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FUZZY STABILITY OF QUADRATIC-CUBIC FUNCTIONAL EQUATIONS

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Abstract. In this paper, we consider the functional equation

$$f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 3f(y) + 3f(-y) = 0$$

and prove the generalized Hyers-Ulam stability for it when the target space is a fuzzy Banach space. The usual method to obtain the stability for mixed type functional equation is to split the cases according to whether the involving mappings are odd or even. In this paper, we show that the stability of a quadratic-cubic mapping can be obtained without distinguishing the two cases.

1. Introduction and preliminaries

Katsaras [12] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Later, some mathematicians have defined fuzzy norms on a vector space in different points of view. In particular, Bag and Samanta [2], following Cheng and Mordeson [3], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [11]. In this paper, we use the definition of fuzzy normed spaces given in [2], [14], and [16].

Definition 1. Let X be a real vector space. A function $N : X \times \mathbb{R} \longrightarrow [0, 1]$ is called *a fuzzy norm on* X if for any $x, y \in X$ and any $s, t \in \mathbb{R}$,

(N1) $N(x, t) = 0$ for $t \leq 0$;
(N2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
(N3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x + y, s + t) \geq min\{N(x, s), N(y, t)\}$;
(N5) $N(x, \cdot)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
(N6) for any $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

In this case, the pair (X, N) is called *a fuzzy normed space.*

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Definition 2. Let (X, N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is said to be *convergent in* (X, N) if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* X and one denotes it by $N - \lim_{n\to\infty} x_n = x$.

Definition 3. Let (X, N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is said to be *Cauchy in* (X, N) if for any $\epsilon > 0$ and any $t > 0$, there exists an $m \in \mathbb{N}$ such that $N(x_{n+p} - x_n, t) > 1 - \epsilon$ for all $n \geq m$ and all positive integer p,.

It is well known that every convergent sequence in a fuzzy normed space is Cauchy. A fuzzy normed space is said to be *complete* if each Cauchy sequence in it is convergent and a complete fuzzy normed space is called *a fuzzy Banach space.*

In 1996, Isac and Rassias [10] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

Definition 4. Let X be a non-empty set. Then a mapping $d : X^2 \longrightarrow [0, \infty]$ is called *a generalized metric on* X if d satisfies the following conditions:
(D1) $d(x, y) = 0$ if and only if $x = y$,
(D2) $d(x, y) = d(y, x)$, and
(D3) $d(x, y) \leq d(x, z) + d(z, y)$.
In case, (X, d) is called *a generalized metric space.*

Theorem 1.1. [7] *Let* (X, d) *be a complete generalized metric space and let* $J : X \longrightarrow X$ *be a strictly contractive mapping with some Lipschitz constant L with* $0 < L < 1$. *Then for each given element* $x \in X$, *either* $d(J^n x, J^{n+1}x) = \infty$ *for all nonnegative integer* n *or there exists a positive integer* n_0 *such that*
(1) $d(J^n x, J^{n+1}x) < \infty$ *for all* $n \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J ;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$ and
(4) $d(y, y^*) \leq \dfrac{1}{1-L}\ d(y, Jy)$ *for all* $y \in Y$.

In 1940, Ulam proposed the following stability problem (cf. [23]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [9] gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki ([1]) for additive mappings and by Rassias [21] for linear mappings to consider the stability problem with unbounded Cauchy differences. During the last decades, the stability problem of functional equations have been extensively investigated by a number of mathematicians (see [4], [5], [6], [8], and [17]).

