\tilde{H}\tilde{b}^i_j\Bigg\} + (1-\beta)H^{\beta+1}\tilde{K}
\end{aligned} \tag{2.12}$$

$$+\beta nH^{\beta-1}\big|\tilde{A}\big|^2\tilde{K}+na(\beta+1)H^{\beta}\tilde{K}+a\beta H^{\beta-1}\big|\tilde{A}\big|^2\tilde{K}\sum_{i=1}^{n}\tilde{b}_i^i,$$

where

$$\left|\tilde{H}\nabla_i\tilde{h}_m^n-\tilde{h}_m^n\nabla_i\tilde{H}\right|^2_{g,\tilde{b}}:=\tilde{b}_n^k\tilde{b}_l^m\left(\tilde{H}\nabla_i\tilde{h}_m^n-\tilde{h}_m^n\nabla_i\tilde{H}\right)\left(\tilde{H}\nabla^i\tilde{h}_k^l-\tilde{h}_k^l\nabla^i\tilde{H}\right).$$

Proof. By using (2.9), one has

$$\begin{aligned}\partial_t\tilde{K}&=\partial_t\det\{\tilde{h}_i^j\}=(\partial_t\tilde{h}_i^j)\tilde{b}_j^i\tilde{K}\\&=\tilde{K}\tilde{b}_j^i\Big(\beta H^{\beta-1}\Delta\tilde{h}_i^j+\beta\left(\beta-1\right)H^{\beta-2}\nabla_i\tilde{H}\nabla^j\tilde{H}-\left(\beta-1\right)H^{\beta}h_i^k\tilde{h}_k^j\\&\quad+\beta\big|\tilde{A}\big|^2H^{\beta-1}h_i^j+a(\beta+1)H^{\beta}\tilde{h}_i^j\Big).\end{aligned}$$

On the other hand, note that

$$\nabla_m\tilde{K}=\tilde{K}\nabla_m\tilde{h}_i^j\tilde{b}_j^i,$$

which implies that

$$\begin{aligned}\Delta\tilde{K}&=\nabla^m\nabla_m\tilde{K}\\&=\frac{\left|\nabla\tilde{K}\right|^2}{\tilde{K}}+\tilde{K}\Delta\tilde{h}_i^j\tilde{b}_j^i+\tilde{K}\nabla_m\tilde{h}_i^j\nabla^m\tilde{b}_j^i.\end{aligned}$$

Therefore

$$\begin{aligned}\partial_t\tilde{K}&=\beta H^{\beta-1}\Delta\tilde{K}-\beta H^{\beta-1}\frac{\left|\nabla\tilde{K}\right|^2}{\tilde{K}}-\beta H^{\beta-1}\tilde{K}\nabla^m\tilde{h}_i^j\nabla_m\tilde{b}_j^i\\&\quad+\beta\left(\beta-1\right)H^{\beta-2}\tilde{K}\nabla_i\tilde{H}\nabla^j\tilde{H}\tilde{b}_j^i+(1-\beta)H^{\beta+1}\tilde{K}\\&\quad+na(\beta+1)H^{\beta}\tilde{K}+\beta H^{\beta-1}\big|\tilde{A}\big|^2\tilde{K}h_i^j\tilde{b}_j^i\\&=\beta H^{\beta-1}\Delta\tilde{K}-\beta H^{\beta-1}\frac{\left|\nabla\tilde{K}\right|^2}{\tilde{K}}-\beta H^{\beta-1}\tilde{K}\nabla^m\tilde{h}_i^j\nabla_m\tilde{b}_j^i\\&\quad+\beta\left(\beta-1\right)H^{\beta-2}\tilde{K}\nabla_i\tilde{H}\nabla^j\tilde{H}\tilde{b}_j^i+(1-\beta)H^{\beta+1}\tilde{K}+\beta nH^{\beta-1}\big|\tilde{A}\big|^2\tilde{K}\\&\quad+na(\beta+1)H^{\beta}\tilde{K}+a\beta H^{\beta-1}\big|\tilde{A}\big|^2\tilde{K}\sum_{i=1}^{n}\tilde{b}_i^i,\end{aligned}$$

where the second equality was derived by using the equations

$$h_i^k\tilde{b}_k^j=a\tilde{b}_i^j+\delta_i^j.$$

The second equality of the lemma then follows from

$$\nabla_m\tilde{b}_j^i=-\tilde{b}_k^i\nabla_m\tilde{h}_l^k\tilde{b}_j^l,$$

$$\frac{\tilde K}{\tilde H^2}\left|\tilde H\nabla_i\tilde h^n_m-\tilde h^n_m\nabla_i\tilde H\right|^2_{g,\tilde b}=\tilde K\tilde b^k_m\nabla_i\tilde h^m_n\tilde b^n_l\nabla^i\tilde h^l_k-2\frac{\left\langle\nabla\tilde K,\nabla\tilde H\right\rangle}{\tilde H}+n\tilde K\frac{|\nabla\tilde H|^2}{\tilde H^2},$$

and

$$-2\frac{\left\langle\nabla\tilde K,\nabla\tilde H\right\rangle}{\tilde H}+n\tilde K\frac{|\nabla\tilde H|^2}{\tilde H^2}=-\frac{\tilde H^{2n}}{n\tilde K}\left|\nabla(\tilde K\tilde H^{-n})\right|^2+\frac{\left|\nabla\tilde K\right|^2}{n\tilde K}.\qquad\square$$

Lemma 2.6. *On any solution M_t of* (1.1) *the quantity $\tilde Q$ satisfies the following evolution equation*:

$$\begin{aligned}\partial_t\tilde Q=\beta H^{\beta-1}\Bigg\{&\Delta\tilde Q+\frac{(n+1)}{n\tilde H^n}\left\langle\nabla\tilde Q,\nabla\tilde H^n\right\rangle-\frac{(n-1)}{n\tilde K}\left\langle\nabla\tilde Q,\nabla\tilde K\right\rangle-\frac{\tilde H^n}{n\tilde K}\left|\nabla\tilde Q\right|^2\\&+\frac{(\beta-1)\,\tilde Q}{H}\nabla_i\tilde H\nabla^j\tilde H\left(\tilde b^i_j-\frac{n}{\tilde H}\delta^i_j\right)+\frac{\tilde Q}{\tilde H^2}\left|\tilde H\nabla_i\tilde h^n_m-\tilde h^n_m\nabla_i\tilde H\right|^2_{g,\tilde b}\\&+\frac{(\beta-1)\tilde QH}{\beta\tilde H}\left(n|\tilde A|^2-\tilde H^2\right)+a\tilde Q|\tilde A|^2\left(\sum_{i=1}^n\tilde b^i_i-\frac{n^2}{\tilde H}\right)\Bigg\}.\end{aligned}\tag{2.13}$$

Proof. By (2.11) with $l=n$ and (2.12)

$$\begin{aligned}\partial_t\tilde Q&=\frac{1}{\tilde H^n}\partial_t\tilde K-\frac{\tilde K}{\tilde H^{2n}}\partial_t\tilde H^n\\&=\beta H^{\beta-1}\Bigg\{\frac{\Delta\tilde K}{\tilde H^n}-\frac{\tilde K}{\tilde H^{2n}}\Delta\tilde H^n-\frac{(n-1)}{n}\frac{\left|\nabla\tilde K\right|^2}{\tilde K\tilde H^n}-\frac{\tilde H^n}{n}\frac{\left|\nabla(\tilde K\tilde H^{-n})\right|^2}{\tilde K}\\&\quad+\frac{(\beta-1)}{H}\frac{\tilde K}{\tilde H^n}\nabla_i\tilde H\nabla^j\tilde H\tilde b^i_j-\frac{n\left((\beta-1)\tilde H-(n-1)H\right)}{H}\frac{\tilde K\left|\nabla(\tilde H)\right|^2}{\tilde H^{n+2}}\\&\quad+\frac{\tilde K}{\tilde H^{n+2}}\left|\tilde H\nabla_i\tilde h^n_m-\tilde h^n_m\nabla_i\tilde H\right|^2_{g,\tilde b}\\&\quad+\frac{\tilde K}{\tilde H^n}\left[\frac{(1-\beta)}{\beta}H^2+n\left(1-\frac{H}{\beta\tilde H}\right)|\tilde A|^2+na\frac{(\beta-1)}{\beta}H+a|\tilde A|^2\sum_{i=1}^n\tilde b^i_i\right]\Bigg\}.\end{aligned}\tag{2.14}$$

Recall the relation $H=\tilde H+na$, so that these terms in square bracket $[\ ]$ can be computed as follows:

$$\begin{aligned}&\frac{(1-\beta)}{\beta}H^2+n\left(1-\frac{H}{\beta\tilde H}\right)|\tilde A|^2+na\frac{(\beta-1)}{\beta}H+a|\tilde A|^2\sum_{i=1}^n\tilde b^i_i\\&=\frac{(1-\beta)}{\beta}H(H-na)+n\left(1-\frac{H}{\tilde H}-\frac{(1-\beta)}{\beta}\frac{H}{\tilde H}\right)|\tilde A|^2+a|\tilde A|^2\sum_{i=1}^n\tilde b^i_i\\&=\frac{(1-\beta)}{\beta}H\tilde H+n\left(-\frac{na}{\tilde H}\right)|\tilde A|^2+n\frac{(\beta-1)}{\beta}\frac{H}{\tilde H}|\tilde A|^2+a|\tilde A|^2\sum_{i=1}^n\tilde b^i_i\end{aligned}$$

$$= \frac{(\beta-1)H}{\beta\tilde{H}}\left(n\left|\tilde{A}\right|^2-\tilde{H}^2\right)+a\left|\tilde{A}\right|^2\left(\sum_{i=1}^{n}\tilde{b}_i^i-\frac{n^2}{\tilde{H}}\right). \tag{2.15}$$

On the other hand, the first derivative and second derivative term in the brace {} can be computed as follows, the equality

$$\nabla_i\left(\frac{\tilde{K}}{\tilde{H}^n}\right)=\frac{\nabla_i\tilde{K}}{\tilde{H}^n}-\frac{\tilde{K}}{\tilde{H}^{2n}}\nabla_i\tilde{H}^n$$

implies

$$\Delta\left(\frac{\tilde{K}}{\tilde{H}^n}\right)=\nabla^i\nabla_i\left(\frac{\tilde{K}}{\tilde{H}^n}\right)$$

$$= \frac{\left|\nabla\tilde{K}\right|^2}{\tilde{H}^n}-2\frac{\left\langle\nabla\tilde{H}^n,\nabla\tilde{K}\right\rangle}{\tilde{H}^{2n}}+2\frac{\tilde{K}}{\tilde{H}^{3n}}\nabla_m\tilde{H}^n\nabla^m\tilde{H}^n-\frac{\tilde{K}}{\tilde{H}^{2n}}\Delta\tilde{H}^n, \tag{2.16}$$

$$\nabla_i\left(\frac{\tilde{K}}{\tilde{H}^n}\right)\nabla^i\tilde{H}^n=\frac{\left\langle\nabla\tilde{H}^n,\nabla\tilde{K}\right\rangle}{\tilde{H}^n}-2\frac{\tilde{K}}{\tilde{H}^{2n}}\left|\nabla\tilde{H}^n\right|^2, \tag{2.17}$$

and

$$\nabla_i\left(\frac{\tilde{K}}{\tilde{H}^n}\right)\nabla^i\tilde{K}=\frac{\left|\nabla\tilde{K}\right|^2}{\tilde{H}^n}-\frac{\tilde{K}}{\tilde{H}^{2n}}\left\langle\nabla\tilde{H}^n,\nabla\tilde{K}\right\rangle. \tag{2.18}$$

From (2.16), (2.17) and (2.18), it follows

$$\begin{aligned}&\frac{\Delta\tilde{K}}{\tilde{H}^n}-\frac{\tilde{K}}{\tilde{H}^{2n}}\Delta\tilde{H}^n-\frac{(n-1)}{n}\frac{\left|\nabla\tilde{K}\right|^2}{\tilde{K}\tilde{H}^n}\\ &=\Delta\left(\frac{\tilde{K}}{\tilde{H}^n}\right)+\frac{(n+1)}{\tilde{H}^n}\left\langle\nabla\left(\frac{\tilde{K}}{\tilde{H}^n}\right),\nabla\tilde{H}^n\right\rangle\\ &\quad-\frac{(n-1)}{n\tilde{K}}\left\langle\nabla\left(\frac{\tilde{K}}{\tilde{H}^n}\right),\nabla\tilde{K}\right\rangle-n(n-1)\frac{\tilde{K}}{\tilde{H}^{n+2}}\left|\nabla\tilde{H}^n\right|^2.\end{aligned} \tag{2.19}$$

Thus, equations (2.19) and (2.15) apply to (2.14) to give

$$\begin{aligned}\partial_t\tilde{Q}=\beta H^{\beta-1}\Bigg\{&\Delta\left(\frac{\tilde{K}}{\tilde{H}^n}\right)+\frac{(n+1)}{\tilde{H}^n}\left\langle\nabla\left(\frac{\tilde{K}}{\tilde{H}^n}\right),\nabla\tilde{H}^n\right\rangle\\ &-\frac{(n-1)}{n\tilde{K}}\left\langle\nabla\left(\frac{\tilde{K}}{\tilde{H}^n}\right),\nabla\tilde{K}\right\rangle-\frac{\tilde{H}^n}{n\tilde{K}}\left|\nabla\left(\frac{\tilde{K}}{\tilde{H}^n}\right)\right|^2\\ &+\frac{(\beta-1)}{H}\frac{\tilde{K}}{\tilde{H}^n}\nabla_i\tilde{H}\nabla^j\tilde{H}\left(\tilde{b}_j^i-\frac{n}{\tilde{H}}\delta_j^i\right)+\frac{\tilde{K}}{\tilde{H}^{n+2}}\left|\tilde{H}\nabla_i\tilde{h}_m^n-\tilde{h}_m^n\nabla_i\tilde{H}\right|_{g,\tilde{b}}^2\end{aligned}$$

$$+\frac{(\beta-1)\,H}{\beta\tilde{H}}\frac{\tilde{K}}{\tilde{H}^n}\left(n\big|\tilde{A}\big|^2-\tilde{H}^2\right)+a\frac{\tilde{K}}{\tilde{H}^n}\big|\tilde{A}\big|^2\left(\sum_{i=1}^n\tilde{b}_i^i-\frac{n^2}{\tilde{H}}\right)\Bigg\},$$

which is (2.13). □

In order to apply the maximum principle to (2.13) and show that

$$\min_{p\in M_t}\tilde{Q}(p,t)$$

is non-decreasing in time some preliminary inequalities are needed in the sequel. The following elementary property is a consequence of ([8], Lemma 4.2) (see also [9] and [22]).

