

## CONTROLLABILITY OF ROLLING BODIES WITH REGULAR SURFACES

S. REZA MOGHADASI

ABSTRACT. A pair of bodies rolling on each other is an interesting example of nonholonomic systems in control theory. There is a geometric condition equivalent to the rolling constraint which enables us to generalize the rolling motions for any two-dimensional Riemannian manifolds. This system has a five-dimensional phase space. In order to study the controllability of the rolling surfaces, we lift the system to a six-dimensional space and show that the lifted system is controllable unless the two surfaces have isometric universal covering spaces. In the non-controllable case there are some three-dimensional orbits each of which corresponds to an isometry of the universal covering spaces.

### 1. Introduction

The nonholonomic systems appear in several branches of science and are interesting due to both their relevance to practical applications and challenges involved in their planning and control. There are many works and approaches to study the nature of such systems and their geometric structures [2, 10, 17]. An interesting example of such systems is a rolling pair of rigid bodies. In a motion of a rigid body touching another one, there may be slipping or spinning. A motion is slipping if the contact point at its contact time has non-zero velocity relative to the other body. The motion is spinning if the angular velocity of the body has nonzero component along the normal vector at the contact point. Since in practice friction resists both slipping and spinning motions, we are interested in the study of non-slipping and non-spinning motions which are called rolling motions. Through this study the controllability, trajectory planning and optimal trajectory planning are important problems specially for robotics and dexterous manipulating purposes. These are challenging problems which have attracted attentions of several researchers. Montana [18] derived a differential geometric model of the rolling constraint between general bodies and discussed the applications to robotic manipulations; Li and Canny [13] showed

---

Received May 14, 2014; Revised June 29, 2015.

2010 *Mathematics Subject Classification*. Primary 93B27; Secondary 70E18, 93B05, 70Q05.

*Key words and phrases*. controllability, rolling bodies, geodesic curvature, isometry group.

that the plate and ball system and also the system of two unequal spheres are controllable; Jurdjevic [9] studied a round ball rolling without slipping between two parallel planes which one of them is movable. He considered the optimal control problem of the system and formulate it using the Lie group  $G = R^2 \times SO(3)$  where the control parameter is the velocity of the movable plate and the cost is kinetic energy of the ball. He showed that trajectories of the center of the ball are in one-to-one correspondence with Euler's elastica. These are the extremals of  $\int k^2 ds$ , where  $k$  is the planar curvature. Levi [12] obtained explicit formulas for evaluating the final configuration of the ball after a circular motion of the plate. Marigo and Bicchi [14] considered a more general case. They claimed that the generated involutive distribution of this system at each point of the phase space is either two-dimensional or the entire five-dimensional space. Moreover, they concluded that in the first case, the two bodies should be specular images of each other, i.e., around the contact point they are locally mirror images of each other. Bryant and Hsu [3] provided an intrinsic formulation of rolling surfaces. Agrachev and Sachkov [2] proved that if the surfaces have the same Gaussian curvature at the contact point the involutive distribution of the system is two-dimensional, otherwise it is five-dimensional. Then they showed that if there is a contact configuration such that the Gaussian curvatures of the surfaces differ at the contact point, its orbit should be open, i.e., the system is locally controllable. Moreover, the two-dimensional orbits correspond to isometries of the surfaces. Chelouah and Chitour [4] studied the motion planning problem of rolling surfaces by two different approaches and as a partial result they provide a more elementary proof of Agrachev and Sachkov's result on controllability problem. A new interpretation of the kinematics of rolling spheres is discussed by Agrachev [1]. In [19], Zimmerman has considered the optimal control of an  $n$ -dimensional sphere rolling on an  $n$ -dimensional plane. Godoy et al. [7] provided an intrinsic formulation of rolling  $n$ -dimensional manifolds and showed that the rolling system of  $S^n$  on  $\mathbb{R}^n$  is controllable and the rolling system of  $SE(3)$  on  $se(3)$  is not controllable. More recent results in controllability of higher dimensional manifolds can be found in [5, 8]. There are several attempts to discretize this problem by considering polyhedral bodies instead of smooth regular bodies. See for example [15] for more details.