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1}$$

is called *a quadratic functional equation* and every solution of the quadratic functional equation is called *a quadratic mapping*. A generalized Hyers-Ulam stability problem for (1) was proved by Skof [22] for mappings from a normed space to a Banach space. Mirmostafaee and Moslehian [15] proved the stability of (1) in fuzzy normed spaces. In 2001, Rassias [20] introduced the following cubic functional equation

$$f(x+2y)-3f(x+y)+3f(x)-f(x-y)-6f(y)=0 \tag{2}$$

and every solution of the cubic functional equation is called *a cubic mapping*. Some authors proved the stability for a cubic functional equation in fuzzy normed spaces via fixed point technique([18]) . In prticular, some authors proved the generalized Hyers-Ulam stability of quadratic-cubic functional equations in fuzzy Banach spaces([19]). In most of papers, authors proved the stability for mixed type functional equations by dividing even mappings caese and odd mappings case.

In this paper, we consider the following new functional equation

$$f(x+2y)-3f(x+y)+3f(x)-f(x-y)-3f(y)+3f(-y)=0. \tag{3}$$

which is a quadratic-cubic functional equation and prove the generalized Hyers-Ulam stability for (3) in fuzzy Banach spaces. The usual method that has been employed by many authors to obtain the stability of mixed type functional equations is to split the cases according to whether the involving mappings are odd or even. In this paper, we show that, in contrast to the usual one, the stability of a quadratic-cubic mapping $f : X \longrightarrow Y$ can be obtained without distinguishing the two cases.

Throughout this paper, we assume that X is a linear space, (Y, N) is a fuzzy Banach space, and (Z, N') is a fuzzy normed space.

2. The Generalized Hyers-Ulam stability for (3)

In this section, we prove the generalized Hyers-Ulam stability for the functional equation (3) in fuzzy Banach spaces. For any mapping $f : X \longrightarrow Y$, let

$$f_o(x)=\frac{f(x)-f(-x)}{2}, \quad f_e(x)=\frac{f(x)+f(-x)}{2}.$$

We start with the following lemma.

Lemma 2.1. *A mapping $f : X \longrightarrow Y$ satisfies (3) and $f(0)=0$ if and only if f is a quadratic-cubic mapping.*

Proof. Suppose that $f : X \longrightarrow Y$ satisfies (3). Then we have

$$f_o(x+2y) - 3f_o(x+y) + 3f_o(x) - f_o(x-y) - 6f_o(y) = 0.$$

for all $x, y \in X$ and so f_o satisfies (2). Similarly, f_e satisfies (1) and thus $f = f_e + f_o$ is a quadratic-cubic mapping. The converse is trivial. $\square$

For any mapping $f : X \longrightarrow Y$, we define the difference operator $Df : X^2 \longrightarrow Y$ by

$$Df(x,y) = f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 3f(y) + 3f(-y)$$

for all $x, y \in X$.

Theorem 2.2. *Assume that $\phi : X^2 \longrightarrow Z$ is a function such that*

$$N'(\phi(x,y),t) \geq N'\Big(\frac{L}{8}\phi(2x,2y),t\Big) \tag{4}$$

for all $x, y \in X$, $t > 0$ and some L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$N(Df(x,y),t) \geq N'(\phi(x,y),t) \tag{5}$$

for all $x, y \in X$ and all $t > 0$. Then there exists a unique quadratic-cubic mapping $F : X \longrightarrow Y$ such that

$$N\Big(F(x) - f(x), \frac{L}{8(1-L)}t\Big) \geq \min\{N'(\phi(0,x),t), N'(\phi(0,-x),t)\} \tag{6}$$

for all $x \in X$ and all $t > 0$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$\begin{aligned} d(g,h) = \inf\{c \in [0,\infty) \mid N(g(x) - h(x), ct) \geq \min\{N'(\phi(0,x),t), N'(\phi(0,-x),t)\},\\ \forall x \in X, \forall t > 0\}. \end{aligned}$$