Lemma 2.7. *For any $\varepsilon\in(0,1/n)$ and any $\tilde{\lambda}=(\tilde{\lambda}_1,\dots,\tilde{\lambda}_n)\in\mathbb{R}^n$ with $\tilde{\lambda}_i>0$ for all $i=1,\dots,n$, there exists a constant $C=C(\varepsilon,n)\in(0,1/n^n)$ satisfies*

$$\tilde{Q}(\tilde{\lambda})>C$$

such that

$$\tilde{\lambda}_{\min}>\varepsilon\tilde{H}(\tilde{\lambda}).$$

The following estimate which is a stronger version of Lemma 2.3(ii) in [14] is also needed.

Lemma 2.8. *If $\tilde{H}>0$ and the inequality $\tilde{h}_i^j>\varepsilon\tilde{H}\delta_i^j$ is valid with some $\varepsilon>0$ at a point on a hypersurface immersed in $\mathbb{H}^{n+1}_\kappa$, then $\varepsilon\le 1/n$ and*

$$\left|\tilde{H}\nabla_i\tilde{h}^n_m-\tilde{h}^n_m\nabla_i\tilde{H}\right|^2\ge\frac{n-1}{2}\varepsilon^2\tilde{H}^2\left|\nabla\tilde{A}\right|^2.$$

Proof. The proof of the lemma can be argued exactly as in ([8], Lemma 4.1), only define $\tilde{h}_i^j:=h_i^j-a\delta_i^j$ at a point on a hypersurface immersed in $\mathbb{H}^{n+1}_\kappa$. □

The preceding two lemmas allow us to prove the pinching estimate for our flow, which is one of the key steps in the proof of our main result.

Theorem 2.9. *There exists a constant $C(n,\beta)\in(0,1/n^n)$ such that if the initial hypersurface M_0 satisfies (1.1) with the constant $C(n,\beta)$, then the inequality $\tilde{Q}>C_0$ is preserved under the H^β-flow in $\mathbb{H}^{n+1}_\kappa$.*

Proof. The evolution equation (2.10) of $\tilde{H}$ implies that if $\tilde{H}>0$ on initial hypersurface $\tilde{H}>0$ for $t>0$ under the H^β-flow, then ensures that the quotient $\tilde{Q}$ is well-defined for $t>0$. For proof of the theorem, it is suffices to prove that the minimum of $\tilde{Q}$ (denote by $\tilde{\mathcal{Q}}$) is nondecreasing in time. To this purpose applying the maximum principle to equation (2.13) for $\tilde{\mathcal{Q}}$ gives

$$\begin{aligned}\partial_t\tilde{\mathcal{Q}}\ge\beta H^{\beta-1}\tilde{\mathcal{Q}}\Bigg\{&\frac{1}{\tilde{H}^2}\left|\tilde{H}\nabla_i\tilde{h}^n_m-\tilde{h}^n_m\nabla_i\tilde{H}\right|^2_{g,\tilde{b}}+\frac{(\beta-1)}{H}\nabla_i\tilde{H}\nabla^j\tilde{H}\left(\tilde{b}^i_j-\frac{n}{\tilde{H}}\delta^i_j\right)\\&+\frac{1}{\beta}\frac{(\beta-1)H}{\tilde{H}}\left(n\big|\tilde{A}\big|^2-\tilde{H}^2\right)+a\big|\tilde{A}\big|^2\left(\sum_{i=1}^n\tilde{b}_i^i-\frac{n^2}{\tilde{H}}\right)\Bigg\}\end{aligned}$$

$$\geq \beta H^{\beta-1}\tilde{\mathcal{Q}}\Bigg\{\frac{1}{\tilde{H}^2}\left|\tilde{H}\nabla_i\tilde{h}^n_m-\tilde{h}^n_m\nabla_i\tilde{H}\right|^2_{g,\tilde{b}}-\frac{(\beta-1)}{\tilde{H}}\left|\nabla\tilde{H}\right|^2\left|\tilde{b}^i_j-\frac{n}{\tilde{H}}\delta^i_j\right|$$

$$+\frac{1}{\beta}\frac{(\beta-1)H}{\tilde{H}}\left(n\left|\tilde{A}\right|^2-\tilde{H}^2\right)+a\left|\tilde{A}\right|^2\left(\sum_{i=1}^n\tilde{b}^i_i-\frac{n^2}{\tilde{H}}\right)\Bigg\}, \tag{2.20}$$

where $\tilde{H}<H$ and Hölder inequality were used to compute the second inequality.

If the ratio of the biggest and smallest principal curvature at everywhere on initial hypersurface is close enough to 1, then the various terms appearing here can be estimated as follows. First, if the initial hypersurface is weakly h-convex, Theorem 1.7 ii) implies that M_t under H^β-flow in $\mathbb{H}^{n+1}_\kappa$ remains strictly h-convex. This implies that the third term of right hand side in inequality (2.20) can be dropped with the strictly h-convexity on M_t. The last term can also be dropped by the arithmetic-harmonic mean inequality,

$$\sum_{i=1}^n\tilde{b}^i_i-\frac{n^2}{\tilde{H}}\geq 0$$

on M_t. It remains to estimate the first two terms of right hand side in inequality (2.20), now proceeding exactly as in [8], [9] and [22], choose orthonormal frame which diagonalises $\tilde{\mathscr{W}}$ so that

$$\left|\tilde{H}\nabla_i\tilde{h}^n_m-\tilde{h}^n_m\nabla_i\tilde{H}\right|^2_{g,\tilde{b}}=\sum_{i,m,n}\frac{1}{\tilde{\lambda}_m}\frac{1}{\tilde{\lambda}_n}\left(\tilde{H}\nabla_i\tilde{h}^n_m-\tilde{h}^n_m\nabla_i\tilde{H}\right)^2$$

$$\geq\frac{1}{\tilde{H}^2}\sum_{i,m,n}\left(\tilde{H}\nabla_i\tilde{h}^n_m-\tilde{h}^n_m\nabla_i\tilde{H}\right)^2, \tag{2.21}$$

where $\tilde{\lambda}_m\leq\tilde{H}$ was used in the last inequality by strictly h-convexity of M_t, i.e., $\tilde{\lambda}_m>0$ for any m. Furthermore, h-convexity of a hypersurface and Lemma 2.8 imply that the inequality (2.21) can be estimated as follows:

$$\left|\tilde{H}\nabla_i\tilde{h}^n_m-\tilde{h}^n_m\nabla_i\tilde{H}\right|^2_{g,\tilde{b}}\geq\frac{n-1}{2}\varepsilon^2\left|\nabla\tilde{A}\right|^2\quad\text{for some}\quad\varepsilon\in(0,1/n). \tag{2.22}$$

A next step is to show that $\left|\tilde{b}^i_j-\frac{n}{\tilde{H}}\delta^i_j\right|$ is small if the principal curvatures are pinched enough. It is clear that

$$\left|\tilde{b}^i_j-\frac{n}{\tilde{H}}\delta^i_j\right|\leq\sqrt{n}\max\left\{\left(\frac{1}{\tilde{\lambda}_{\min}}-\frac{n}{\tilde{H}}\right),\left(\frac{n}{\tilde{H}}-\frac{1}{\tilde{\lambda}_{\max}}\right)\right\}.$$

Since for some $\varepsilon\in(0,1/n)$

$$\tilde{\lambda}_{\min}\geq\varepsilon\tilde{H}, \tag{2.23}$$

then

$$\frac{1}{\tilde{\lambda}_{\min}}-\frac{n}{\tilde{H}}\leq\frac{1-\varepsilon n}{\varepsilon\tilde{H}}. \tag{2.24}$$

On other hand, (2.23) gives

$$\tilde{\lambda}_{\max} \leq (1-(n-1)\varepsilon)\,\tilde{H} \tag{2.25}$$

which implies that

$$\frac{n}{\tilde{H}} - \frac{1}{\tilde{\lambda}_{\max}} \leq \frac{(n-1)\,(1-n\varepsilon)}{\tilde{H}\,(1-(n-1)\varepsilon)}. \tag{2.26}$$

This combines with estimate (2.24) to give

$$\left|\tilde{b}^i_j - \frac{n}{\tilde{H}}\delta^i_j\right| \leq \mathscr{N}(\varepsilon), \tag{2.27}$$

where

$$\mathscr{N}(\varepsilon) = \begin{cases} \dfrac{\sqrt{n}(1-\varepsilon n)}{\varepsilon\tilde{H}}, & 0<\varepsilon\leq\dfrac{1}{2(n-1)}, \\ \dfrac{\sqrt{n}(n-1)\,(1-n\varepsilon)}{\tilde{H}\,(1-(n-1)\varepsilon)}, & \dfrac{1}{2(n-1)}<\varepsilon<\dfrac{1}{n}. \end{cases}$$

Thus, the inequality $\left|\nabla\tilde{H}\right|^2 \leq n\left|\nabla\tilde{A}\right|^2$, estimations (2.22) and (2.27) give:

$$\begin{aligned} &\frac{1}{\tilde{H}^2}\left|\tilde{H}\nabla_i\tilde{h}^n_m - \tilde{h}^n_m\nabla_i\tilde{H}\right|^2_{g,\tilde{b}} - \frac{(\beta-1)}{\tilde{H}}\left|\nabla\tilde{H}\right|^2\left|\tilde{b}^i_j - \frac{n}{\tilde{H}}\delta^i_j\right| \\ &\geq \frac{\left|\nabla\tilde{A}\right|^2}{\left|\tilde{H}\right|^2}\left(\frac{n-1}{2}\varepsilon^2 - n\,(\beta-1)\,\mathscr{N}(\varepsilon)\right). \end{aligned} \tag{2.28}$$

To achieve our purpose by application of the maximum principle, it is needed that $\mathscr{N}'(\varepsilon) := \left(\frac{n-1}{2}\varepsilon^2 - n\,(\beta-1)\,\mathscr{N}(\varepsilon)\right)$ is non-negative on M_t. In fact, $\mathscr{N}(\varepsilon)$ is a strictly decreasing function of ε; in addition, $\mathscr{N}(\varepsilon)$ is arbitrarily large as ε goes to zero and tends to zero as ε goes to $1/n$ by its definition. Therefore, $\mathscr{N}'(\varepsilon)$ is a strictly increasing function of ε, it is negative as ε goes to zero and positive as ε goes to $1/n$. So there exists a unique value $\varepsilon_0 \in (0,1/n)$ such that

$$\mathscr{N}'(\varepsilon_0) = 0. \tag{2.29}$$

By Lemma 2.7 there exists a constant $C_0 \in (0,1/n^n)$ satisfies $\tilde{Q}(\tilde{\lambda}) > C_0$ such that $\tilde{\lambda}_{\min} > \varepsilon\tilde{H}(\tilde{\lambda})$ with a $\varepsilon_0 \in (0,1/n)$ given by (2.29). Thus, if $\tilde{Q} > C_0 \geq 0$ everywhere on the initial hypersurface, applying the maximum principle for $\tilde{Q}$ implies that $\partial_t\tilde{\mathcal{Q}} \geq 0$, i.e., $\tilde{\mathcal{Q}}$ is nondecreasing in time. This guarantees that $\tilde{Q} > C_0$ is preserved under the H^β-flow in $\mathbb{H}^{n+1}_\kappa$. □

3. The pinching estimate

This section will show that the principal curvature comes close together, at least at those points where the mean curvature tends to infinity (for the un-normalized equation (1.1)). To do so, let

$$f = \frac{1}{n^n} - \frac{\tilde{K}}{\tilde{H}^n}.$$

Then as remarked in Section 2, $f \geq 0$ with equality holding only at umbilic points.