It is not difficult to verify that the results of the non-controllable case in [14] contradict those of Agrachev and Sachkov's in [2]. One can consider surfaces in  $\mathbb{R}^3$  with constant Gaussian curvature equal to one. Unit sphere is an example of such surfaces and it is the only one which is also complete embedded surface. But there are several non-complete embedded surfaces with constant Gaussian curvature equal to one [11, page 66]. These surfaces are locally isometric to the unit sphere and have positive but different principal curvatures at each point. Thus, they are strictly convex surfaces in  $\mathbb{R}^3$  but they are not congruent to the unit sphere even locally, i.e., there is no isometry of  $\mathbb{R}^3$  which map them to the sphere even locally. Therefore, in each contact configuration of such

surfaces with the unit sphere, the surfaces cannot be specular images of the sphere. By the result in [14] the rolling system should be locally controllable, but by the result in [2] the rolling system is not locally controllable and local two-dimensional orbits correspond to local isometries of two surfaces, i.e., all orbits have two dimensions.

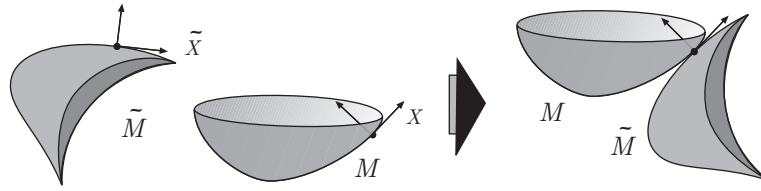
In this paper, we consider the controllability of rolling surfaces in the general case and obtain the results of Agrachev and Sachkov by lifting the system to the more appropriate space which has six dimensions. This method simplifies computations and proposes a way to consider the controllability of  $n$ -dimensional surfaces. At the end, with the aid of rigidity theorem for strictly convex surfaces in  $\mathbb{R}^3$ , we show that if the surfaces are embedded in  $\mathbb{R}^3$  and one of them is strictly convex, at the nonholonomic contact configuration, the surfaces should coincide or be mirror images of each other with respect to the common tangent plane. This global result is a true version of Marigo and Bicchi's claim in [14].

## 2. Description of rolling surfaces

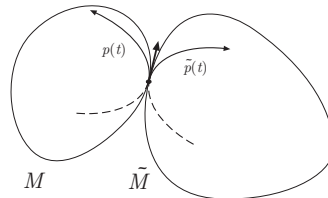
In this section we first give a brief review of basic facts about rolling of two smooth strictly convex bodies in  $\mathbb{R}^3$ . Then we follow Agrachev and Sachkov [2] to generalize the concept of rolling for any pair of 2D Riemannian manifolds. After that we lift the control system which is defined on the five-dimensional contact configuration space to the more natural phase space which is the product of unit tangent bundles of the surfaces. At the end of this section we give a clear description of this lifted system. The study of controllability is postponed to the next section.

Suppose that  $A$  and  $B$  are two closed and smooth strictly convex surfaces in  $\mathbb{R}^3$ . These are the boundary of compact strictly convex regions in  $\mathbb{R}^3$  with non-empty interior. In order to study the motion of  $A$  and  $B$  on each other we need a clear description of contact configurations.

Each contact configuration of these surfaces can be specified by identifying a tangent plane of  $A$  with a tangent plane of  $B$  which can be done in two different ways according to different orientations of tangent planes. In one of these two types of contact configurations the two surfaces touch each other from the outside and in the other case they touch from the inside. The type of contact configuration can be specified by choosing orientations for  $A$  and  $B$ . Once we fix orientations for  $A$  and  $B$ , the identification of two tangent planes can be done by identifying a unit vector in one of them with a unit vector in the other. From now on, we assume that all our surfaces are oriented. A smooth motion of  $A$  and  $B$  on each other gives two smooth curves  $p(t)$  and  $q(t)$  on  $A$  and  $B$ , respectively. Note that in general this motion cannot be recovered from these two curves, but in the non-slipping motions these curves describe the motion uniquely. One can find physical description of such motions and their equivalent mathematical explanations in [16]. Here, we only review some results which we use in this paper. If the motion is not slipping, then



at each instant  $t$ , the velocity vectors  $\dot{p}(t)$  and  $\dot{q}(t)$  coincide. Conversely, two smooth curves  $p(t)$  on  $A$  and  $q(t)$  on  $B$  with the same speed at each instant  $t$ , uniquely determine a smooth non-slipping motion of  $A$  and  $B$  on each other. If the motion is not slipping nor spinning, then  $p$  and  $q$  have the same speed and geodesic curvature. Conversely, two smooth curves  $p(t)$  and  $q(t)$  with the same speed and geodesic curvature at each instant  $t$ , uniquely determine a smooth rolling of  $A$  and  $B$  on each other. Therefore, starting from a contact configuration, a smooth curve on one of the surfaces initiating from the contact point identify a unique rolling of  $A$  and  $B$ .