Then (S,d) is a generalized complete metric space([13]). Define a mapping $J : S \longrightarrow S$ by

$$Jg(x) = 6g(\frac{x}{2}) - 2g(-\frac{x}{2})$$

for all $g \in S$ and all $x \in X$. Let $g, h \in S$ and $d(g,h) \leq c$ for some $c \in [0,\infty)$. Then by (4), we have

$$\begin{aligned} &N(Jg(x) - Jh(x), cLt) \\ &= N\Big(6g\Big(\frac{x}{2}\Big) - 6h\Big(\frac{x}{2}\Big) - 2g\Big(-\frac{x}{2}\Big) + 2h\Big(-\frac{x}{2}\Big), cLt\Big) \\ &\geq \min\Big\{N\Big(g\Big(\frac{x}{2}\Big) - h\Big(\frac{x}{2}\Big), \frac{cL}{8}t\Big), N\Big(g\Big(-\frac{x}{2}\Big) - h\Big(-\frac{x}{2}\Big), \frac{cL}{8}t\Big)\Big\} \\ &\geq \min\{N'(\phi(0,x),t), N'(\phi(0,-x),t)\} \end{aligned}$$

for all $x \in X$ and all $t > 0$. Hence we have $d(Jg, Jh) \leq Ld(g,h)$ for any $g, h \in S$ and so J is a strictly contractive mapping.

Next, we claim that $d(Jf, f) < \infty$. Putting $x = 0$ and $y = x$ in (5), we get

$$N(f(2x) - 6f(x) + 2f(-x), t) \geq N'(\phi(0, x), t)$$

for all $x \in X$, $t > 0$ and hence

$$\begin{aligned} N(f(x) - Jf(x), \frac{L}{8}t) &= N\Big(f(x) - 6f\Big(\frac{x}{2}\Big) + 2f\Big(-\frac{x}{2}\Big), \frac{L}{8}t\Big) \\ &\geq N'\Big(\phi\Big(0, \frac{x}{2}\Big), \frac{L}{8}t\Big) \geq N'(\phi(0, x), t) \\ &\geq \min\{N'(\phi(0, x), t), N'(\phi(0, -x), t)\} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So we have $d(f, Jf) \leq \frac{L}{8} < \infty$. By Theorem 1.1, there exists a mapping $F : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, F) \to 0$ as $n \to \infty$. By induction, there are sequences $\{a_n\}$ and $\{b_n\}$ in $\mathbb{R}^+$ such that

$$J^n f(x) = a_n f\Big(\frac{x}{2^n}\Big) - b_n f\Big(-\frac{x}{2^n}\Big)$$

for all $x \in X$ and $n \in \mathbb{N}$, where

$$\begin{cases} 6a_n + 2b_n = a_{n+1} \\ 2a_n + 6b_n = b_{n+1} \end{cases} \tag{7}$$

for all $n \in \mathbb{N}$. Then we get

$$F(x) = N - \lim_{n \to \infty} \Big[a_n f\Big(\frac{x}{2^n}\Big) - b_n f\Big(-\frac{x}{2^n}\Big)\Big] \tag{8}$$

for all $x \in X$. Moreover, by (7), we get

$$4(a_n - b_n) = a_{n+1} - b_{n+1}, \ \ 8(a_n + b_n) = a_{n+1} + b_{n+1}$$

for all $n \in \mathbb{N}$ and by (8), we have

$$\begin{aligned} F_e(x) &= N - \lim_{n \to \infty} (a_n - b_n) f_e\Big(\frac{x}{2^n}\Big) = N - \lim_{n \to \infty} 2^{2n} f_e\Big(\frac{x}{2^n}\Big), \\ F_o(x) &= N - \lim_{n \to \infty} (a_n + b_n) f_o\Big(\frac{x}{2^n}\Big) = N - \lim_{n \to \infty} 2^{3n} f_o\Big(\frac{x}{2^n}\Big) \end{aligned} \tag{9}$$

for all $x \in X$. Replacing x, y by $\frac{x}{2^n}$, $\frac{x}{2^n}$ in (5), respectively, by (4), we have