Theorem 3.1. *If $\beta > 1$ and the initial hypersurface of $\mathbb{H}^{n+1}_{\kappa}$ is pinched in the sense that*

$$\frac{\tilde{K}(p)}{\tilde{H}^n(p)} > C(n,\beta) \quad \text{for all} \quad p \in M^n,$$

then there exist constants $\sigma > 0$ and $C < +\infty$ depending only on M_0 such that

$$\frac{1}{n^n} - \frac{\tilde{K}}{\tilde{H}^n} \leq C\tilde{H}^{-\sigma}$$

for all time $0 \leq t \leq T$ under the flow (1.1).

The rest of this section will consist of proving Theorem 3.1. Our goal is to bound the function $f_\sigma := \tilde{H}^\sigma f$ for sufficiently small σ. First, an evolution equation for the quantity f_σ is needed.

Lemma 3.2. *For any σ, f_σ has the evolution equation*

$$\begin{aligned}
\partial_t f_\sigma = \beta H^{\beta-1}\Bigg\{ & \Delta f_\sigma + 2\left(1-\sigma\left(1+f\frac{\tilde{H}^n}{\tilde{K}}\right)\right)\left\langle \nabla f_\sigma, \frac{\nabla \tilde{H}}{\tilde{H}}\right\rangle + \frac{\tilde{H}^{n-\sigma}}{\tilde{K}}\left|\nabla f_\sigma\right|^2 \\
& + \sigma\left(-1+(\beta-1)\frac{\tilde{H}}{H}+\sigma\left(1+f\frac{\tilde{H}^n}{\tilde{K}}\right)\right) f_\sigma \frac{\left|\nabla \tilde{H}\right|^2}{\tilde{H}^2} \\
& + \tilde{H}^\sigma\Bigg[-\frac{(\beta-1)}{H}\frac{\tilde{K}}{\tilde{H}^n}\nabla_i\tilde{H}\nabla^j\tilde{H}\left(\tilde{b}^i_j - \frac{n}{\tilde{H}}\delta^i_j\right) \\
& - \frac{\tilde{K}}{\tilde{H}^{n+2}}\left|\tilde{H}\nabla_i\tilde{h}^n_m - \tilde{h}^n_m\nabla_i\tilde{H}\right|^2_{g,\tilde{b}} + \frac{\sigma H}{\beta\tilde{H}} f\left|\tilde{A}\right|^2 \\
& - \frac{\tilde{K}}{\tilde{H}^n}\frac{(\beta-1)H}{\beta\tilde{H}}\left(n\left|\tilde{A}\right|^2 - \tilde{H}^2\right) \\
& - a\frac{\tilde{K}}{\tilde{H}^n}\left|\tilde{A}\right|^2\left(\sum_{i=1}^n \tilde{b}^i_i - \frac{n^2}{\tilde{H}}\right) + \frac{2a\sigma H}{\beta} f\Bigg]\Bigg\}.
\end{aligned} \tag{3.1}$$

Proof. From Lemma 2.6 and the evolution equation (2.11) of $\tilde{H}^\sigma$ it follows that

$$\begin{aligned}
\partial_t f_\sigma &= \tilde{H}^\sigma \partial_t f + f\partial_t \tilde{H}^\sigma \\
&= \beta H^{\beta-1}\Bigg\{ \tilde{H}^\sigma \Delta f + f\Delta\tilde{H}^\sigma + \tilde{H}^\sigma\Bigg[\frac{(n+1)}{\tilde{H}^n}\left\langle \nabla f, \nabla \tilde{H}^n\right\rangle \\
&\quad - \frac{(n-1)}{n\tilde{K}}\left\langle \nabla f, \nabla\tilde{K}\right\rangle - \frac{\tilde{H}^n}{n\tilde{K}}\left|\nabla f\right|^2 + \frac{\sigma\left((\beta-1)\tilde{H} - (\sigma-1)H\right)}{H} f\frac{\left|\nabla\tilde{H}\right|^2}{\tilde{H}^2} \\
&\quad - \frac{(\beta-1)}{H}\frac{\tilde{K}}{\tilde{H}^n}\nabla_i\tilde{H}\nabla^j\tilde{H}\left(\tilde{b}^i_j - \frac{n}{\tilde{H}}\delta^i_j\right) - \frac{\tilde{K}}{\tilde{H}^{n+2}}\left|\tilde{H}\nabla_i\tilde{h}^n_m - \tilde{h}^n_m\nabla_i\tilde{H}\right|^2_{g,\tilde{b}}
\end{aligned} \tag{3.2}$$

$$+\frac{(\beta-1)H}{\beta\tilde{H}}\frac{\tilde{K}}{\tilde{H}^n}\left(n\left|\tilde{A}\right|^2-\tilde{H}^2\right)+\frac{\sigma H}{\beta\tilde{H}}f\left|\tilde{A}\right|^2$$
$$-a\frac{\tilde{K}}{\tilde{H}^n}\left|\tilde{A}\right|^2\left(\sum_{i=1}^{n}\tilde{b}_i^i-\frac{n^2}{\tilde{H}}\right)+\frac{2a\sigma H}{\beta}f\Bigg]\Bigg\}.$$

Furthermore

$$\nabla_i f_\sigma=\tilde{H}^\sigma\nabla_i f+\sigma f\tilde{H}^{\sigma-1}\nabla_i\tilde{H},$$
$$\nabla_i\tilde{K}=\left(\frac{n\tilde{K}}{\tilde{H}}+\sigma f\tilde{H}^{\sigma-1}\right)\nabla_i\tilde{H}-f\tilde{H}^{n-\sigma}\nabla_i f_\sigma,$$

$$\Delta f_\sigma=\tilde{H}^\sigma\Delta f+f\Delta\tilde{H}^\sigma+2\sigma\tilde{H}^{\sigma-1}\left\langle\nabla f,\nabla\tilde{H}\right\rangle, \tag{3.3}$$

$$\tilde{H}^\sigma\left\langle\nabla f,\nabla\tilde{H}\right\rangle=\left\langle\nabla f_\sigma,\nabla\tilde{H}\right\rangle-\frac{\sigma f_\sigma}{\tilde{H}^2}\left|\nabla\tilde{H}\right|^2, \tag{3.4}$$

$$\left\langle\nabla f,\nabla\tilde{K}\right\rangle=\tilde{H}^{-\sigma}\left(\frac{n\tilde{K}}{\tilde{H}}+2\sigma f\tilde{H}^{\sigma-1}\right)\left\langle\nabla f_\sigma,\nabla\tilde{H}\right\rangle, \tag{3.5}$$
$$-\left(n\tilde{K}+\sigma f\tilde{H}^n\right)\frac{\sigma f}{\tilde{H}^2}\left|\nabla\tilde{H}\right|^2-\tilde{H}^{n-\sigma}\left|\nabla f_\sigma\right|^2,$$

$$|\nabla f|^2=\tilde{H}^{-2\sigma}\left|\nabla f_\sigma\right|^2-2\sigma f_\sigma\tilde{H}^{-2\sigma-1}\left\langle\nabla f_\sigma,\nabla\tilde{H}\right\rangle \tag{3.6}$$
$$+n\sigma^2 f_\sigma^2\tilde{H}^{-2\sigma}\left|\nabla\tilde{H}\right|^2.$$

Using identities (3.3), (3.4), (3.5), and (3.6) a direct calculation gives

$$\tilde{H}^\sigma\Delta f+f\Delta\tilde{H}^\sigma+\tilde{H}^\sigma\Bigg[\frac{(n+1)}{\tilde{H}^n}\left\langle\nabla f,\nabla\tilde{H}^n\right\rangle-\frac{(n-1)}{n\tilde{K}}\left\langle\nabla f,\nabla\tilde{K}\right\rangle$$
$$-\frac{\tilde{H}^n}{n\tilde{K}}|\nabla f|^2+\frac{\sigma\left((\beta-1)\tilde{H}-(\sigma-1)H\right)}{H}f\frac{\left|\nabla\tilde{H}\right|^2}{\tilde{H}^2}\Bigg]$$
$$=\Delta f_\sigma+2\left(1-\sigma\left(1+f\frac{\tilde{H}^n}{\tilde{K}}\right)\right)\left\langle\nabla f_\sigma,\frac{\nabla\tilde{H}}{\tilde{H}}\right\rangle+\frac{\tilde{H}^{n-\sigma}}{\tilde{K}}\left|\nabla f_\sigma\right|^2 \tag{3.7}$$
$$+\sigma\left(-1+(\beta-1)\frac{\tilde{H}}{H}+\sigma\left(1+f\frac{\tilde{H}^n}{\tilde{K}}\right)\right)f_\sigma\frac{\left|\nabla\tilde{H}\right|^2}{\tilde{H}^2}.$$

Identity (3.7) applies to (3.2) to give (3.1). □

Proof of Theorem 3.1. In order to apply the maximum principle to (3.1) for f_σ, the following inequality is needed:

$$
\begin{aligned}
&\sigma\left(-1+(\beta-1)\frac{\tilde{H}}{H}+\sigma\left(1+f\frac{\tilde{H}^n}{\tilde{K}}\right)\right)f\frac{\left|\nabla\tilde{H}\right|^2}{\tilde{H}^2}\\
&-\frac{(\beta-1)}{H}\frac{\tilde{K}}{\tilde{H}^n}\nabla_i\tilde{H}\nabla^j\tilde{H}\left(\tilde{b}^i_j-\frac{n}{\tilde{H}}\delta^i_j\right)-\frac{\tilde{K}}{\tilde{H}^{n+2}}\left|\tilde{H}\nabla_i\tilde{h}^n_m-\tilde{h}^n_m\nabla_i\tilde{H}\right|^2_{g,\tilde{b}}\\
&+\frac{\sigma H}{\beta\tilde{H}}f\left|\tilde{A}\right|^2-\frac{\tilde{K}}{\tilde{H}^n}\frac{(\beta-1)H}{\beta\tilde{H}}\left(n\left|\tilde{A}\right|^2-\tilde{H}^2\right)\\
&-a\frac{\tilde{K}}{\tilde{H}^n}\left|\tilde{A}\right|^2\left(\sum_{i=1}^n\tilde{b}^i_i-\frac{n^2}{\tilde{H}}\right)+\frac{2a\sigma H}{\beta}f\le 0.
\end{aligned}
\tag{3.8}
$$

Since on the initial hypersurface $\tilde{Q}=\tilde{K}(p)/\tilde{H}^n(p)>C(n,\beta)$, then this remains so on M_t for any $t>0$ by Theorem 2.9. Thus, as in the proof of Theorem 2.9 (see inequality (2.28)), there exists a constant $\varepsilon(>\varepsilon_0)$ such that

$$
\begin{aligned}
&-\frac{(\beta-1)}{H}\frac{\tilde{K}}{\tilde{H}^n}\nabla_i\tilde{H}\nabla^j\tilde{H}\left(\tilde{b}^i_j-\frac{n}{\tilde{H}}\delta^i_j\right)-\frac{\tilde{K}}{\tilde{H}^{n+2}}\left|\tilde{H}\nabla_i\tilde{h}^n_m-\tilde{h}^n_m\nabla_i\tilde{H}\right|^2_{g,\tilde{b}}\\
&\le -C(n,\beta)\mathscr{N}'(\varepsilon)\frac{\left|\nabla\tilde{A}\right|^2}{\left|\tilde{H}\right|^2}\\
&\le -\frac{1}{n}C(n,\beta)\mathscr{N}'(\varepsilon)\frac{\left|\nabla\tilde{H}\right|^2}{\left|\tilde{H}\right|^2},
\end{aligned}
\tag{3.9}
$$

where the inequality $\frac{1}{n}\left|\nabla\tilde{H}\right|^2\le\left|\nabla\tilde{A}\right|^2$ on a convex hypersurface was used to obtain the last inequality. Thus, since

$$
\left|-1+(\beta-1)\frac{\tilde{H}}{H}+\sigma\left(1+f\frac{\tilde{H}^n}{\tilde{K}}\right)\right|f
$$

always can be bounded on the hypersurfaces M_t for any $t>0$, choosing σ small enough implies that the first order derivative terms in the left hand side of (3.8) can be made non-positive. With the aid of $H=\tilde{H}+na$, $\tilde{H}^2\le n\left|\tilde{A}\right|^2$, the arithmetic-harmonic mean inequality, Lemma 2.3 and Theorem 2.9, the rest of terms in the left hand side of (3.8) can be estimated as follows:

$$
\begin{aligned}
&\frac{\sigma H}{\beta\tilde{H}}f\left|\tilde{A}\right|^2-\frac{(\beta-1)H}{\beta\tilde{H}}\frac{\tilde{K}}{\tilde{H}^n}\left(n\left|\tilde{A}\right|^2-\tilde{H}^2\right)-a\frac{\tilde{K}}{\tilde{H}^n}\left|\tilde{A}\right|^2\left(\sum_{i=1}^n\tilde{b}^i_i-\frac{n^2}{\tilde{H}}\right)+\frac{2a\sigma H}{\beta}f\\
&\le\frac{\sigma}{\beta}\left(1+\frac{na}{\tilde{H}}\right)f\left|\tilde{A}\right|^2-C_0(n,\beta)\delta\frac{(\beta-1)}{\beta}\left(1+\frac{na}{\tilde{H}}\right)f\left|\tilde{H}\right|^2+\frac{2a\sigma H}{\beta\left|\tilde{A}\right|^2}f\left|\tilde{A}\right|^2
\end{aligned}
$$

$$\leq \frac{\sigma}{\beta}\left(1+\frac{na}{\tilde{H}}\right)f\left|\tilde{A}\right|^2-C_0(n,\beta)\delta\frac{(\beta-1)}{\beta}\left(1+\frac{na}{\tilde{H}}\right)f\left|\tilde{A}\right|^2$$
$$+\frac{2a\sigma}{\beta}\left(\frac{\sqrt{n}}{\left|\tilde{A}\right|}+\frac{na}{\left|\tilde{A}\right|^2}\right)f\left|\tilde{A}\right|^2$$
$$\leq \frac{1}{\beta}\left\{-C_0(n,\beta)\delta(\beta-1)\left(1+\frac{na}{\tilde{H}}\right)+\sigma\left[\left(1+\frac{na}{\tilde{H}}\right)+2a\left(\frac{\sqrt{n}}{\left|\tilde{A}\right|}+\frac{na}{\left|\tilde{A}\right|^2}\right)\right]\right\}f\left|\tilde{A}\right|^2.$$