Knowing the above point, we are able to generalize the concept of rolling to any pair of oriented 2D Riemannian manifolds. Suppose that  $M$  and  $\tilde{M}$  are oriented 2D Riemannian manifolds. Any orientation preserving identification of a tangent space of  $M$  with a tangent space of  $\tilde{M}$  is specified by a pair of unit vectors  $X$  and  $\tilde{X}$  tangent to  $M$  and  $\tilde{M}$  respectively. Note that the contact configuration doesn't change under simultaneous rotation of  $X$  and  $\tilde{X}$ , so the space of contact configurations is  $(SM \times S\tilde{M})/S^1$  where  $SM$  is the unit tangent bundle of  $M$ . A smooth rolling of these surfaces on each other is specified by a pair of smooth curves  $p(t)$  on  $M$  and  $\tilde{p}(t)$  on  $\tilde{M}$  with the same speed and geodesic curvature at each instant. In this situation  $p$  and  $\tilde{p}$  are the curves on  $M$  and  $\tilde{M}$  traced out by the rolling, and at each instant  $t$ , the contact configuration is specified by identifying  $\dot{p}(t)/|\dot{p}(t)|$  with  $\dot{\tilde{p}}(t)/|\dot{\tilde{p}}(t)|$ .

Starting from a contact configuration specified by  $(X, \tilde{X}) \in SM \times S\tilde{M}$ , any smooth curve on  $M$  initiating from the base point of  $X$ , uniquely identifies a rolling of  $M$  and  $\tilde{M}$ .

*Remark 1.* The concept of rolling is also defined when  $M$  and  $\tilde{M}$  are not geodesically complete. But in order to roll the surfaces along a curve  $p$  on one of them, one should guaranty the existence of a curve  $\tilde{p}$  on the other with the

same speed and geodesic curvature and with the initial conditions given by the starting contact configuration.

*Remark 2.* Rolling along piecewise smooth curves can be defined in a similar manner. Moreover, for non-orientable manifolds one can consider their orientable covering spaces. It is not difficult to check that the controllability of rolling for covering spaces is equivalent to the controllability of rolling for the base surfaces.

It is more convenient to work with  $SM \times S\tilde{M}$  as the phase space instead of the contact configuration space  $(SM \times S\tilde{M})/S^1$ . Since simultaneous rotations are allowed in this new phase space, rolling is defined by the curves  $\eta(t) = (R_t X, R_t \tilde{X})$  and  $\eta(t) = (\dot{p}(t)/|\dot{p}(t)|, \dot{\tilde{p}}(t)/|\dot{\tilde{p}}(t)|)$ , where  $R_t$  is rotation by  $t$  and  $p$  and  $\tilde{p}$  have the same speed and geodesic curvature.

In the rest of this section we give a clear description of rolling on  $SM \times S\tilde{M}$ . Let  $R_t : SM \rightarrow SM$  be the rotation by  $t$ , and  $\varphi^t : SM \rightarrow SM$  be the geodesic flow on  $SM$ , i.e.,  $\pi(\varphi^t(X))$  is the unique geodesic in  $M$  starting with velocity  $X$  and  $\frac{d}{dt}\pi(\varphi^t(X)) = \varphi^t(X)$ . Since  $R$  and  $\varphi$  are smooth semigroups, there are smooth vector fields  $V_1$  and  $V_2$  on  $SM$  such that  $\frac{d}{dt}R_t(X) = V_1(R_t(X))$  and  $\frac{d}{dt}\varphi^t(X) = V_2(\varphi^t(X))$ . Let  $\tilde{V}_1$  and  $\tilde{V}_2$  be the similar vector fields on  $S\tilde{M}$ . The following proposition gives the required tools to describe the rolling as a control-affine system on  $SM \times S\tilde{M}$ .

Consider the following control-affine system on  $SM$

$$(1) \quad \dot{X} = u_1(t)V_1(X) + u_2(t)V_2(X), \quad X \in SM$$

with locally integrable controls  $u_1(t)$  and  $u_2(t)$ . Note that by Nagano-Sussmann orbit Theorem [2] one can assume  $u_1(t)$  and  $u_2(t)$  are piecewise smooth or piecewise constant. The above assumption on controls is a weak condition for existence and uniqueness of solutions of equation (1).

