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OPTIMAL CONTROL FOR SOME REACTION DIFFUSION MODEL

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Abstract. This paper is concerned with the optimal control problem for some reaction diffusion model. That is, we show the existence of the global weak solution for the Field-Noyes model. We also show the existence of the optimal control.

1. Introduction

In this paper we are concerned with the following optimal control problem:

$$(\mathbf{P}) \qquad \text{minimize} \;\; J(u)$$

with the cost functional $J(u)$ of the form

$$\begin{aligned} J(u) = &\int_0^T \|y(u) - y_d\|^2_{H^1(I)} dt + \int_0^T \|\rho(u) - \rho_d\|^2_{H^1(I)} dt \\ &+ \int_0^T \|w(u) - w_d\|^2_{H^1(I)} + \gamma \int_0^T \|u\|^2_{L^2(I)} dt, \quad u \in L^2(0,T; L^2(I)), \end{aligned}$$

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where $y = y(u)$, $\rho = \rho(u)$ and $w = w(u)$ are governed by the Field-Noyes model

$$
\begin{aligned}
&\frac{\partial y}{\partial t} = a\frac{\partial^2 y}{\partial x^2} + \frac{1}{\epsilon}\big(qw - yw + y - y^2\big) \qquad \text{in } I \times (0,T],\\
&\frac{\partial \rho}{\partial t} = b\frac{\partial^2 \rho}{\partial x^2} + y - \rho \qquad \text{in } I \times (0,T],\\
&\frac{\partial w}{\partial t} = d\frac{\partial^2 w}{\partial x^2} + \frac{1}{\delta}\big(- qw - yw + c\rho + u\big) \qquad \text{in } I \times (0,T],\\
&\frac{\partial y}{\partial x}(0,t) = \frac{\partial y}{\partial x}(l,t) = 0 \qquad \text{on } (0,T],\\
&\frac{\partial \rho}{\partial x}(0,t) = \frac{\partial \rho}{\partial x}(l,t) = 0 \qquad \text{on } (0,T],\\
&\frac{\partial w}{\partial x}(0,t) = \frac{\partial w}{\partial x}(l,t) = 0 \qquad \text{on } (0,T],\\
&y(x,0) = y_0(x), \quad \rho(x,0) = \rho_0(x), \quad w(x,0) = w_0(x) \qquad \text{in } I.
\end{aligned}
\tag{1.1}
$$

Here, $I = (0,l)$ is a bounded interval in $\mathbb{R}$. $y(x,t)$ denotes the concentrations of $HBrO_2$, $\rho(x,t)$ the concentrations of Ce^{4+}, and $w(x,t)$ the concentrations of Br^- at $x \in I$ and a time $t \in [0,T]$, respectively. $a > 0$, $b > 0$, and $d > 0$ represent the diffusion rate of each species. Finally, δ, ϵ, q, c and γ are positive constants. The term $u(x,t)$ denotes the control fuction at $x \in I$ and a time $t \in [0,T]$ ([1], [9], [11]).

The model (1.1) was introduced by Field and Noyes is the simple mathematical model for describing the complicated mechanism from a global point of view([3], [9]). In [9] and [10], the authors studied for global dynamics of the Field-Noyes model including the global attractor and exponential attractor.

Many authors have been studied the optimal control problem for the reaction diffusion model([2], [4], [5]). In [6], the optimal control problem for the chemotaxis model studied. Ryu([7], [8]) studied the optimal control problem for B-Z reaction model of two variables. In this paper, we show the existence of the global weak solution of (1.1). We also show the existence of the optimal control.

The paper is organized as follows. Section 2 show the existence of the global weak solutions. In Section 3 we show the existence of the optimal control.

Notations: For simplicity, we shall use a universal constant C to denote various constants which are determined in each occurrence in a specific way by $a, b, c, d, \epsilon, \delta, \gamma$, l. In a case when C depends also on some parameter, say θ, it will be denoted by C_θ.