Furthermore, choosing σ even smaller implies that

$$-C_0(n,\beta)\delta(\beta-1)\left(1+\frac{na}{\tilde{H}}\right)+\sigma\left[\left(1+\frac{na}{\tilde{H}}\right)+2a\left(\frac{\sqrt{n}}{\left|\tilde{A}\right|}+\frac{na}{\left|\tilde{A}\right|^2}\right)\right]\leq 0.$$

Now, applying the maximum principal to (3.1) for f_σ shows that there exists a $\sigma>0$ such that

$$f_\sigma(p,t)\leq \max_{p\in M^n} f_\sigma(p,0), \quad \forall\,(p,t)\in M\times[p,T),$$

which concludes the proof. □

4. The normalized equation

As we have seen in [13], the solution of the un-normalized equation (1.1) shrinks down to a single point q_0 in $\mathbb{H}^{n+1}_\kappa$ after a finite time. Note that q_0 lays in the region enclosed by M_t for all times $0\leq t\leq T$. Since it is shown in Sections 2 and 3 that the initial pinching is preserved under the initial conditions and that it improves as the curvature becomes large, thus for t close to T all quantities arising from the metric of the hyperbolic space are negligible compared to the curvatures of the hypersurface. Then in this section we can follow the same way as in the Euclidean case ([1], [14] and [22]) to consider a natural normalized equation of the evolution equation (1.1) by keeping some geometrical quantity fixed, for example the total area of the normalized hypersurfaces which equals to the same area of the unit geodesic sphere of $\mathbb{H}^{n+1}_\kappa$, i.e., the total area $\mathrm{A}(\hat{M}_t)=\mathrm{A}(\hat{M}_0)$, where the geometric quantities associated with the normalized immersions are distinguished by a hat. The next section follows closely Section 7 of [1] and Section 6 of [2].

Let $\alpha=(\beta+1)\,n^\beta$, $\psi(t)=(\alpha(T-t))^{-1/(\beta+1)}$, multiply the solution X of (1.1) at each time $0\leq t\leq T$ with a positive constant $\psi(t)$ such that the hypersurface $\hat{M}_t$ is given by

$$\hat{\mathrm{X}}(\cdot,t)=\psi(t)(\mathrm{X}(\cdot,t)-q_0).$$

Define a new time parameter $\hat{t}$ by $\hat{t}:=-\frac{1}{\alpha}\ln\left(1-\frac{t}{T}\right)$ such that

$$\frac{\mathrm{d}\hat{t}}{\mathrm{d}t}=(\psi(t))^{\beta+1}. \tag{4.1}$$

A direct calculation shows that the following equation is satisfied for $\hat{t}$ in the interval $[0,+\infty)$:

$$\partial_{\hat{t}}\hat{\mathrm{X}}(\cdot,\hat{t}) = -\hat{H}^{\beta}\cdot\hat{\nu} + n^{\beta}\hat{\mathrm{X}}(p,\hat{t}). \tag{4.2}$$

The following evolution equations for various geometric quantities along the normalized flow can be obtained.

Theorem 4.1. *On any solution $\hat{M}_{\hat{t}}$ of* (4.2) *the following hold*:

$$\partial_{\hat{t}}\hat{g}_{ij} = -2\hat{H}^{\beta}\hat{h}_{ij} + 2n^{\beta}\hat{g}_{ij}, \tag{4.3}$$

$$\partial_{\hat{t}}\hat{\nu} = \beta\hat{H}^{\beta-1}\hat{\nabla}\hat{H}, \tag{4.4}$$

$$\partial_{\hat{t}}(\mathrm{d}\hat{\mu}_{\hat{t}}) = -\hat{H}^{\beta+1}\mathrm{d}\hat{\mu}_{\hat{t}} + n^{\beta+1}\mathrm{d}\hat{\mu}_{\hat{t}}, \tag{4.5}$$

$$\begin{aligned}\partial_{\hat{t}}\hat{h}_{ij} = {}& \beta\hat{H}^{\beta-1}\hat{\Delta}\hat{h}_{ij} + \beta(\beta-1)\hat{H}^{\beta-2}\hat{\nabla}_i\hat{H}\hat{\nabla}_j\hat{H} - (\beta+1)\hat{H}^{\beta}\hat{h}_i^k\hat{h}_{kj}\\ & + \beta\left(\left|\hat{A}\right|^2 + na^2\right)\hat{H}^{\beta-1}\hat{h}_{ij} - a^2(\beta+1)\hat{H}^{\beta}\hat{g}_{ij}\\ & + n^{\beta}\hat{h}_{ij} + a^2n^{\beta}\left\langle\hat{\mathrm{X}},\hat{\nu}\right\rangle\hat{g}_{ij},\end{aligned} \tag{4.6}$$

$$\begin{aligned}\partial_{\hat{t}}\hat{h}_i^j = {}& \beta\hat{H}^{\beta-1}\hat{\Delta}\hat{h}_i^j + \beta(\beta-1)\hat{H}^{\beta-2}\hat{\nabla}_i\hat{H}\nabla^j\hat{H} - (\beta-1)\hat{H}^{\beta}\hat{h}_i^k\hat{h}_k^j\\ & + \beta\left(\left|\hat{A}\right|^2 + na^2\right)\hat{H}^{\beta-1}\hat{h}_i^j - a^2(\beta+1)\hat{H}^{\beta}\delta_i^j,\\ & - n^{\beta}\hat{h}_i^j + a^2n^{\beta}\left\langle\hat{\mathrm{X}},\hat{\nu}\right\rangle\delta_i^j,\end{aligned} \tag{4.7}$$

$$\begin{aligned}\partial_{\hat{t}}\hat{H} = {}& \beta\hat{H}^{\beta-1}\hat{\Delta}\hat{H} + \beta(\beta-1)\hat{H}^{\beta-2}|\hat{\nabla}\hat{H}|^2 + \left(\left|\hat{A}\right|^2 - na^2\right)\hat{H}^{\beta}\\ & - n^{\beta}\hat{H} + a^2n^{\beta+1}\left\langle\hat{\mathrm{X}},\hat{\nu}\right\rangle,\\ = {}& \hat{\Delta}\hat{H}^{\beta} + \left(\left|\hat{A}\right|^2 - na^2\right)\hat{H}^{\beta} - n^{\beta}\hat{H} + a^2n^{\beta+1}\left\langle\hat{\mathrm{X}},\hat{\nu}\right\rangle,\end{aligned} \tag{4.8}$$

$$\begin{aligned}\partial_{\hat{t}}\hat{H}^l = {}& \beta\hat{H}^{\beta-1}\hat{\Delta}\hat{H}^l + l\beta(\beta-l)\hat{H}^{\beta+l-3}\left|\hat{\nabla}\hat{H}\right|^2 + l\left(\left|\hat{A}\right|^2 - na^2\right)\hat{H}^{\beta+l-1}\\ & - ln^{\beta}\hat{H}^l + a^2ln^{\beta+1}\left\langle\hat{\mathrm{X}},\hat{\nu}\right\rangle\hat{H}^{l-1}, \quad l\in\mathbb{R}.\end{aligned} \tag{4.9}$$

Proof. Since the metric $\langle\cdot,\cdot\rangle$ on $\mathbb{H}^{n+1}_{\kappa}$ is independent of time,

$$\begin{aligned}\partial_{\hat{t}}\hat{g}_{ij} &= \partial_{\hat{t}}\left\langle\bar{\nabla}_i\hat{\mathrm{X}},\bar{\nabla}_j\hat{\mathrm{X}}\right\rangle\\ &= \left\langle\partial_{\hat{t}}\bar{\nabla}_i\hat{\mathrm{X}},\bar{\nabla}_j\hat{\mathrm{X}}\right\rangle + (i\leftrightarrow j)\\ &= \left\langle\bar{\nabla}_i\left(-\hat{H}^{\beta}\cdot\hat{\nu} + n^{\beta}\hat{\mathrm{X}}\right),\bar{\nabla}_j\hat{\mathrm{X}}\right\rangle + (i\leftrightarrow j)\\ &= -\hat{H}^{\beta}\left\langle\bar{\nabla}_i\hat{\nu},\bar{\nabla}_j\hat{\mathrm{X}}\right\rangle + n^{\beta}\left\langle\bar{\nabla}_i\hat{\mathrm{X}},\bar{\nabla}_j\hat{\mathrm{X}}\right\rangle + (i\leftrightarrow j)\\ &= -2\hat{H}^{\beta}\hat{h}_{ij} + 2n^{\beta}\hat{g}_{ij}\end{aligned}$$

using the definitions of the first and the second fundamental form. The evolution of the unit normal to $\hat{M}_{\hat{t}}$ is a straightforward computation:

$$\begin{aligned}
\partial_{\hat{t}}\hat{\nu} &= \left\langle \bar{\nabla}_i \hat{\mathrm{X}}, \partial_{\hat{t}}\hat{\nu} \right\rangle \bar{\nabla}_j \hat{\mathrm{X}} \hat{g}^{ij} \\
&= -\left\langle \partial_{\hat{t}} \bar{\nabla}_i \hat{\mathrm{X}}, \hat{\nu} \right\rangle \bar{\nabla}_j \hat{\mathrm{X}} \hat{g}^{ij} \\
&= -\left\langle \bar{\nabla}_i \left(-\hat{H}^\beta \cdot \hat{\nu} + n^\beta \hat{\mathrm{X}} \right), \hat{\nu} \right\rangle \bar{\nabla}_j \hat{\mathrm{X}} \hat{g}^{ij} \\
&= \bar{\nabla}_i \hat{H}^\beta \bar{\nabla}_j \hat{\mathrm{X}} \hat{g}^{ij} \\
&= \hat{\nabla} \hat{H}^\beta \\
&= \beta \hat{H}^{\beta-1} \hat{\nabla} \hat{H}.
\end{aligned}$$

The time derivative of the measure $\mathrm{d}\hat{\mu}_{\hat{t}} = \sqrt{\det \hat{g}_{ij}}\mathrm{d}\hat{x}$ on $\hat{M}_{\hat{t}}$ can be easily derived from (4.3).

Denote $\bar{\nabla}_i \hat{\mathrm{X}}$ by $\hat{e}_i$ for all $i = 1, \ldots, n$. First, the evolution of the normalized second fundamental $\{\hat{h}_{ij}\}$ can be calculated from the definition of $\{\hat{h}_{ij}\}$ and the evolving equation (4.4) as follows:

$$\begin{aligned}
\partial_{\hat{t}}\hat{h}_{ij} =& -\partial_{\hat{t}} \left\langle \bar{\nabla}_{\hat{e}_i} \hat{e}_j, \hat{\nu} \right\rangle \\
=& -\left\langle \bar{\nabla}_{\hat{\mathrm{X}}_*(\partial_{\hat{t}})} \bar{\nabla}_{\hat{e}_i} \hat{e}_j, \hat{\nu} \right\rangle - \left\langle \bar{\nabla}_{\hat{e}_i} \hat{e}_j, \partial_{\hat{t}}\hat{\nu} \right\rangle \\
=& -\left\langle \bar{\nabla}_{\hat{e}_i} \bar{\nabla}_{\hat{\mathrm{X}}_*(\partial_{\hat{t}})} \hat{e}_j, \hat{\nu} \right\rangle - \left\langle \bar{\mathrm{R}}_{\hat{e}_i, \hat{\mathrm{X}}_*(\partial_{\hat{t}})} \hat{e}_j, \hat{\nu} \right\rangle - \left\langle \bar{\nabla}_{\hat{e}_i} \hat{e}_j, \hat{\nabla} \hat{H}^\beta \right\rangle \\
=& -\left\langle \bar{\nabla}_{\hat{e}_i} \bar{\nabla}_{\hat{e}_j} \left(-\hat{H}^\beta \cdot \hat{\nu} + n^\beta \hat{\mathrm{X}} \right), \hat{\nu} \right\rangle - \left\langle \bar{\mathrm{R}}_{\hat{e}_i, -\hat{H}^\beta \cdot \hat{\nu} + n^\beta \hat{\mathrm{X}}(p,\hat{t})} \hat{e}_j, \hat{\nu} \right\rangle \\
& - \left\langle \bar{\nabla}_{\hat{e}_i} \hat{e}_j, \hat{e}_k \right\rangle \hat{\nabla}_l \hat{H}^\beta \hat{g}^{kl} \\
=& \hat{\nabla}_i \hat{\nabla}_j \hat{H}^\beta - \left\langle \hat{\nabla}_{\hat{e}_i} \hat{e}_j, \hat{e}_k \right\rangle \hat{\nabla}_l \hat{H}^\beta \hat{g}^{kl} - \hat{H}^\beta \left\langle \bar{\nabla}_i \hat{\nu}, \bar{\nabla}_j \hat{\nu} \right\rangle \\
& - n^\beta \left\langle \bar{\nabla}_{\hat{e}_i} \hat{e}_j, \hat{\nu} \right\rangle + \hat{H}^\beta \bar{\mathrm{R}}_{i0j0} - n^\beta \hat{\mathrm{X}}^\gamma \hat{H}^\beta \bar{\mathrm{R}}_{i\gamma j0} \\
=& \mathrm{Hess}_{\hat{\nabla}} \hat{H}^\beta(\hat{e}_i, \hat{e}_j) - \hat{H}^\beta \hat{h}_{ik} \hat{h}^k_j + n^\beta \hat{h}_{ij} + \hat{H}^\beta \bar{\mathrm{R}}_{i0j0} + n^\beta \hat{\mathrm{X}}^\gamma \hat{H}^\beta \bar{\mathrm{R}}_{i\gamma j0},
\end{aligned}$$