**Proposition 1.** *For the above control system we have the following properties.*

- (a) *If  $p(t) = \pi(X(t))$  is the projection of a solution  $X(t)$  of (1) on  $M$ , then  $\dot{p}(t) = u_2(t)X(t)$ . If in addition  $u_2 \neq 0$ , then  $|u_2(t)|k(t) = u_1(t)$  where  $k(t)$  is the geodesic curvature of  $p(t)$ .*
- (b)  *$V_1, V_2$  and  $V_3 = [V_1, V_2]$  are linearly independent at each point and*

$$[V_3, V_1] = V_2, \quad [V_3, V_2] = -K^M V_1,$$

*where  $K^M(X)$  is the Gaussian curvature of  $M$  at  $\pi(X)$ .*

Inspired by the above proposition, the rolling in  $SM \times S\tilde{M}$  is given by the following control-affine system.

$$(2) \quad \frac{d}{dt}(X, \tilde{X}) = u_1(t)(V_1(X), \tilde{V}_1(\tilde{X})) + u_2(t)(V_2(X), \tilde{V}_2(\tilde{X})) \quad (X, \tilde{X}) \in SM \times S\tilde{M}$$

By applying a control with  $u_2(t) \equiv 0$  we obtain a simultaneous rotation. A control with  $u_1 \equiv 0$  gives us two geodesics on  $M$  and  $\tilde{M}$  with the same speed.

In general, by the above proposition, any solution to the control system (2) gives two curves on  $M$  and  $\tilde{M}$  with the same speed and geodesic curvature.

*Remark 3.* The above construction can be generalized to the rolling problem for a pair of smooth  $n$ -dimensional oriented Riemannian manifolds. Similar to the 2D rolling problem, it is more convenient to consider  $FM \times F\tilde{M}$  as the phase space of the control system, where  $FM$  and  $F\tilde{M}$  are the orthogonal frame bundles on  $M$  and  $\tilde{M}$ , respectively. In this phase space, rolling is defined by simultaneous rotations and geodesic flows. There are  $n(n-1)/2$  degrees of freedom to control the simultaneous rotations and  $n$  degrees of freedom for the geodesic flows.

*Proof of Proposition 1.* This proposition is well known and the second part is the structure equations for unit tangent bundle. One can also obtain it by direct computation in a geodesic orthogonal coordinate system of  $M$  with positive orientation. In this coordinate system, the coordinate lines  $x = 0$  and  $y = \text{const.}$  are unit speed geodesics, the metric is given by  $ds^2 = dx^2 + g(x, y)dy^2$ , the Gaussian curvature is  $K = -\frac{(\sqrt{g})_{xx}}{\sqrt{g}}$  and  $V_1$  and  $V_2$  at the tangent vector  $X = (x, y, u, v)$  are given by

$$V_1(X) = \frac{d}{d\theta}|_{\theta=0} R_\theta(X) = (0, 0, -v\sqrt{g}, u/\sqrt{g}),$$

$$V_2(X) = (u, v, \frac{g_x}{2}v^2, -(\frac{g_x}{g})uv - (\frac{g_y}{2g})v^2).$$

See ([6]) for more details. □

### 3. Controllability results

In this section we study the controllability of (2) which is a symmetric control-affine system. In order to prove the local controllability of (2), it is enough to show that the involutive distribution generated by  $(V_1, \tilde{V}_1)$  and  $(V_2, \tilde{V}_2)$  at  $(X, \tilde{X})$  has six dimensions. Proposition 1 helps us to compute the Lie brackets of these vector fields. As a consequence we obtain the following controllability result.

**Theorem 1.** *If  $K^M(X_0) \neq K^{\tilde{M}}(\tilde{X}_0)$ , then the control system (2) is locally controllable at the point  $(X_0, \tilde{X}_0) \in SM \times S\tilde{M}$ .*

*Proof.* Let  $F = K^{\tilde{M}}(\tilde{X}_0) - K^M(X_0)$ ,  $V_3 = [V_1, V_2]$ ,  $\tilde{V}_3 = [\tilde{V}_1, \tilde{V}_2]$ ,  $V_4 = [V_3, V_2]$  and  $\tilde{V}_4 = [\tilde{V}_3, \tilde{V}_2]$ . By Proposition 1, we have

$$[V_2, V_4] = -K^M[V_2, V_1] - (V_2 \cdot K^M)V_1 = K^M V_3 + DV_1,$$

$$[V_3, V_4] = -K^M[V_3, V_1] - (V_3 \cdot K^M)V_1 = -K^M V_2 - EV_1$$

for a suitable choice of functions  $D$  and  $E$ . According to these relations and similar relations for  $\tilde{V}_i$ , if  $F \neq 0$ , then