$$\begin{aligned} & N\Big(Df_e\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big), \frac{t}{2^{2n}}\Big) \\ & \geq \min\Big\{N\Big(Df\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big), \frac{t}{2^{2n}}\Big), N\Big(Df\Big(-\frac{x}{2^n}, -\frac{y}{2^n}\Big), \frac{t}{2^{2n}}\Big)\Big\} \\ & \geq \min\Big\{N'\Big(\phi\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big), \frac{t}{2^{2n}}\Big), N'\Big(\phi\Big(-\frac{x}{2^n}, -\frac{y}{2^n}\Big), \frac{t}{2^{2n}}\Big)\Big\} \\ & \geq \min\Big\{N'\Big(\phi(x, y), \frac{2^n}{L^n}t\Big), N'\Big(\phi(-x, -y), \frac{2^n}{L^n}t\Big)\Big\} \end{aligned}$$

for all $x, y \in X$, $t > 0$, and all $n \in \mathbb{N}$. Since $0 < L < 1$, letting $n \to \infty$ in the last inequality, F_e is a solution of (3). Similarly, F_o is a solution of (3) and hence $F = F_e + F_o$ is a solution of (3). By Lemma 2.1, F is a quadratic-cubic mapping. Since $d(f, Jf) \leq \frac{L}{8}$, by Theorem 1.1, we have (6).

Now, we will show the uniqueness of F. Let G be a quadratic-cubic mapping with (6). Then clearly, G is a fixed point of J and by (6), we get

$$d(Jf,G) = d(Jf,JG) \le Ld(f,G) \le \frac{L^2}{8(1-L)} < \infty \tag{10}$$

and hence by (3) in Theorem 1.1, $F = G$. $\square$

Related with Theorem 2.2, we can also have the following theorem. And the proof is similar to that of Theorem 2.2.

Theorem 2.3. *Assume that $\phi : X^2 \longrightarrow Z$ is a function such that*

$$N'(\phi(2x,2y),t) \ge N'(4L\phi(x,y),t) \tag{11}$$

for all $x, y \in X$, $t > 0$ and some L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$N(Df(x,y),t) \ge N'(\phi(x,y),t) \tag{12}$$

for all $x, y \in X$ and all $t > 0$. Then there exists a unique quadratic-cubic mapping $F : X \longrightarrow Y$ such that

$$N\Big(F(x) - f(x), \frac{1}{4(1-L)}t\Big) \ge \min\{N'(\phi(0,x),t), N'(\phi(0,-x),t)\} \tag{13}$$

for all $x \in X$ and all $t > 0$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$\begin{aligned} d(g,h) = inf\{c \in [0,\infty) \mid N(g(x)-h(x),ct) \ge \min\{N'(\phi(0,x),t), N'(\phi(0,-x),t)\},\\ \forall x \in X, \forall t > 0\}. \end{aligned}$$

Then (S,d) is a generalized complete metric space(See [13]). Define a mapping $J : S \longrightarrow S$ by

$$Jg(x) = \frac{3}{16}g(2x) + \frac{1}{16}g(-2x)$$

for all $g \in S$ and all $x \in X$. Let $g, h \in S$ and $d(g,h) \le c$ for some $c \in [0,\infty)$. Then by (11), we have

$$\begin{aligned} N(Jg(x)-Jh(x),cLt) &= N\Big(\frac{3}{16}g(2x) - \frac{3}{16}h(2x) + \frac{1}{16}g(-2x) - \frac{1}{16}h(-2x), cLt\Big)\\ &\ge \min\{N(g(2x)-h(2x),4cLt), N(g(-2x)-h(-2x),4cLt)\}\\ &\ge \min\{N'(\phi(0,x),t), N'(\phi(0,-x),t)\} \end{aligned}$$

for all $x \in X$ and all $t > 0$. Hence we have $d(Jg,Jh) \le Ld(g,h)$ for any $g, h \in S$ and so J is a strictly contractive mapping.