where $\hat{\mathrm{X}} = \hat{\mathrm{X}}^\gamma \hat{e}_\gamma$, $\gamma = 0, 1, \ldots, n$ and $\hat{\nu}$ is arranged to be $\hat{e}_0$. Thus, the desired equation (4.6) follows from Simons' identity (see [23] for a proof or see also [20] for a simple derivation) which is a consequence of the equations of Gauß and Codazzi:

$$\begin{aligned}
\hat{\Delta}\hat{h}_{ij} = \hat{\nabla}_i \hat{\nabla}_j \hat{H} + \hat{H} \hat{h}_{ik} \hat{h}^k_j - \left|\hat{A}\right|^2 \hat{h}_{ij} + \hat{H} \bar{\mathrm{R}}_{i0j0} - \hat{h}_{ij} \bar{\mathrm{R}}^k{}_{0k0} \\
+ \hat{h}_{jk} \bar{\mathrm{R}}^k{}_{li}{}^l + \hat{h}_{ik} \bar{\mathrm{R}}^k{}_{lj}{}^l - 2\hat{h}_{kl} \bar{\mathrm{R}}_i{}^k{}_j{}^l + \bar{\nabla}_j \bar{\mathrm{R}}^k{}_{0ki} + \bar{\nabla}_k \bar{\mathrm{R}}_{i0}{}^k{}_j
\end{aligned}$$

and having into account that in our case the background space is a hyperbolic space, the ambient space is locally symmetric ($\bar{\nabla}\bar{\mathrm{R}} = 0$) and the Riemann curvature tensor of the ambient space takes the form

$$\bar{\mathrm{R}}_{\alpha\beta\gamma\delta} = -a^2(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}).$$

The evolution equations (4.7), (4.8) and (4.9) can be easily derived from the evolution equations (4.3) and (4.6). □

From the definition of $\psi(t)$ and the relation (4.1) it follows that

$$\frac{\mathrm{d}\psi(\hat{t})}{\mathrm{d}\hat{t}} = n^{\beta}\psi(\hat{t}) \tag{4.10}$$

and

$$\frac{\mathrm{d}\psi^{-1}(\hat{t})}{\mathrm{d}\hat{t}} = -n^{\beta}\psi^{-1}(\hat{t}). \tag{4.11}$$

Thus, the normalized turbulent second fundamental form is given by

$$\hat{\mathring{h}}_{ij} = \hat{h}_{ij} - a\psi^{-1}\hat{g}_{ij}, \tag{4.12}$$

which implies that

$$\hat{\mathring{H}} = \hat{H} - na\psi^{-1} \tag{4.13}$$

and

$$\left|\hat{\mathring{A}}\right|^2 = \left|\hat{A}\right|^2 + na^2\psi^{-2} - 2a\psi^{-1}\hat{H}. \tag{4.14}$$

From (4.10), (4.11), (4.12), (4.13), (4.14) and Theorem 4.1 the following theorem can be obtained by computations similar to those in Theorem 4.1 above.

Theorem 4.2. *On any solution $\hat{M}_{\hat{t}}$ of* (4.2) *the following hold*:

$$\begin{aligned}\partial_{\hat{t}}\hat{\mathring{h}}_{ij} &= \beta\hat{H}^{\beta-1}\hat{\Delta}\hat{\mathring{h}}_{ij} + \beta(\beta-1)\hat{H}^{\beta-2}\hat{\nabla}_i\hat{\mathring{H}}\hat{\nabla}_j\hat{\mathring{H}} - (\beta+1)\hat{H}^{\beta}\hat{h}_i^k\hat{\mathring{h}}_{kj}\\ &\quad+\beta\left(\left|\hat{\mathring{A}}\right|^2 + na^2\left(1-\psi^{-2}\right)\right)\hat{H}^{\beta-1}\hat{h}_{ij} + a(\beta+1)\psi^{-1}\hat{H}^{\beta}\hat{\mathring{h}}_{ij}\\ &\quad+n^{\beta}\hat{\mathring{h}}_{ij} + a^2n^{\beta}\left\langle\hat{\mathrm{X}},\hat{\nu}\right\rangle\hat{g}_{ij} + a^2(\beta+1)\left(\psi^{-2}-1\right)\hat{H}^{\beta}\hat{g}_{ij},\end{aligned} \tag{4.15}$$

$$\begin{aligned}\partial_{\hat{t}}\hat{\mathring{h}}_i^j &= \beta\hat{H}^{\beta-1}\hat{\Delta}\hat{\mathring{h}}_i^j + \beta(\beta-1)\hat{H}^{\beta-2}\hat{\nabla}_i\hat{\mathring{H}}\nabla^j\hat{\mathring{H}} - (\beta-1)\hat{H}^{\beta}\hat{h}_i^k\hat{\mathring{h}}_k^j\\ &\quad+\beta\left(\left|\hat{\mathring{A}}\right|^2 + na^2\left(1-\psi^{-2}\right)\right)\hat{H}^{\beta-1}\hat{h}_i^j + a(\beta+1)\psi^{-1}\hat{H}^{\beta}\hat{\mathring{h}}_i^j\\ &\quad-n^{\beta}\hat{\mathring{h}}_i^j + a^2n^{\beta}\left\langle\hat{\mathrm{X}},\hat{\nu}\right\rangle\delta_i^j + a^2(\beta+1)\left(\psi^{-2}-1\right)\hat{H}^{\beta}\delta_i^j,\end{aligned} \tag{4.16}$$

$$\begin{aligned}\partial_{\hat{t}}\hat{\mathring{H}} &= \beta\hat{H}^{\beta-1}\hat{\Delta}\hat{\mathring{H}} + \beta(\beta-1)\hat{H}^{\beta-2}|\hat{\nabla}\hat{\mathring{H}}|^2 + \hat{H}^{\beta}\left|\hat{\mathring{A}}\right|^2 + 2a\psi^{-1}\hat{H}^{\beta}\hat{\mathring{H}}\\ &\quad-na^2\left(1-\psi^{-2}\right)\hat{H}^{\beta} - n^{\beta}\hat{\mathring{H}} + a^2n^{\beta+1}\left\langle\hat{\mathrm{X}},\hat{\nu}\right\rangle,\end{aligned} \tag{4.17}$$

$$\begin{aligned}\partial_{\hat{t}}\hat{\mathring{H}}^l &= \beta\hat{H}^{\beta-1}\hat{\Delta}\hat{\mathring{H}}^l + l\beta[(\beta-1)\tilde{\mathring{H}} - (l-1)\hat{H}]\hat{\mathring{H}}^{l-2}H^{\beta-2}|\hat{\nabla}\hat{\mathring{H}}|^2\\ &\quad+l\hat{H}^{\beta}\hat{\mathring{H}}^{l-1}\left|\hat{\mathring{A}}\right|^2 + 2al\psi^{-1}\hat{H}^{\beta}\hat{\mathring{H}}^l - na^2l\left(1-\psi^{-2}\right)\hat{H}^{\beta}\\ &\quad-ln^{\beta}\hat{\mathring{H}}^l + a^2ln^{\beta+1}\left\langle\hat{\mathrm{X}},\hat{\nu}\right\rangle\hat{\mathring{H}}^{l-1},\quad l\in\mathbb{R}.\end{aligned} \tag{4.18}$$

Denote the normalized quantity $\det\{\hat{\tilde{h}}_i^j\}$ for the turbulent Gauß curvature $\tilde{K} = \det\{\tilde{h}_i^j\}$ by $\hat{\tilde{K}}$. Since the $\hat{M}_{\hat{t}}$ remains strictly convex along the normalized flow (4.2) for $t > 0$, i.e., $\hat{\tilde{h}}_{ij} > 0$, the inverse of the normalized turbulent Weingarten map $\{\hat{\tilde{b}}_j^i\} = \{\hat{\tilde{h}}_j^i\}^{-1}$ is well-defined. From (4.16) the evolution equation for $\hat{\tilde{K}}$ along the normalized flow is easily derived in the same way as the un-normalized equation.

Lemma 4.3. *On any solution $\hat{M}_{\hat{t}}$ of* (4.2) *the quantity $\hat{\tilde{K}}$ satisfies the following evolution equation*:

(4.19)

$$\begin{aligned}\partial_t \hat{\tilde{K}} = \beta \hat{H}^{\beta-1}\Bigg\{ & \hat{\Delta}\hat{\tilde{K}} - \frac{(n-1)}{n}\frac{\left|\hat{\nabla}\hat{\tilde{K}}\right|^2}{\hat{\tilde{K}}} + \frac{\hat{\tilde{K}}}{\hat{\tilde{H}}^2}\left|\hat{\tilde{H}}\hat{\nabla}_i\hat{\tilde{H}}^n_m - \hat{\tilde{H}}^n_m\hat{\nabla}_i\hat{\tilde{H}}\right|^2_{\hat{g},\hat{\tilde{b}}} \\ & - \frac{\hat{\tilde{H}}^{2n}}{n\hat{\tilde{K}}}\left|\hat{\nabla}(\hat{\tilde{K}}\hat{\tilde{H}}^{-n})\right|^2 + \frac{(\beta-1)}{\hat{H}}\hat{\tilde{K}}\hat{\nabla}_i\hat{\tilde{H}}\hat{\nabla}^j\hat{\tilde{H}}\hat{\tilde{b}}_j^i + \frac{(1-\beta)}{\beta}\hat{H}^2\hat{\tilde{K}} \\ & + a\left(\left|\hat{\tilde{A}}\right|^2 + na^2\left(1-\psi^{-2}\right)\right)\hat{\tilde{K}}\left(n + a\psi^{-1}\sum_{i=1}^n \hat{\tilde{b}}_i^i\right) \\ & + \frac{na\psi^{-1}(\beta+1)}{\beta}\hat{H}\hat{\tilde{K}}\Bigg\} - n^{\beta+1}\hat{\tilde{K}} + a^2 n^\beta\left\langle \hat{\mathrm{X}}, \hat{\nu}\right\rangle\hat{\tilde{K}}\sum_{i=1}^n \hat{\tilde{b}}_i^i.\end{aligned}$$

5. Convergence to a unit geodesic sphere

To finish the proof of Theorem 1.8 observe that it remains to deal with the issues related to the convergence of the normalized flow. This section will show that the normalized hypersurfaces $\hat{M}_{\hat{t}}$ along the normalized flow (4.2) converge to a unit geodesic sphere of $\mathbb{H}^{n+1}_\kappa$ in the C^∞-topology as $\hat{t} \to +\infty$.

It is useful to bound the normalized inner radius $\hat{\rho}_-(\hat{t})$ and the normalized circumradius radius $\hat{\rho}_+(\hat{t})$ of $\hat{M}_{\hat{t}}$ for $\hat{t} > 0$.