$$\begin{aligned} & \text{rank} \begin{bmatrix} V_1 & V_2 & V_3 & V_4 & [V_2, V_4] & [V_4, V_3] \\ \tilde{V}_1 & \tilde{V}_2 & \tilde{V}_3 & \tilde{V}_4 & [\tilde{V}_2, \tilde{V}_4] & [\tilde{V}_4, \tilde{V}_3] \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} V_1 & V_2 & V_3 & K^M V_1 & DV_1 + K^M V_3 & EV_1 + K^M V_2 \\ \tilde{V}_1 & \tilde{V}_2 & \tilde{V}_3 & K^{\tilde{M}} \tilde{V}_1 & \tilde{D}\tilde{V}_1 + K^{\tilde{M}} \tilde{V}_3 & \tilde{E}\tilde{V}_1 + K^{\tilde{M}} \tilde{V}_2 \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} V_1 & V_2 & V_3 & 0 & DV_1 & EV_1 \\ \tilde{V}_1 & \tilde{V}_2 & \tilde{V}_3 & F\tilde{V}_1 & \tilde{D}\tilde{V}_1 + F\tilde{V}_3 & \tilde{E}\tilde{V}_1 + F\tilde{V}_2 \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} V_1 & V_2 & V_3 & 0 & 0 & 0 \\ \tilde{V}_1 & \tilde{V}_2 & \tilde{V}_3 & F\tilde{V}_1 & \tilde{D}\tilde{V}_1 + F\tilde{V}_3 & \tilde{E}\tilde{V}_1 + F\tilde{V}_2 \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} V_1 & V_2 & V_3 & 0 & 0 & 0 \\ \tilde{V}_1 & \tilde{V}_2 & \tilde{V}_3 & F\tilde{V}_1 & F\tilde{V}_3 & F\tilde{V}_2 \end{bmatrix} = 6. \end{aligned}$$

This concludes the local controllability of the system at the point  $(X_0, \tilde{X}_0)$ .  $\square$

By the above theorem if there is a point  $(X_0, \tilde{X}_0)$  in an orbit with  $K^M(X_0) \neq K^{\tilde{M}}(\tilde{X}_0)$ , then this orbit should be open. Therefore, at each point on a non-open orbit, the Gaussian curvatures of  $M$  and  $\tilde{M}$  are equal. This fact implies that the two surfaces are locally isometric. The following theorem describes this local isometry.

**Theorem 2.** *If the orbit of a point  $(X_0, \tilde{X}_0)$  is not open in  $SM \times S\tilde{M}$ , then there exist two open neighborhoods  $U \subset M$  and  $\tilde{U} \subset \tilde{M}$  of the points  $\pi(X_0)$  and  $\tilde{\pi}(\tilde{X}_0)$  and an isometry  $h : U \rightarrow \tilde{U}$  such that  $h_*(X_0) = \tilde{X}_0$ .*

*Proof.* Consider the geodesic orthogonal coordinate systems of  $M$  and  $\tilde{M}$  such that  $X_0$  and  $\tilde{X}_0$  correspond to  $(0, 0, 0, 1)$ , and suppose that in these coordinate systems the metrics are  $ds = dx^2 + g(x, y)dy^2$  and  $d\tilde{s} = dx^2 + \tilde{g}(x, y)dy^2$ . Since the coordinate lines are orthogonal and  $x = 0$  and  $y = \text{cons.}$  are unit geodesics, for each  $(x, y)$ ,  $((x, y, 0, 1), (x, y, 0, 1))$  is accessible from  $(X_0, \tilde{X}_0)$ . Therefore, by Theorem 1,  $K^M(x, y) = K^{\tilde{M}}(x, y)$ , i.e.,  $(\sqrt{g})_{xx}/\sqrt{g} = (\sqrt{\tilde{g}})_{xx}/\sqrt{\tilde{g}}$ . Moreover, in a geodesic orthogonal coordinate,  $g(0, y) = 1$  and  $g_x(0, y) = 0$  (see [6]), thus  $\sqrt{g}$  and  $\sqrt{\tilde{g}}$  are the solutions of a second order ODE with the same initial values. Therefore,  $g = \tilde{g}$  and these coordinate systems give us the isometry  $h$ .  $\square$

Note that if  $h : U \rightarrow \tilde{U}$  is an isometry, then by restricting the system (2) to  $SU \times S\tilde{U}$ ,  $\{(X, h_*(X)) : X \in SU\}$  is a three-dimensional orbit in the six-dimensional phase space. This shows that at each point  $\eta = (X, h_*(X))$ ,  $\dim \text{Lie}\{(V_1, \tilde{V}_1), (V_2, \tilde{V}_2)\}(\eta) = 3$ , and by a rolling in  $SU \times S\tilde{U}$  which starts from a contact configuration of the form  $(X_0, h_*(X_0))$ , each point  $x \in U$  can only touch the point  $h(x) \in \tilde{U}$ .