Next, we claim that $d(Jf,f) < \infty$. Putting $x = 0$ and $y = x$ in (12), we get

$$N(f(2x) - 6f(x) + 2f(-x), t) \ge N'(\phi(0,x),t)$$

for all $x \in X$, $t > 0$ and hence

$$\begin{aligned} & N(Jf(x) - f(x), \frac{1}{4}t) \\ &= N\Big(\frac{3}{16}\Big[f(2x) - 6f(x) + 2f(-x)\Big] + \frac{1}{16}\Big[f(-2x) - 6f(-x) + 2f(x)\Big], \frac{1}{4}t\Big) \\ &\geq \min\{N'(\phi(0,x),t), N'(\phi(0,-x),t)\} \end{aligned}$$

for all $x \in X$, $t > 0$ and so we have $d(f, Jf) \leq \frac{1}{4} < \infty$. By Theorem 1.1, there exists a mapping $F : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, F) \to 0$ as $n \to \infty$ and by (4) in Theorem 1.1, we have (13). By induction, there are sequences $\{a_n\}$ and $\{b_n\}$ in $\mathbb{R}^+$ such that

$$J^n f(x) = a_n f(2^n x) + b_n f(-2^n x)$$

for all $x \in X$ and all $n \in \mathbb{N}$, where

$$\begin{cases} \frac{3}{16}a_n + \frac{1}{16}b_n = a_{n+1} \\ \frac{1}{16}a_n + \frac{3}{16}b_n = b_{n+1} \end{cases} \tag{14}$$

for all $n \in \mathbb{N}$. Then we get

$$F(x) = N - \lim_{n\to\infty} [a_n f(2^n x) + b_n f(-2^n x))] \tag{15}$$

for all $x \in X$ and by (14), we get

$$\frac{1}{4}(a_n + b_n) = a_{n+1} + b_{n+1}, \quad \frac{1}{8}(a_n - b_n) = a_{n+1} - b_{n+1}$$

for all $n \in \mathbb{N}$. By (15), we obtain

$$\begin{aligned} F_e(x) &= N - \lim_{n\to\infty} (a_n + b_n) f_e(2^n x) = N - \lim_{n\to\infty} \frac{1}{2^{2n}} f_e(2^n x), \\ F_o(x) &= N - \lim_{n\to\infty} (a_n - b_n) f_o(2^n x) = N - \lim_{n\to\infty} \frac{1}{2^{3n}} f_o(2^n x) \end{aligned} \tag{16}$$

for all $x \in X$. By (12), we have

$$\begin{aligned} & N(Df_e(2^n x, 2^n y), 2^{2n} t) \\ &\geq \min\{N(Df(2^n x, 2^n y), 2^{2n} t), N(Df(-2^n x, -2^n y), 2^{2n} t)\} \\ &\geq \min\{N'(\phi(2^n x, 2^n y), 2^{2n} t), N'(\phi(-2^n x, -2^n y), 2^{2n} t)\} \\ &\geq \min\Big\{N'\Big(\phi(x,y), \frac{1}{L^n}t\Big), N'\Big(\phi(-x,-y), \frac{1}{L^n}t\Big)\Big\} \end{aligned}$$

for all $x, y \in X$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, F_e satisfies (3). Similarly, F_o satisfies (3) and so $F = F_e + F_o$ satisfies (3). By Lemma 2.1, F is a quadratic-cubic mapping.