Lemma 5.1. *There exists a constant $0 < C_1 = C_1(a, M_0)$ such that on $\hat{M}_{\hat{t}}$ along the normalized flow* (4.2) *for all $\hat{t} \geq 0$*

$$\frac{1}{C_1} \leq \hat{\rho}_- \leq 1 \leq \hat{\rho}_+ \leq C_1.$$

Proof. Let $B_r(o)$ be the geodesic ball of radius r with centered at o. Since along the normalized flow the total area of $\hat{M}_{\hat{t}}$ satisfies that $\mathrm{A}(\hat{M}_{\hat{t}}) = \mathrm{A}(B_1(o)) = \mathrm{A}_0$, the formulas for the total area of $B_{\hat{r}}(o)$

$$\mathrm{A}(B_r(o)) = \mathrm{s}^n_\kappa(r)\mathrm{A}(\mathbb{S}^n),$$

where $\mathrm{A}(\mathbb{S}^n)$ is the total area of the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, and the inequality $\mathrm{A}(B_{\rho_-}(o)) \le \mathrm{A}_0 \le \mathrm{A}(B_{\rho_+}(o))$ implies that

$$\hat{\rho}_- \le 1 \le \hat{\rho}_+. \tag{5.1}$$

Firstly, choosing a bigger $\hat{t}_0$ for $\hat{t} > \hat{t}_0$, by grouping (5.1) and Lemma 2.4 iii) it follows that there exists a positive constant $\hat{C} = C(a\psi^{-1}(\hat{t})) < C(a)$ (argue similarly as in Theorem 3.1 of [6] by replacing $a\psi^{-1}$ to a) and

$$1 \le \hat{\rho}_+ \le \hat{C}\left(\hat{\rho}_- + \sqrt{\hat{\rho}_-}\right) \le 2C\sqrt{\hat{\rho}_-}, \tag{5.2}$$

which is

$$\hat{\rho}_- \ge \left(\frac{1}{2C}\right)^2. \tag{5.3}$$

Combining (5.1) and (5.2) it shows that

$$\hat{\rho}_+ \le 2C. \tag{5.4}$$

Note that $\psi^{-1}(\hat{t}) \le \psi^{-1}(0)$, the same argument on $[0, \hat{t}_0]$ also gives (5.3) and (5.4). Therefore taking the constant C_1 as

$$C_1 = \max\left\{4C^2, 2C\right\}$$

gives our conclusion. □

The above lemma allows us to obtain a uniform bound on the normalized mean curvature $\hat{H}(\hat{t})$ for all $\hat{t} \ge 0$. For this purpose, following the procedure of [1], [7], [18] and [24], we consider again the evolution under (4.2) of the function

$$\hat{Z}_{\hat{t}} = \frac{\hat{H}^\beta}{\Psi - \epsilon'}, \tag{5.5}$$

where $\Psi = \left\langle \hat{\nu}, \hat{\mathrm{X}} \right\rangle$ and ϵ' is a constant to be chosen later. Using Theorem 4.1 a routine computation in the same way as the un-normalized equation gives the evolution equation for $\hat{Z}_{\hat{t}}$ along the normalized flow (4.2).

Lemma 5.2. *For $\hat{t} \in [0, +\infty)$ and any constant ϵ',*

$$\begin{aligned} \partial_{\hat{t}}\hat{Z} = {} & \beta\hat{H}^{\beta-1}\hat{\Delta}\hat{Z} + \frac{2\beta\hat{H}^{\beta-1}}{\Psi - \epsilon'}\left\langle \hat{\nabla}\hat{Z}, \hat{\nabla}\Psi \right\rangle + \left((\beta+1) - \epsilon'\beta\frac{\left|\hat{A}\right|^2}{\hat{H}}\right)\hat{Z}^2 \\ & - n\beta a^2\hat{H}^{\beta-1}\hat{Z} - \beta n^\beta \hat{Z} + \left(\frac{n\beta a^2}{\hat{H}} - \frac{1}{\Psi - \epsilon'}\right) n^\beta \Psi \hat{Z}. \end{aligned} \tag{5.6}$$

Proof. The equations (4.2) and (4.4) imply that

$$\partial_{\hat{t}}\Psi = -\hat{H}^\beta + n^\beta\left\langle \hat{\mathrm{X}}, \hat{\nu} \right\rangle + \beta\hat{H}^{\beta-1}\left\langle \hat{\mathrm{X}}, \hat{\nabla}\hat{H} \right\rangle. \tag{5.7}$$

A straightforward computation gives

$$\hat{\Delta}\Psi = \hat{H} + \left\langle \hat{\mathrm{X}}, \hat{\nabla}\hat{H} \right\rangle - \left|\hat{A}\right|^2\Psi. \tag{5.8}$$

By substitution of this expression in (5.7), the evolution equation of Ψ can be obtained:

$$\partial_{\hat{t}}\Psi = \beta\hat{H}^{\beta-1}\hat{\Delta}\Psi - (\beta+1)\hat{H}^{\beta} + n^{\beta}\Psi + \beta\hat{H}^{\beta-1}\left|\hat{A}\right|^2\Psi. \tag{5.9}$$

Thus, from (4.9) with $l=\beta$ and (5.9), it follows

$$\begin{aligned}\partial_{\hat{t}}\hat{Z} = {} & \frac{\beta\hat{H}^{\beta-1}}{\Psi-\epsilon'}\hat{\Delta}\hat{H}^{\beta} - \frac{\beta\hat{H}^{2\beta-1}}{(\Psi-\epsilon')^2}\hat{\Delta}\Psi + \left((\beta+1) - \epsilon'\beta\frac{\left|\hat{A}\right|^2}{\hat{H}}\right)\hat{Z}^2 \\ & - n\beta a^2\hat{H}^{\beta-1}\hat{Z} - \beta n^{\beta}\hat{Z} + \left(\frac{n\beta a^2}{\hat{H}} - \frac{1}{\Psi-\epsilon'}\right)n^{\beta}\Psi\hat{Z}.\end{aligned} \tag{5.10}$$

Another computation shows that

$$\frac{\beta\hat{H}^{\beta-1}\hat{\Delta}\hat{H}^{\beta}}{\Psi-\epsilon'} - \frac{\beta\hat{H}^{2\beta-1}\hat{\Delta}\Psi}{(\Psi-\epsilon')^2} = \beta\hat{H}^{\beta-1}\hat{\Delta}Z + 2\frac{\beta\hat{H}^{\beta-1}}{\Psi-\epsilon'}\langle\hat{\nabla}Z,\hat{\nabla}\Psi\rangle. \tag{5.11}$$

Therefore the equation (5.10) can be manipulated into the desired form (5.6) by combining (5.11) with (5.10). □

Proposition 5.3. *Let $\hat{M}_{\hat{t}}$ be a solution of the normalized flow (4.2) in $\mathbb{H}^{n+1}_{\kappa}$. There exists a positive constant such that*

$$\hat{H}(p,\hat{t}) \le C(\hat{M}_0,\beta,n,a) \quad \textit{for all } (p,\hat{t}) \in M^n\times[0,+\infty).$$

Proof. For any fixed $\hat{t}\in[0,+\infty)$, the convexity of the $\hat{M}_{\hat{t}}$ implies

$$\hat{\rho}_- \le \Psi = \left\langle\hat{\nu},\hat{\mathrm{X}}\right\rangle \le \hat{\rho}_+.$$

Moreover, having into account Lemma 5.1, this gives

$$\frac{1}{C_1} \le \Psi \le C_1.$$

Thus, taking the constant ϵ' in the definition (5.1) of $\hat{Z}$ as

$$\epsilon' = \frac{1}{2C_1}$$

implies

$$C_1 - \epsilon' > \Psi - \epsilon' \ge \epsilon' > 0. \tag{5.12}$$

Combining this, convexity of $\hat{M}_{\hat{t}}$ implies that

$$\hat{Z} \ge 0, \quad \text{and} \quad \left|\hat{A}\right|^2 \ge \frac{1}{n}\hat{H}^2. \tag{5.13}$$

From Lemma 5.2, (5.12) and (5.13) the following inequality can be obtained:

$$\begin{aligned}\partial_{\hat{t}}\hat{Z} \le {} & \beta\hat{H}^{\beta-1}\hat{\Delta}\hat{Z} + \frac{2\beta\hat{H}^{\beta-1}}{\Psi-\epsilon'}\left\langle\hat{\nabla}\hat{Z},\hat{\nabla}\Psi\right\rangle + \left((\beta+1) - \epsilon'\beta\frac{\hat{H}}{n}\right)\hat{Z}^2 \\ & + \left(\frac{n\beta a^2}{\hat{H}} - \frac{1}{C_1-\epsilon'}\right)n^{\beta}\Psi\hat{Z}.\end{aligned} \tag{5.14}$$

Assume that in $(p_0, \hat{t}_0)$, Z attains a big maximum $C \gg 0$ for the first time. Then

$$H^\beta(p_0, \hat{t}_0) \geq C(\Psi - \epsilon')(p_0, \hat{t}_0) \geq \epsilon' C,$$

which gives a contradiction if

$$C \geq \max_{p \in M^n} \left\{ \hat{Z}(p,0), \frac{1}{\epsilon'} \left(\frac{n(\beta+1)}{\epsilon'\beta} \right)^\beta, \frac{1}{\epsilon'} \left(n\beta a^2 \left(C_1 - \epsilon' \right) \right)^\beta \right\}. \qquad \square$$

To prove our result, a key step is to obtain a Harnack inequality or a Hölder estimate which is uniform in time on the curvature. Such an estimate implies that if the curvature is positive at a given point of our hypersurface, then it also satisfies a uniform positive lower bound in a whole neighborhood and so guarantees uniform parabolicity. However, it is hard to derive this type of inequalities, which has been pointed out by Schulze in [22], since the speed for the flow has a homogeneity degree larger than one in the curvatures. Schulze in [22] (see also [8] for details) has also realized that in contrast with the standard Laplacian $\hat{\Delta}$, the operators $\beta \hat{H}^{\beta-1} \hat{\Delta}$ which appear in the evolution equations become degenerate if the curvatures approach zero. In fact, following [22], the evolution equation for $\hat{H}$ can be view as a porous medium equation, see the second form in the evolution equation (4.8), and an interior Hölder estimate for solutions of such equations has been established by DiBenedetto and Friedman ([11], Theorem 1.2).

Lemma 5.4. *For a normalized flow* (4.2)*, there exists a constant* $\hat{C}$ *satisfying*

$$\int_{\hat{t}_1}^{\hat{t}_2} \int_{\hat{M}_{\hat{t}}} \left| \hat{\nabla} \hat{H}^\beta \right|^2 \mathrm{d}\hat{\mu}_{\hat{t}} \mathrm{d}\hat{t} \leq \hat{C} \left(1 + \hat{t}_2 - \hat{t}_1 \right) \quad \textit{for} \quad 0 \leq \hat{t}_1 < \hat{t}_2 < +\infty.$$

Proof. Applying the evolution equations (4.5), (4.9) with $l = \beta + 1$ and integration by parts gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\hat{t}} \int_{\hat{M}_{\hat{t}}} \hat{H}^{\beta+1} \mathrm{d}\hat{\mu}_{\hat{t}} =& \int_{\hat{M}_{\hat{t}}} \partial_{\hat{t}} \hat{H}^{\beta+1} \mathrm{d}\hat{\mu}_{\hat{t}} + \int_{\hat{M}_{\hat{t}}} \hat{H}^{\beta+1} \partial_{\hat{t}} (\mathrm{d}\hat{\mu}_{\hat{t}}) \\
=& -(\beta+1) \int_{\hat{M}_{\hat{t}}} \left| \hat{\nabla} \hat{H}^\beta \right|^2 \mathrm{d}\hat{\mu}_{\hat{t}} \\
&+ \int_{\hat{M}_{\hat{t}}} \hat{H}^{2\beta} \left((\beta+1) \left(\left| \hat{A} \right|^2 - na^2 \right) - \hat{H}^2 \right) \mathrm{d}\hat{\mu}_{\hat{t}} \\
&+ (n-\beta-1) n^\beta \int_{\hat{M}_{\hat{t}}} \hat{H}^{\beta+1} \mathrm{d}\hat{\mu}_{\hat{t}} + (\beta+1) n^{\beta+1} a^2 \int_{\hat{M}_{\hat{t}}} \hat{H}^\beta \Psi \mathrm{d}\hat{\mu}_{\hat{t}},
\end{aligned}$$

which is

(5.15)

$$\int_{\hat{M}_{\hat{t}}} \left| \hat{\nabla} \hat{H}^\beta \right|^2 \mathrm{d}\hat{\mu}_{\hat{t}} = -\frac{1}{\beta+1} \frac{\mathrm{d}}{\mathrm{d}\hat{t}} \int_{\hat{M}_{\hat{t}}} \hat{H}^{\beta+1} \mathrm{d}\hat{\mu}_{\hat{t}}$$

$$+\frac{1}{\beta+1}\int_{\hat{M}_{\hat{t}}}\hat{H}^{2\beta}\left((\beta+1)\left(\left|\hat{A}\right|^2-na^2\right)-\hat{H}^2\right)\mathrm{d}\hat{\mu}_{\hat{t}}$$
$$+\frac{(n-\beta-1)n^\beta}{\beta+1}\int_{\hat{M}_{\hat{t}}}\hat{H}^{\beta+1}\mathrm{d}\hat{\mu}_{\hat{t}}+n^{\beta+1}a^2\int_{\hat{M}_{\hat{t}}}\hat{H}^\beta\Psi\mathrm{d}\hat{\mu}_{\hat{t}}.$$

Since $\hat{M}_{\hat{t}}$ is convex along the normalized flow the last term on the right hand side of (5.15) can be estimated:

$$n^{\beta+1}a^2\int_{\hat{M}_{\hat{t}}}\hat{H}^\beta\Psi\mathrm{d}\hat{\mu}_{\hat{t}}\le n^{\beta+1}a^2\hat{H}^\beta_{\max}\int_{\hat{M}_{\hat{t}}}\Psi\mathrm{d}\hat{\mu}_{\hat{t}}$$
$$\le C_1n^{\beta+1}a^2\hat{H}^\beta_{\max}\mathrm{s}^n_\kappa(1)\mathrm{A}(\mathbb{S}^n).$$

As remarked earlier, $\left|\hat{A}\right|\le\left|\hat{H}\right|$ and $\hat{H}$ is uniformly bounded, so that the second, the third and the last term on the right hand side of (5.15) can be bounded by a positive constant. Therefore, after integrating (5.15) from $\hat{t}_1$ to $\hat{t}_2$ applying these facts above gives the conclusion of Lemma 5.4. □

Proposition 5.5. *For any point $(p,\hat{t})\in M\times(0,+\infty)$, there exists a space-time neighborhood $\mathcal{U}\subset M\times(0,+\infty)$, whose diameter does not depend on the point $(p,\hat{t})$ and such that*

$$\left\|\hat{H}\right\|_{C^\alpha(\mathcal{U})}\le C$$

for some positive constants $C=C(n,a,\beta,\hat{M}_0)$ and $\alpha\in(0,1)$.