By the Nagano-Sussmann orbit Theorem [2], each orbit of the control system (2) is an immersed submanifold of  $SM \times S\tilde{M}$ . Consider the restriction

of the control system (2) to a non-open orbit  $\mathcal{O}$ . By the above theorem,  $\text{Lie}\{(V_1, \tilde{V}_1), (V_2, \tilde{V}_2)\}$  is a three-dimensional involutive distribution on  $\mathcal{O}$ . Since the new control system is controllable, its unique orbit,  $\mathcal{O}$ , has three dimensions. Therefore, every non-open orbit of control system (2) is a three-dimensional immersed submanifold of  $SM \times \tilde{SM}$ .

Since we have three degrees of freedom for choosing the first component of a point in  $\mathcal{O}$ , we have no more degrees of freedom for choosing the second component. In other words, for each  $X \in SM$ ,  $\{\tilde{X} \in \tilde{SM} : (X, \tilde{X}) \in \mathcal{O}\}$  has zero dimension, hence it is a discrete subset of  $\tilde{SM}$ . This shows that if we reach  $\eta' = (X, \tilde{X})$  by a rolling from the initial contact configuration  $\eta \in \mathcal{O}$ , then  $X$  (resp.  $\pi(X)$ ) can only coincide with  $\tilde{X}$  (resp.  $\pi(\tilde{X})$ ) during a continuous change of the rolling. The following homotopy theorem is a more precise statement of these facts.

**Theorem 3.** *Suppose that the orbit of  $(X_0, \tilde{X}_0)$  is not open in  $SM \times \tilde{SM}$  and  $h : [0, T] \times [0, 1] \rightarrow M$  is a continuous homotopy of piecewise regular admissible curves of  $M$  for contact configuration  $(X_0, \tilde{X}_0)$ . Then  $h$  corresponds to a homotopy  $\tilde{h}$  of piecewise regular admissible curves in  $\tilde{M}$  for contact configuration  $(X_0, \tilde{X}_0)$ . In other words, by rolling from initial contact configuration  $(X_0, \tilde{X}_0)$  along  $p^s = h(\cdot, s)$  the final contact points on  $M$  and  $\tilde{M}$  are independent of  $s$ .*

*Proof.* It is enough to show that each  $s_0 \in I$  has a neighborhood for which the above theorem is true. Let  $\tilde{p}^s$  in  $\tilde{M}$  corresponds to  $p^s$ . Since all contact configurations  $(X^s(t), \tilde{X}^s(t)) = (\dot{p}^s(t)/|\dot{p}^s(t)|, \dot{\tilde{p}}^s(t)/|\dot{\tilde{p}}^s(t)|)$  belong to the orbit of  $(X_0, \tilde{X}_0)$  and this orbit is not open in  $SM \times \tilde{SM}$ , by Theorem 2 there exist neighborhoods  $U^t \subset M$  and  $\tilde{U}^t \subset \tilde{M}$  of  $p^{s_0}(t)$  and  $\tilde{p}^{s_0}(t)$  and an isometry  $h^t : U^t \leftrightarrow \tilde{U}^t$  such that  $h_*^t(X^{s_0}(t)) = \tilde{X}^{s_0}(t)$ . One can find  $0 = t_0 < t_1 < \dots < t_{n+1} = T$  such that for each  $i$ ,  $p^{s_0}([t_i, t_{i+1}])$  lies in one of  $U^t$ 's which we denote by  $U_i$ . Since  $h$  is continuous,  $p^s([t_i, t_{i+1}]) \subset U_i$  for each  $s$  sufficiently close to  $s_0$ . One can also assume that  $p^{s_0}(t_i)$  and  $p^s(t_i)$  are connected by a piecewise regular curve  $\alpha_i$  in  $U_{i-1} \cap U_i$  (maybe by passing through a smaller neighborhood of  $s_0$  or by choosing appropriate  $U^t$ 's like small geodesically convex neighborhoods). Let us denote  $p^s([t_{i-1}, t_i])$  by  $p_i^s$ . Since  $h_i : U_i \leftrightarrow \tilde{U}_i$  is an isometry, the final contact configuration of rolling along a curve in  $U_i$  from an initial contact configuration of the form  $(X, h_{i*}(X))$  is of the same form and depends only on the end point of that curve. Thus both the results of rolling along  $p_i^{s_0}$  and rolling along  $\alpha_{i+1}^- * p_i^s * \alpha_i$  from initial contact configuration  $(X^{s_0}(t_i), \tilde{X}^{s_0}(t_i))$  are  $(X^{s_0}(t_{i+1}), \tilde{X}^{s_0}(t_{i+1}))$ . Therefore, the results of rolling from initial contact configuration  $(X_0, \tilde{X}_0)$  along  $p^{s_0} = p_n^{s_0} * \dots * p_0^{s_0}$  and rolling along  $p^s = (p_n^s * \alpha_n) * (\alpha_n^- * p_{n-1}^s * \alpha_{n-1}) * \dots * (\alpha_1 * p_0^s)$  are the same.  $\square$