2. Global weak solutions

Let us set three product Hilbert spaces $\mathcal{V} \subset \mathcal{H} = \mathcal{H}' \subset \mathcal{V}'$ as

$$\begin{aligned}\mathcal{V} &= H^1(I) \times H^1(I) \times H^1(I),\\ \mathcal{H} &= L^2(I) \times L^2(I) \times L^2(I),\\ \mathcal{V}' &= (H^1(I))' \times (H^1(I))' \times (H^1(I))'.\end{aligned}$$

Also we set a symmetric bilinear form on $\mathcal{V} \times \mathcal{V} \times \mathcal{V}$:

$$\begin{aligned}a(Y, \widetilde{Y}) = \big(A_1^{1/2} y, A_1^{1/2} \tilde{y}\big)_{L^2(I)} &+ \big(A_2^{1/2} \rho, A_2^{1/2} \tilde{\rho}\big)_{L^2(I)}\\ &+ \big(A_3^{1/2} w, A_3^{1/2} \tilde{w}\big)_{L^2(I)}, \quad Y = \begin{pmatrix} y \\ \rho \\ w \end{pmatrix}, \widetilde{Y} = \begin{pmatrix} \tilde{y} \\ \tilde{\rho} \\ \tilde{w} \end{pmatrix} \in \mathcal{V},\end{aligned}$$

where $A_1 = -a\frac{\partial^2}{\partial x^2} + \epsilon^{-1}$, $A_2 = -b\frac{\partial^2}{\partial x^2} + 1$, and $A_3 = -d\frac{\partial^2}{\partial x^2} + \delta^{-1}q$ with the same domain $\mathcal{D}(A_i) = H_n^2(I) = \{z \in H^2(I); \frac{\partial z}{\partial x}(0) = \frac{\partial z}{\partial x}(l) = 0\}$ $(i = 1, 2, 3)$. Obviously, the form satisfies

$$|a(Y, \widetilde{Y})| \le M\|Y\|_{\mathcal{V}}\|\widetilde{Y}\|_{\mathcal{V}}, \quad Y, \widetilde{Y} \in \mathcal{V}, \tag{a.i}$$

$$a(Y, Y) \ge m\|Y\|_{\mathcal{V}}^2, \quad Y \in \mathcal{V} \tag{a.ii}$$

with some m and $M > 0$. This form then defines a linear isomorphism $A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix}$ from $\mathcal{V}$ to $\mathcal{V}'$, and the part of A in $\mathcal{H}$ is a positive definite self-adjoint operator in $\mathcal{H}$. Let us set the space of initial values as

$$\mathcal{K} = \Big\{ \begin{pmatrix} y_0 \\ \rho_0 \\ w_0 \end{pmatrix} \in \mathcal{H}; 0 \le y_0 \in L^2(I),\ 0 \le \rho_0 \in L^2(I),\ 0 \le w_0 \in L^2(I) \Big\}.$$

We also set the control space $\mathcal{U} = L^2(0, T; \mathcal{H})$ and

$$\mathcal{U}_{ad} = \Big\{ \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \in \mathcal{U}; u \in L^2(0, T; L^2(I)), u \ge 0, \|u\|_{L^2(0,T;L^2(I))} \le C \Big\}.$$

Then (1.1) is formulated as the following problem

$$\begin{aligned}&\frac{dY}{dt} + AY = F(Y) + U, \quad 0 < t \le T,\\ &Y(0) = Y_0\end{aligned} \tag{2.1}$$

in the space $\mathcal{V}'$. Here, Y_0 is defined by $Y_0 = \begin{pmatrix} y_0 \\ \rho_0 \\ w_0 \end{pmatrix}$ and $U = \begin{pmatrix} 0 \\ 0 \\ \delta^{-1}u \end{pmatrix}$.