Proof. For any point $(p_0,\hat{t}_0)\in M\times(0,+\infty)$, Lemma 5.1 implies that there is a $q_0\in M\times(0,+\infty)$ such that $\hat{M}_{\hat{t}_0}$ encloses $B_{1/C_1}(q_0)$, where $B_{1/C_1}(q_0)$ is the geodesic ball of radius $1/C_1$ centered at q_0 in $\mathbb{H}^{n+1}_\kappa$. Proposition 5.3 tells us that the speed of the normalized flow (4.2) is uniformly bounded. Then there exists $\eta>0$ (depending only on $H_{\max}$) such that $\hat{M}_{\hat{t}_0}$ encloses $B_{1/2C_1}(q_0)$ for all $\hat{t}\in\big(\max\{\hat{t}_0-\eta,0\},\min\{\hat{t}_0-\eta,T\}\big)$. Since along the flow (4.2) $\hat{h}_{ij}>a\psi^{-1}\hat{g}_{ij}\ge0$, i.e., hypersurfaces $\hat{M}_{\hat{t}}$ are strictly convex for $\hat{t}>0$, which yields

$$\left\langle\hat{\nu},\hat{\mathrm{X}}-q_0\right\rangle\ge\frac{1}{2C_1}$$

for all $(p,\hat{t})\in M\times\big(\max\{\hat{t}_0-\eta,0\},\min\{\hat{t}_0-\eta,T\}\big)$. As a consequence of the fact $\hat{\rho}_+\le C_1$ and Hadamard's theorem in the hyperbolic space, for a proof see [12, Theorem 10.3.1]. there is a constant $\zeta>0$, $2\zeta<\eta$, such that such evolving hypersurfaces $\hat{M}_{\hat{t}}\cap\exp_{q_0}\mathbb{S}^n_\zeta(q_0)$, where $\mathbb{S}^n_\zeta(q_0)\subset\mathrm{T}_{q_0}\mathbb{H}^{n+1}_\kappa$ with radius ζ, can be represent as a graph over a function $r(u,\hat{t})$ on $\mathbb{S}^n_\zeta(q_0)$, which is uniformly bounded in C^2. Here recall some basic formulae relating quantities over $\hat{M}_{\hat{t}}$ satisfying $\hat{M}_{\hat{t}}=\mathrm{graph}r(\hat{t})$. For each $\hat{t}$, regard r_{q_0} as a function on $\mathbb{S}^n$. Let D be the Levi-Civita connection on $\mathbb{S}^n$. Then a local coordinate vector field of $\hat{M}_{\hat{t}}$ has the following representation

$$\hat{\mathrm{X}}_{\hat{\mathrm{t}}*}(\frac{\partial}{\partial u^i})=D_ir(u)\partial_{r_p}+\mathrm{s}_\kappa(r(u))e_i,\qquad 1\le i\le n,\tag{5.16}$$

and the outward unit normal vector of $\hat{M}_{\hat{t}}$ can be expressed as

$$\hat{\nu} = \frac{1}{|\xi|}\Big(\mathrm{s}_\kappa(r)\partial_{r_p} - \sum_{i=1}^{n} D_i r e_i\Big) \tag{5.17}$$

with

$$|\xi| = \sqrt{\mathrm{s}_\kappa^2(r) + |Dr|^2}. \tag{5.18}$$

After a standard computation, the second fundamental form of $\hat{M}_{\hat{t}}$ can be expressed as

$$\hat{h}_{ij} = -\frac{1}{|\xi|}\Big(\mathrm{s}_\kappa(r) D_j D_i r - \mathrm{s}_\kappa^2(r)\mathrm{c}_\kappa(r)\sigma_{ij} - 2\mathrm{c}_\kappa(r) D_i r D_j r\Big), \tag{5.19}$$

and the metric $\hat{g}_{ij}$ is

$$\hat{g}_{ij} = D_i r D_j r + \mathrm{s}_\kappa^2(r)\sigma_{ij}. \tag{5.20}$$

From this, the inverse metric can be expressed as

$$\hat{g}^{ij} = \frac{1}{\mathrm{s}_\kappa^2(r)}\Big(\sigma^{ij} - \frac{1}{|\xi|^2} D^i r D^j r\Big), \tag{5.21}$$

where $D^i r = \sigma^{ij} D_j r$. Then equations (5.19) and (5.21) imply that

$$\hat{H} = -\frac{1}{|\xi|\,\mathrm{s}_\kappa(r)}\Big(\Delta_{\mathbb{S}} r - \frac{1}{|\xi|^2}\nabla^2_{\mathbb{S}} r(Dr, Dr)\Big) + \frac{\mathrm{c}_\kappa(r)}{|\xi|}\Big(n + \frac{|Dr|^2}{|\xi|^2}\Big). \tag{5.22}$$

Using (5.20) and (5.21) the Christoffel symbols have the expression:

$$\begin{aligned}\hat{\Gamma}_{ij}^k = \frac{1}{\mathrm{s}_\kappa^2(r)}\Big[&D_i D_j r D_l r + \mathrm{s}_\kappa(r)\mathrm{c}_\kappa(r)\big(D_i r \sigma_{lj} + D_j r \sigma_{il} - D_l r \sigma_{ij}\big)\Big] \\ &\cdot \Big(\sigma^{kl} - \frac{1}{|\xi|^2} D^k r D^l r\Big).\end{aligned} \tag{5.23}$$

Thus, in local coordinates the Laplacian $\hat{\Delta}$ can be represented as:

$$\begin{aligned}\hat{\Delta} &= \hat{g}^{ij}\left(D_i D_j - \hat{\Gamma}_{ij}^k D_k\right) \\ &= D_i\left(\hat{g}^{ij} D_j\right) - D_i\left(\hat{g}^{ij}\right) D_j - \hat{g}^{ij}\hat{\Gamma}_{ij}^k D_k \\ &= D_i\left(\hat{g}^{ij} D_j\right) + \left(\hat{\Gamma}_{ik}^i \hat{g}^{kj} + \hat{\Gamma}_{ik}^j \hat{g}^{ki}\right) D_j - \hat{g}^{ij}\hat{\Gamma}_{ij}^k D_k \\ &= D_i\left(\hat{g}^{ij} D_j\right) + \hat{\Gamma}_{ik}^i \hat{g}^{kj} D_j.\end{aligned}$$

Therefore, the second identity of the evolution equation (4.8) of $\hat{H}$ can be rewritten as

$$\partial_{\hat{t}}\hat{H} = D_i\left(\frac{1}{\mathrm{s}_\kappa^2(r)}\Big(\sigma^{ij} - \frac{1}{|\xi|^2} D^i r D^j r\Big) D_j \hat{H}^\beta\right) + b^j D_j \hat{H}^\beta + c \tag{5.24}$$

with coefficients given by

$$\begin{aligned} b^j &= \hat{\Gamma}^i_{ik}\hat{g}^{kj} := b^j\left(u, r, Dr, D^2r\right), \\ c &= \left(\left|\hat{A}\right|^2 - na^2\right)\hat{H}^\beta - n^\beta\hat{H} + a^2n^{\beta+1}\left\langle \hat{\mathrm{X}}, \hat{\nu}\right\rangle := c\left(u, r, Dr, D^2r\right), \end{aligned} \tag{5.25}$$

which are uniformly bounded coming from the bounds on curvature and the preserving h-convexity of evolving hypersurface. Moreover, Lemma 5.4 gives

$$\int_{\hat{M}_{\hat{t}}\cap\exp_{q_0}\mathbb{S}^n_\zeta(q_0)\times\left[\max\{\hat{t}_0-\zeta,0\},\min\{\hat{t}_0-\zeta,T\}\right]}\left|\hat{\nabla}\hat{H}^\beta\right|^2\mathrm{d}\hat{\mu}_{\hat{t}}\mathrm{d}\hat{t} \le \hat{C}(\zeta).$$

Therefore applying Theorem 1.2 in [11] to (5.24) with $\zeta' = \zeta/2$ gives that

$$\left\|\hat{H}\right\|_{C^\alpha\left(\hat{M}_{\hat{t}}\cap\exp_{q_0}\mathbb{S}^n_{\zeta'}(q_0)\times\left[\max\{\hat{t}_0-\zeta',0\},\min\{\hat{t}_0-\zeta',T\}\right]\right)} \le C$$

for suitable $\alpha \in (0,1)$ and some positive constants $C = C(n, a, \beta, \hat{M}_{\hat{0}})$. □

Theorem 5.6. *Under the conditions of Theorem 1.8, the normalized hypersurfaces $\hat{M}_{\hat{t}}$ converge smoothly as $\hat{t} \to +\infty$ to a unit geodesic sphere of $\mathbb{H}^{n+1}_\kappa$.*

Proof. Take a sequence of time $\{\hat{t}_i\} \subset [0, +\infty)$ with $\hat{t}_i \to +\infty$. The uniform bounds on the curvatures imply that there exists a subsequence of $\{\hat{t}_i\}$, again denoted by $\{\hat{t}_i\}$, such that, depending only on distance from the origin,

$$\hat{M}_{\hat{t}_i} \to \hat{M}_{+\infty} \quad \text{in the } C^{1,\alpha}\text{-topology for any } \alpha < 1,$$

and $\hat{M}_{+\infty}$ is a convex $C^{1,1}$-hypersurface. Since $\hat{\rho}_+ \le C_1$, at each time $\hat{t}_i$, there exists a point $p_i \in M$ satisfying

$$\hat{H}(p_i, \hat{t}_i) \ge n\mathrm{co}_\kappa(C_1). \tag{5.26}$$

Proposition 5.5 implies that $\hat{H}$ cannot decrease too fast in the sense that there exists a $\delta > 0$, independently of $(p_i, \hat{t}_i)$, satisfying

$$\hat{H}\Big|_{\hat{M}_{\hat{t}_i}\cap\exp_{q_i}\mathbb{S}^n_\delta(q_i)\times\left[\max\{\hat{t}_i-\delta,0\},\min\{\hat{t}_i-\delta,T\}\right]} \ge \frac{n\mathrm{co}_\kappa(C_1)}{2},$$

where $q_i = \hat{\mathrm{X}}(p_i)$. Furthermore, choosing δ small enough implies that, proceeding as in Proposition 5.5, $\hat{M}_{\hat{t}_i} \cap \exp_{q_i}\mathbb{S}^n_\delta(q_i)$ can be written as the graph of a function r_i for any $\hat{t} \in \left[\max\{\hat{t}_i - \delta, 0\}, \min\{\hat{t}_i - \delta, T\}\right]$. Using parabolic Schauder theory on any space-time neighborhood

$$\hat{M}_{\hat{t}_i} \cap \exp_{q_i}\mathbb{S}^n_\delta(q_i) \times \left[\max\{\hat{t}_i - \delta, 0\}, \min\{\hat{t}_i - \delta, T\}\right]$$

gives uniform C^∞-estimates on the functions r_i in a neighborhoods even smaller with a suitable radius, denoted by δ'. Therefore,

$$\hat{M}_{\hat{t}_i} \cap \exp_{q_i}\mathbb{S}^n_{\delta'}(q_i) \to \hat{M}_{+\infty} \cap \exp_{q_{+\infty}}\mathbb{S}^n_{\delta'}(q_{+\infty}) \quad \text{in } C^\infty,$$

where $\hat{\mathrm{X}}(p_i, \hat{t}_i) \to q_{+\infty} \in \hat{M}_{+\infty}$.

On the other hand, Theorem 3.1 implies that $\hat{M}_{+\infty} \cap \exp_{q_{+\infty}}\mathbb{S}^n_{\delta'}(q_{+\infty})$ must be totally umbilic, and therefore is a part of a geodesic sphere. From (5.26)

it follows that $\hat{M}_{+\infty} \cap \exp_{q_{+\infty}} \mathbb{S}^n_{\delta'}(q_{+\infty})$ has mean curvature at least $n\mathrm{co}_\kappa(C_1)$. Then, using again the uniform Hölder continuity shows that (5.25) holds for i larger enough in $\exp_{q_i} \mathbb{S}^n_\delta(q_i)/\exp_{q_i} \mathbb{S}^n_{\delta'}(q_i)$. Thus, the region where $\hat{M}_{+\infty}$ is known to be spherical can be extended. After finitely many iterations it can be shown that $\hat{M}_{+\infty}$ is a unit geodesic sphere, centered at the origin.