The set of homotopy classes of admissible piecewise regular curves of  $M$  for any initial contact configuration  $\eta = (X, \tilde{X})$  is a simply connected two-dimensional manifold (may be with boundary) which is denoted by  $\mathcal{P}_\eta$ . It has



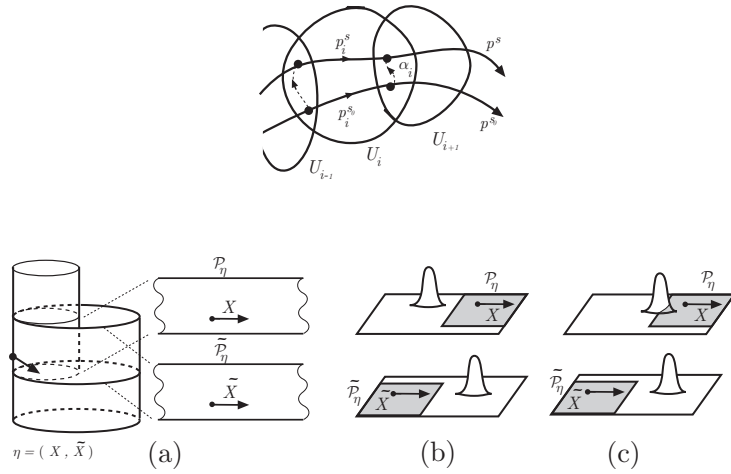


FIGURE 1.  $\mathcal{P}_\eta$  and  $\tilde{\mathcal{P}}_\eta$  for several surfaces. In (a) and (b) there is an isometry  $\tau : \mathcal{P}_\eta \rightarrow \tilde{\mathcal{P}}_\eta$  such that  $\tau_*(X) = \tilde{X}$ , so the orbit of  $\eta = (X, \tilde{X})$  has three dimensions. But in (c) they are not isometric, hence the orbit of  $\eta$  is open.

the natural projection  $\iota : \mathcal{P}_\eta \rightarrow M$  which makes it an oriented Riemannian manifold. Note that when  $\tilde{M}$  is complete,  $\mathcal{P}_\eta$  is the universal covering space of  $M$ . By the above theorem, if the orbit of  $\eta$  is not open in  $SM \times \tilde{SM}$ , there is an orientation preserving isometry  $\tau : \mathcal{P}_\eta \leftrightarrow \tilde{\mathcal{P}}_\eta$  such that  $\tau_*(X) = \tilde{X}$ . The converse is obvious.

- Theorem 4.**
- (a) *Orbits of the control system (2) are open or three-dimensional immersed submanifolds of  $SM \times \tilde{SM}$ . Furthermore the orbit of  $\eta = (X, \tilde{X})$  has three dimensions if and only if there exists an orientation preserving isometry  $\tau : \mathcal{P}_\eta \rightarrow \tilde{\mathcal{P}}_\eta$  such that  $\tau_*(X) = \tilde{X}$ .*
  - (b) *Suppose that  $M$  and  $\tilde{M}$  are complete. Then three-dimensional orbits are exactly  $\{(X, h_*(X)) : X \in SM\}$  where  $h$  is an orientation preserving isometry from universal covering space of  $M$  to universal covering space of  $\tilde{M}$ .*
  - (c) *If  $M$  and  $\tilde{M}$  are complete embedded surfaces in  $\mathbb{R}^3$  and one of them is closed and has positive Gaussian curvature every where, then either (2) is controllable or there is a rigid transformation of  $\mathbb{R}^3$  which brings  $M$  to  $\tilde{M}$  (i.e., they have the same shape in  $\mathbb{R}^3$ ). In the former case all holonomic contact configurations are exactly those in which the surfaces coincide or they are mirror images of each other with respect to the common tangent plane.*

*Proof.* Part (a) is an immediate consequence of the above theorems. In part (b),  $P_\eta$  and  $\tilde{P}_\eta$  are universal covering spaces of  $M$  and  $\tilde{M}$ . Therefore, this part is special case of part (a).