Now, we show the uniqueness of F. Let G be a quadratic-cubic mapping with (13). Then clearly, G is a fixed point of J and

$$d(Jf, G) = d(Jf, JG) \leq Ld(f, G) \leq \frac{L}{4(1-L)} < \infty \tag{17}$$

and hence by (3) in Theorem 1.1, $F = G$. □

We can use Theorem 2.2 and Theorem 2.3 to get a classical result in the framework of normed spaces. For example, it is well known that for any normed space $(X, ||\cdot||)$, the mapping $N_X : X \times \mathbb{R} \longrightarrow [0,1]$, defined by

$$N_X(x,t) = \begin{cases} 0, & \text{if } t \le 0 \\ \frac{t}{t+||x||}, & \text{if } t > 0 \end{cases}$$

is a fuzzy norm on X. In [14], [15] and [16], some examples are provided for the fuzzy norm N_X and other fuzzy norms. Here especially using the fuzzy norm $N_X = N'_X$ and taking $\phi(x,y) = \|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p}$, we have the following example.

Corollary 2.4. *Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$\|Df(x,y)\| \le \|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p}. \tag{18}$$

for all $x, y \in X$ and a fixed positive real number p with $0 < p < 1$ or $\frac{3}{2} < p$. Then there exists a unique quadratic-cubic mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \le \begin{cases} \dfrac{1}{2^{2p}-8}\|x\|^{2p}, & \text{if } \frac{3}{2} < p, \\ \\ \dfrac{1}{8-2^{2p}}\|x\|^{2p}, & \text{if } 0 < p < 1 \end{cases}$$

for all $x \in X$.

The following example shows that the inequality (18) is not stable for $p = 1$.

Remark 1. *Define mappings $s, t : \mathbb{R} \longrightarrow \mathbb{R}$ by*

$$s(x) = \begin{cases} x^2, & \text{if } |x| < 1 \\ 1, & \text{ortherwise}, \end{cases} \qquad t(x) = \begin{cases} x^3, & \text{if } |x| < 1 \\ -1, & \text{if } x \le -1 \\ 1, & \text{if } 1 \le x \end{cases}$$

and a mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty}\left[\frac{s(2^n x)}{4^n} + \frac{t(2^n x)}{8^n}\right]$$

We will show that there is a positive real number M such that

$$|Df(x,y)| \le M(|x||y| + |x|^2 + |y|^2) \tag{19}$$

for all $x, y \in \mathbb{R}$. But there do not exist a quadratic-cubic mapping $F : \mathbb{R} \longrightarrow \mathbb{R}$ and a non-negative constant K such that

$$|F(x) - f(x)| \le K|x|^2 \tag{20}$$

for all $x \in \mathbb{R}$.

Proof. Note that $s_o(x) = 0$ and $t_o(x) = t(x)$ for all $x \in \mathbb{R}$. Hence $|f_o(x)| \le \frac{8}{7}$. First, suppose that $\frac{1}{64} \le |x||y| + |x|^2 + |y|^2$. Then $|Df_o(x, y)| \le 1024(|x||y| + |x|^2 + |y|^2)$. Now, suppose that $\frac{1}{64} > |x||y| + |x|^2 + |y|^2$. Then there is a non-negative integer m such that

$$\frac{1}{2^{3m+9}} \le |x||y| + |x|^2 + |y|^2 < \frac{1}{2^{3m+6}}$$

and since $|x| < 1$ and $|y| < 1$,

$$2^{3m}|x|^3 < 2^{3m}|x|^2 < \frac{1}{64}, \quad 2^{3m}|y|^3 < 2^{3m}|y|^2 < \frac{1}{64}.$$

Hence we have

$$\{2^m(x + 2y),\ 2^m(x \pm y),\ 2^m x,\ 2^m y\} \subseteq \{z \in \mathbb{R} \mid |z|^3 < 1\}$$

and so for any $n = 0, 1, 2, \cdots, m$,

$$Dt_0(2^n x, 2^n y) = 0.$$

Thus

$$\begin{aligned} |Df_o(x, y)| &= \Big|\sum_{n=0}^{\infty} \frac{1}{8^n} Dt_o(2^n x, 2^n y)\Big| \le \sum_{n=m+1}^{\infty} \Big|\frac{1}{8^n} Dt_o(2^n x, 2^n y)\Big| \le \frac{2}{2^{3m}} \\ &\le 2^{10}(|x||y| + |x|^2 + |y|^2). \end{aligned}$$