$F(\cdot): \mathcal{V} \to \mathcal{V}'$ is the mapping

$$F(Y) = \begin{pmatrix} \epsilon^{-1}\big(qw - yw + 2y - y^2\big) \\ y \\ \delta^{-1}\big(- yw + c\rho\big) \end{pmatrix} \tag{2.2}$$

and $F(\cdot)$ satisfies the following conditions:

(f.i) For each $\eta > 0$, there exists an increasing continuous function ϕ_η : $[0,\infty) \to [0,\infty)$ such that

$$\|F(Y)\|_{\mathcal{V}'} \le \eta\|Y\|_{\mathcal{V}} + \phi_\eta(\|Y\|_{\mathcal{H}}), \quad Y \in \mathcal{V}, \text{ a.e. } (0,T);$$

(f.ii) For each $\eta > 0$, there exists an increasing continuous function ψ_η : $[0,\infty) \to [0,\infty)$ such that

$$\begin{aligned} &\|F(\widetilde{Y}) - F(Y)\|_{\mathcal{V}'} \le \eta\|\widetilde{Y} - Y\|_{\mathcal{V}} \\ &+ (\|\widetilde{Y}\|_{\mathcal{V}} + \|Y\|_{\mathcal{V}} + 1)\psi_\eta(\|\widetilde{Y}\|_{\mathcal{H}} + \|Y\|_{\mathcal{H}})\|\widetilde{Y} - Y\|_{\mathcal{H}}, \quad \widetilde{Y}, Y \in \mathcal{V}, \text{ a.e. } (0,T). \end{aligned}$$

Indeed, it is seen as in [7] that

$$\|y^2\|_{(H^1(I))'} \le \eta\|y\|_{H^1(I)} + C_\eta\|y\|^4_{L^2(I)}, \quad y \in H^1(I)$$

and

$$\begin{aligned} \|yw\|_{(H^1(I))'} \le & C\|yw\|_{L^{\frac{3}{2}}(I)} \\ \le & C\|y\|_{L^2(I)}\|w\|^{\frac{3}{4}}_{L^2(I)}\|w\|^{\frac{1}{4}}_{H^1(I)} \\ \le & \eta\|w\|_{H^1(I)} + C_\eta\Big(\|y\|_{L^2(I)}\|w\|^{\frac{3}{4}}_{L^2(I)}\Big)^{\frac{4}{3}} \\ \le & \eta\|w\|_{H^1(I)} + C_\eta\Big(\|y\|^2_{L^2(I)} + \|w\|^3_{L^2(I)}\Big), \quad y, w \in H^1(I). \end{aligned}$$

Hence, the condition (f.i) is fulfilled.

On the other hand, for $\tilde{y}, y, \tilde{w}, w \in H^1(I)$,

$$\begin{aligned} \|\tilde{y}^2 - y^2\|_{(H^1(I))'} &\le C\|\tilde{y}^2 - y^2\|_{L^2(I)} \\ &\le C\big(\|\tilde{y}\|_{L^\infty(I)} + \|y\|_{L^\infty(I)}\big)\|\tilde{y} - y\|_{L^2(I)} \\ &\le C\big(\|\tilde{y}\|_{H^1(I)} + \|y\|_{H^1(I)}\big)\|\tilde{y} - y\|_{L^2(I)} \end{aligned}$$

and

$$\begin{aligned} &\|\tilde{y}\tilde{w} - yw\|_{(H^1(I))'} \\ &\le C\Big(\|(\tilde{y} - y)\tilde{w}\|_{L^2(I)} + \|y(\tilde{w} - w)\|_{L^2(I)}\Big) \\ &\le C\Big(\|\tilde{w}\|_{L^\infty(I)}\|\tilde{y} - y\|_{L^2(I)} + \|y\|_{L^\infty(I)}\|\tilde{w} - w\|_{L^2(I)}\Big) \\ &\le C\big(\|\tilde{w}\|_{H^1(I)} + \|y\|_{H^1(I)}\big)\big(\|\tilde{y} - y\|_{L^2(I)} + \|\tilde{w} - w\|_{L^2(I)}\big). \end{aligned}$$

Hence, the condition (f.ii) is fulfilled.

We then obtain the local existence of the weak solution ([6]).