Since the above argument can be applied to any time sequence $\{\hat{t}_i\}$, it can be concluded that the whole family $\hat{M}_{\hat{t}}$ converges to a unit geodesic sphere as $\hat{t} \to +\infty$ in C^∞, depending only on distance from the origin. □

6. Exponential convergence

The last step towards the proof of Theorem 1.8 is to show that under the normalized flow (4.2) the $\hat{M}_{\hat{t}}$ converges to $\hat{M}_{+\infty}$ exponentially. A natural quantity to control the pinching of the principal curvature along the normalized flow (4.2) is the quotient $\hat{\tilde{K}}/\hat{\tilde{H}}^n$ denoted by $\hat{\tilde{Q}}$. From (4.13), (4.18) and (4.19) the evolution equation for $\hat{\tilde{Q}}$ along the normalized flow can be derived in the same way as the un-normalized equation.

Lemma 6.1. *On any solution $\hat{M}_{\hat{t}}$ of (4.2) the quantity $\hat{\tilde{Q}}$ satisfies the following evolution equation:*

$$\begin{aligned}\partial_{\hat{t}}\hat{\tilde{Q}} = \beta\hat{H}^{\beta-1}\Bigg\{&\hat{\Delta}\hat{\tilde{Q}} + \frac{(n+1)}{n\hat{\tilde{H}}}\left\langle\hat{\nabla}\hat{\tilde{Q}}, \hat{\nabla}\hat{\tilde{H}}^n\right\rangle - \frac{(n-1)}{n\hat{\tilde{K}}}\left\langle\hat{\nabla}\hat{\tilde{Q}}, \hat{\nabla}\hat{\tilde{K}}\right\rangle - \frac{\hat{\tilde{H}}^n}{n\hat{\tilde{K}}}\left|\hat{\nabla}\hat{\tilde{Q}}\right|^2\\ &+ \frac{(\beta-1)}{\hat{H}}\hat{\tilde{Q}}\hat{\nabla}_i\hat{\tilde{H}}\hat{\nabla}^j\hat{\tilde{H}}\left(\hat{\tilde{b}}^i_j - \frac{n}{\hat{\tilde{H}}}\delta^i_j\right) + \frac{\hat{\tilde{Q}}}{\hat{\tilde{H}}^2}\left|\hat{\tilde{H}}\hat{\nabla}_i\hat{\tilde{h}}^n_m - \hat{\tilde{h}}^n_m\hat{\nabla}_i\hat{\tilde{H}}\right|^2_{\hat{\tilde{g}},\hat{\tilde{b}}}\\ &+ \hat{\tilde{Q}}\frac{(\beta-1)\hat{H}}{\beta\hat{\tilde{H}}}\left(n\left|\hat{\tilde{A}}\right|^2 - \hat{\tilde{H}}^2\right) + a\psi^{-1}\hat{\tilde{Q}}\left(\left|\hat{\tilde{A}}\right|^2\sum_{i=1}^n\hat{\tilde{b}}^i_i - n\hat{\tilde{H}}\right)\\ &+ na^2\left(1-\psi^{-2}\right)\hat{\tilde{Q}}\left(a\psi^{-1}\sum_{i=1}^n\hat{\tilde{b}}^i_i + n\right)\Bigg\}\\ &+ a^2n^\beta\left\langle\hat{\mathrm{X}}, \hat{\nu}\right\rangle\hat{\tilde{Q}}\sum_{i=1}^n\left(\hat{\tilde{b}}^i_i - \frac{n^2}{\hat{\tilde{H}}}\right) + a^2n^2\left(1-\psi^{-2}\right)\hat{H}^\beta\frac{\hat{\tilde{Q}}}{\hat{\tilde{H}}}.\end{aligned}$$

Thus the above lemma gives the evolution equation of $\hat{f} = 1/n^n - \hat{\tilde{Q}}$ as follows.

Proposition 6.2. *On any solution $\hat{M}_{\hat{t}}$ of (4.2) the quantity $\hat{f}$ satisfies the following evolution equation:*

$$\partial_{\hat{t}}\hat{f} = \beta\hat{H}^{\beta-1}\Bigg\{\hat{\Delta}\hat{f} + \frac{(n+1)}{n\hat{\tilde{H}}}\left\langle\hat{\nabla}\hat{f}, \hat{\nabla}\hat{\tilde{H}}^n\right\rangle - \frac{(n-1)}{n\hat{\tilde{K}}}\left\langle\hat{\nabla}\hat{f}, \hat{\nabla}\hat{\tilde{K}}\right\rangle + \frac{\hat{\tilde{H}}^n}{n\hat{\tilde{K}}}\left|\hat{\nabla}\hat{f}\right|^2$$

$$
\begin{aligned}
&- \hat{\tilde{Q}}\left[\frac{(\beta-1)}{\hat{H}}\hat{\nabla}_i\hat{\tilde{H}}\hat{\nabla}^j\hat{\tilde{H}}\left(\hat{\tilde{b}}^i_j-\frac{n}{\hat{\tilde{H}}}\delta^i_j\right)+\frac{1}{\hat{\tilde{H}}^2}\left|\hat{\tilde{H}}\hat{\nabla}_i\hat{\tilde{h}}^n_m-\hat{\tilde{h}}^n_m\hat{\nabla}_i\hat{\tilde{H}}\right|^2_{\hat{g},\hat{\tilde{b}}}\right]\\
&- \hat{\tilde{Q}}\frac{(\beta-1)\hat{H}}{\beta\hat{\tilde{H}}}\left(n\left|\hat{\tilde{A}}\right|^2-\hat{\tilde{H}}^2\right)-a\psi^{-1}\hat{\tilde{Q}}\left(\left|\hat{\tilde{A}}\right|^2\sum_{i=1}^n\hat{\tilde{b}}^i_i-n\hat{\tilde{H}}\right)\\
&+ na^2\left(1-\psi^{-2}\right)\hat{\tilde{Q}}\left(a\psi^{-1}\sum_{i=1}^n\hat{\tilde{b}}^i_i+n\right)\Bigg\}\\
&- a^2n^\beta\left\langle\hat{\mathrm{X}},\hat{\nu}\right\rangle\hat{\tilde{K}}\sum_{i=1}^n\left(\hat{\tilde{b}}^i_i-\frac{n^2}{\hat{\tilde{H}}}\right)-a^2n^2\left(1-\psi^{-2}\right)\hat{H}^\beta\frac{\hat{\tilde{Q}}}{\hat{\tilde{H}}}.
\end{aligned}
$$

From the preceding proposition the same arguments as in Theorem 3.5 of [22] shows that the normalized flow (4.2) converges exponentially in C^∞ to a unit geodesic sphere of $\mathbb{H}^{n+1}_\kappa$.

Theorem 6.3. *There are $\hat{t}_0$, positive constants δ_i and C_i for every $i\geq 0$ such that for all $\hat{t}\in[\hat{t}_0,+\infty)$ the following estimates hold*

$$
\begin{aligned}
\left|\ln\left(\frac{\hat{g}_{+\infty}(u,u)}{\hat{g}_{\hat{t}}(u,u)}\right)\right|&\leq C_0\mathrm{e}^{-\delta\hat{t}}\quad \textit{for every non-zero tangent vector } u,\\
\left|\hat{\nabla}^i\left(\hat{A}_{\hat{t}}-\hat{A}_{+\infty}\right)\right|&\leq C_i\mathrm{e}^{-\delta\hat{t}},\\
\left|\hat{H}_{\hat{t}}-n\right|&\leq C_0\mathrm{e}^{-\delta\hat{t}}.
\end{aligned}
$$

Hence, under the normalized flow (4.2) *the $\hat{M}_{\hat{t}}$ converges exponentially in to in C^∞ to a unit geodesic sphere of $\mathbb{H}^{n+1}_\kappa$.*

Proof. For a time $\hat{t}_0$ big enough, the following estimate can be obtain:

$$1-\psi^{-2}>0\quad\text{for all }\hat{t}\in[\hat{t}_0,+\infty).$$

Applying the similar argument as in Theorem 2.9 for Proposition 6.2 with Lemma 2.3 gives

$$
\begin{aligned}
\partial_{\hat{t}}\hat{f}\leq&\beta\hat{H}^{\beta-1}\Bigg\{\hat{\Delta}\hat{f}+\frac{(n+1)}{n\hat{\tilde{H}}}\left\langle\hat{\nabla}\hat{f},\hat{\nabla}\hat{\tilde{H}}^n\right\rangle-\frac{(n-1)}{n\hat{\tilde{K}}}\left\langle\hat{\nabla}\hat{f},\hat{\nabla}\hat{\tilde{K}}\right\rangle\\
&+\frac{\hat{\tilde{H}}^n}{n\hat{\tilde{K}}}\left|\hat{\nabla}\hat{f}\right|^2-\delta\hat{\tilde{H}}^2\hat{f}\Bigg\}
\end{aligned}
\tag{6.1}
$$

for some constant $\delta>0$ small enough. Note that $\hat{H}$ can be bounded from below by the positive constant $n\mathrm{co}_\kappa(C_1)$ and the positive factor $\psi(\hat{t})$ is increasing in time, thus there exists a time $\hat{t}_1$ even bigger such that $\hat{\tilde{H}}>n(\mathrm{co}_\kappa(C_1)-a)$ for all $\hat{t}\in[\hat{t}_1,+\infty)$. Then applying the maximum principle to (6.1) implies that for a time $\hat{t}$ even bigger there exist a positive constant δ' and a positive constant C satisfying

$$\hat{f}(\hat{t})\leq C\mathrm{e}^{-\delta'\hat{t}}.$$

Once this is established one can derive as in Theorem 3.5 of [22] the desired estimates of Theorem 6.3. □

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References

[1] B. H. Andrews, *Contraction of convex hypersurfaces in Euclidean space*, Calc. Var. Partial Differential Equations **2** (1994), no. 2, 151–171.
[2] ———, *Contraction of convex hypersurfaces in Riemannian spaces*, J. Differential Geom. **39** (1994), no. 2, 407–431.
[3] ———, *Gauss curvature flow: the fate of the rolling stones*, Invent. Math. **138** (1999), no. 1, 151–161.
[4] ———, *Motion of hypersurfaces by Gauss curvature*, Pacific J. Math. **195** (2000), no. 1, 1–34.
[5] ———, *Moving surfaces by non-concave curvature functions*, preprint (2004), available at arXiv:math.DG/0402273.
[6] A. Borisenko and V. Miquel, *Total curvatures of convex hypersurfaces in hyperbolic space*, Illinois J. Math. **43** (1999), no. 1, 61–78.
[7] E. Cabezas-Rivas and V. Miquel, *Volume preserving mean curvature flow in the hyperbolic space*, Indiana Univ. Math. J. **56** (2007), no. 5, 2061–2086.
[8] E. Cabezas-Rivas and C. Sinestrari, *Volume-preserving flow by powers of the m-th mean curvature*, Calc. Var. Partial Differential Equations **38** (2009), no. 3-4, 441–469.
[9] B. Chow, *Deforming convex hypersurfaces by the nth root of the Gaussian curvature*, J. Differential Geom. **22** (1985), no. 1, 117–138.
[10] ———, *Deforming convex hypersurfaces by the square root of the scalar curvature*, Invent. Math. **87** (1987), no. 1, 63–82.
[11] E. DiBenedetto and A. Friedman, *Hölder estimates for nonlinear degenerate parabolic systems*, J. Reine Angew. Math. **357** (1985), 1–22.
[12] C. Gerhardt, *Curvature Problems*, Series in Geometry and Topology, 39, International Press, Somerville, MA, Series in Geometry and Topology, 2006.
[13] S. Z. Guo, G. H. Li, and C. X. Wu, *Contraction of horosphere-convex hypersurfaces by powers of the mean curvature in the hyperbolic state*, J. Korean Math. Soc. **50** (2013), no. 6, 1311–1332.
[14] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom. **20** (1984), no. 1, 237–266.
[15] ———, *Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature*, Invent. Math. **84** (1986), no. 3, 463–480.
[16] G. H. Li, L. J. Yu, and C. X. Wu, *Curvature flow with a general forcing term in Euclidean spaces*, J. Math. Anal. Appl. **353** (2009), no. 2, 508–520.
[17] M. Makowski, *Mixed volume preserving curvature flows in hyperbolic space*, preprint, arxiv:12308.1898v1, [math.DG] 9 Aug 2012.
[18] J. A. McCoy, *Mixed volume preserving curvature flows*, Calc. Var. Partial Differential Equations **24** (2005), no. 2, 131–154.
[19] O. C. Schnürer, *Surfaces contracting with speed* $|A|^2$, J. Differential Geom. **71** (2005), no. 4, 347–363.

[20] R. Schoen, L. Simon, and S. T. Yau, *Curvature estimates for minimal hypersurfaces*, Acta Math. **134** (1975), no. 3-4, 275–288.
[21] F. Schulze, *Evolution of convex hypersurfaces by powers of the mean curvature*, Math. Z. **251** (2005), no. 4, 721–733.
[22] ———, *Convexity estimates for flows by powers of the mean curvature*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **5** (2006), no. 2, 261–277.
[23] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. (2) **88** (1968), no. 1, 62–105.
[24] K. Tso, *Deforming a Hypersurface by Its Gauss-Kronecker Curvature*, Comm. Pure Appl. Math. **38** (1985), no. 6, 867–882.

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