Note that if  $M$  is a closed orientable 2D Riemannian manifold and its universal covering space has positive curvature everywhere, then  $M$  is simply connected, so it is equal to its universal covering space. Thus, in the case of part (c), if the control system is not controllable, then  $M$  and  $\tilde{M}$  should be isometric closed surfaces with positive Gaussian curvature. Moreover, at a holonomic contact configuration, there is an isometry between two surfaces which induces the identity map on the common tangent plane. According to rigidity theorem for closed surfaces with positive Gaussian curvature in  $\mathbb{R}^3$  (see [11, page 136]), this isometry can be extended to an isometry of  $\mathbb{R}^3$  which is identity on the common tangent plane. Such an isometry is either the identity map or a reflection with respect to this plane. In the first case, the surfaces are coincide and in the second case, they are mirror images of each other with respect to the common tangent plane.  $\square$

### References

- [1] A. Agrachev, *Rolling balls and octonions*, Proc. Steklov Inst. Math. **258** (2007), no. 1, 13–22.
- [2] A. Agrachev and Y. Sachkov, *Control theory from the geometric viewpoint*, Encyclopaedia of Mathematical Sciences, 87, Control Theory and Optimization II, Springer-Verlag, Berlin, 2004.
- [3] R. L. Bryant and L. Hsu, *Rigidity of integral curves of rank 2 distributions*, Invent. Math. **114** (1993), no. 2, 435–461.
- [4] A. Chelouah and Y. Chitour, *On the motion planning of rolling surfaces*, Forum Math. **15** (2003), no. 5, 727–758.
- [5] Y. Chitour and P. Kokkonen, *Rolling of manifolds and controllability in dimension three*, in Mémoires de la Société Mathématique de France, 2015.
- [6] M. P. Do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall Englewood Cliffs, 1976.
- [7] M. Godoy Molina, E. Grong, I. Markina, and F. Silva Leite, *An intrinsic formulation of the problem on rolling manifolds*, J. Dyn. Control Syst. **18** (2012), no. 2, 181–214.
- [8] E. Grong, *Controllability of rolling without twisting or slipping in higher dimensions*, SIAM J. Control Optim. **50** (2012), no. 4, 2462–2485.
- [9] V. Jurdjevic, *The geometry of the plate-ball problem*, Archive for Rational Mechanics and Analysis **124** (1993), no. 4, 305–328.
- [10] ———, *Geometric Control Theory*, Cambridge university press, 1997.
- [11] W. Klingenberg, *A Course in Differential Geometry*, Springer Science & Business Media, 51, 2013.
- [12] M. Levi, *Geometric phases in the motion of rigid bodies*, Arch. Rational Mech. Anal. **123** (1993), no. 3, 305–328.
- [13] Z. Li and J. Canny, *Motion of two rigid bodies with rolling constraint*, IEEE Trans. Automat. Control **6** (1990), no. 1, 62–72.
- [14] A. Marigo and A. Bicchi, *Rolling bodies with regular surface: Controllability theory and applications*, IEEE Trans. Automat. Control **45** (2000), no. 9, 1586–1599.
- [15] A. Marigo, M. Ceccarelli, S. Piccinocchi, and A. Bicchi, *Planning motions of polyhedral parts by rolling*, Algorithmica **26** (2000), no. 3-4, 560–576.

- [16] S. R. Moghadasi, *Rolling of a body on a plane or a sphere: A geometric point of view*, Bull. Austral. Math. Soc. **70** (2004), no. 2, 245–256.
- [17] J. Monforte, *Geometric, control, and numerical aspects of nonholonomic systems*, 1793, Springer Verlag, 2002.
- [18] D. J. Montana, *The kinematics of contact and grasp*, Int. J. Rob. Res. **7** (1988), no. 3, 17–32.
- [19] J. A. Zimmerman, *Optimal control of the sphere  $S^n$  rolling on  $E^n$* , Math. Control Signals Systems **17** (2005), no. 1, 14–37.

S. REZA MOGHADASI  
DEPARTMENT OF MATHEMATICAL SCIENCE  
SHARIF UNIVERSITY OF TECHNOLOGY  
P.O. BOX 11155-9415, TEHRAN, IRAN  
*E-mail address:* moghadasi@sharif.edu