Note that $t_e(x) = 0$ and $s_e(x) = s(x)$ for all $x \in X$. Hence $|f_e(x)| \le \frac{4}{3}$ for all $x \in \mathbb{R}$, First, suppose that $\frac{1}{64} \le |x||y| + |x|^2 + |y|^2$. Then $|Df_e(x, y)| \le \frac{2048}{3}(|x||y|+|x|^2+|y|^2)$ for all $x, y \in \mathbb{R}$. Now, suppose that $\frac{1}{64} > |x||y|+|x|^2+|y|^2$. Then there is a non-negative integer k such that

$$\frac{1}{2^{2k+8}} \le |x||y| + |x|^2 + |y|^2 < \frac{1}{2^{2k+6}} \tag{21}$$

and so

$$2^k|x| < \frac{1}{8}, \quad 2^k|y| < \frac{1}{8}.$$

Hence we have

$$\{2^k(x + 2y),\ 2^k(x \pm y),\ 2^k x,\ 2^k y\} \subseteq \{z \in \mathbb{R} \mid |z|^2 < 1\}$$

and so for any $n = 0, 1, 2, \cdots, k$,

$$Ds_e(2^n x, 2^n y) = 0.$$

Thus

$$\begin{aligned} \Big|Df_e(x, y)\Big| &= \Big|\sum_{n=0}^{\infty} \frac{1}{4^n} Ds_e(2^n x, 2^n y)\Big| = \Big|\sum_{n=k+1}^{\infty} \frac{1}{4^n} Ds_e(2^n x, 2^n y)\Big| \le \frac{8}{3} \cdot \frac{1}{2^{2k}} \\ &= \frac{2048}{3} \cdot \frac{1}{2^{2k+8}} \le \frac{2048}{3}(|x||y| + |x|^2 + |y|^2). \end{aligned}$$

Hence there is a positive real number M with (19).

Suppose that there exist a quadratic mapping $Q : \mathbb{R} \longrightarrow \mathbb{R}$, a cubic mapping $C : \mathbb{R} \longrightarrow \mathbb{R}$, and a non-negative constant K with (20).

Since $|f(x)| \leq \frac{60}{21}$, by (20), we have

$$-\frac{60}{21 \cdot n^3} - \frac{K|x|^2}{n} \leq \frac{Q(x)}{n} + C(x) \leq \frac{60}{21 \cdot n^3} + \frac{K|x|^2}{n}$$

for all $x \in X$ and all positive integers n. Hence

$$C(x) = 0$$

for all $x \in X$. Hence we get

$$|f(x) - Q(x)| \leq K|x|^2 \tag{22}$$

for all $x \in X$. Since $|f(x)| \leq \frac{60}{21}$, by (22), we have

$$-\frac{60}{21 \cdot n^2} - K|x|^2 \leq Q(x) \leq \frac{60}{21 \cdot n^2} + K|x|^2$$

for all $x \in X$ and all positive integers n. Hence we obtain

$$|Q(x)| \leq K|x|^2 \tag{23}$$

for all $x \in X$. By (22) and (23), we get

$$|f(x)| \leq 2K|x|^2 \tag{24}$$

for all $x \in X$ and hence we get

$$|f_e(x)| \leq 2K|x|^2 \tag{25}$$

for all $x \in X$. Take a positive integer l such that $l > 2K$ and pick $x \in \mathbb{R}$ with $0 < 2^l x < 1$. Since $0 < 2^l x < 1$ and so

$$f_e(x) = \sum_{n=0}^{\infty} \frac{s(2^n x)}{4^n} \geq \sum_{n=0}^{l-1} \frac{s(2^n x)}{4^n} \geq lx^2 > 2Kx^2$$

which contradicts to (25). $\square$

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