Theorem 2.1. *Let* (a.i)*,* (a.ii)*,* (f.i)*, and* (f.ii) *be satisfied. Then, for any* $Y_0 \in \mathcal{K}$ *and* $U \in \mathcal{U}_{ad}$*,* (2.1) *has a unique weak solution*

$$Y \in H^1(0,T(Y_0,U);\mathcal{V}') \cap \mathcal{C}([0,T(Y_0,U)];\mathcal{H}) \cap L^2(0,T(Y_0,U);\mathcal{V}).$$

Here, $T(Y_0,U) > 0$ *is determined by* $\|Y_0\|_{\mathcal{H}}$ *and* $\|U\|_{L^2(0,T;\mathcal{H})}$*.*

Theorem 2.2. *For any* $Y_0 \in \mathcal{K}$ *and* $U \in \mathcal{U}_{ad}$*, the weak solution* Y *of* (2.1) *is nonnegative. Therefore* Y *is a weak solution of* (1.1)*.*

Proof. We show nonnegativity of solutions, which is proved by the same method in Yagi([9]). We consider an auxiliary problem

$$\begin{aligned} &\frac{d\overline{Y}}{dt} + A\overline{Y} = \overline{F}(\overline{Y}) + U, \quad 0 < t \le T, \\ &\overline{Y}(0) = Y_0. \end{aligned} \tag{2.3}$$

Here, $\overline{F}(Y) = \begin{pmatrix} \epsilon^{-1}\big(qw - yw + 2y - y^2\big) \\ |y| \\ \delta^{-1}\big(-yw + c\rho\big) \end{pmatrix}$ is modified nonlinear operator to (2.2). Then, we also know that $\overline{Y} = \begin{pmatrix} \bar{y} \\ \bar{\rho} \\ \bar{w} \end{pmatrix} \in H^1(0,\overline{T}(Y_0,U);\mathcal{V}') \cap L^2(0,\overline{T}(Y_0,U);\mathcal{V})$.

Let us verify first that $\bar{\rho} \ge 0$ by the truncation method. Consider $H(\bar{\rho})$ is $\mathcal{C}^{1.1}$ cutoff function for $-\infty < \bar{\rho} < \infty$ given by $H(\bar{\rho}) = \frac{\bar{\rho}^2}{2}$ for $-\infty \le \bar{\rho} < 0$ and $H(\bar{\rho}) = 0$ for $0 \le \bar{\rho} < \infty$. Since $\bar{\rho} \in L^2(0,\overline{T}(Y_0,U);H^1(I))$, we see $H'(\bar{\rho}) \in L^2(0,\overline{T}(Y_0,U);H^1(I))$.

Therefore, if we take $H'(\bar{\rho})$ as the test function for the second equation in (2.3), we obtain

$$\begin{aligned} &\langle \bar{\rho}'(t), H'(\bar{\rho}(t))\rangle_{(H^1(I))',H^1(I)} \\ =&\Big\langle b\frac{\partial^2\bar{\rho}}{\partial x^2} + |\bar{y}| - \bar{\rho}, H'(\bar{\rho}(t))\Big\rangle_{(H^1(I))',H^1(I)} \\ =&b\Big\langle \frac{\partial^2\bar{\rho}}{\partial x^2}, H'(\bar{\rho}(t))\Big\rangle_{(H^1(I))',H^1(I)} + \Big\langle |\bar{y}| - \bar{\rho}, H'(\bar{\rho}(t))\Big\rangle_{(H^1(I))',H^1(I)} \\ =&I_1 + I_2. \end{aligned}$$

Since $I_1 = -b\int_0^l |\frac{\partial H'(\bar{\rho}(t))}{\partial x}|^2 dx$, we see that $I_1 \le 0$. Since $H'(\bar{\rho}) \le 0$, $H'(\bar{\rho})\bar{\rho} \ge 0$, it follows that $I_2 \le 0$. Therefore, if we put

$$\psi(t) = \int_0^l H(\bar{\rho}(t))dx, \quad 0 \le t \le \overline{T}(Y_0,U),$$

then we see

$$\frac{d}{dt}\psi(t) = \langle \bar{\rho}'(t), H'(\bar{\rho}(t))\rangle_{(H^1(I))',H^1(I)} \le 0.$$

Therefore, $\psi(t) \le \psi(0)$ for $0 \le t \le \overline{T}(Y_0,U,)$. Thus, $\psi(0) = 0$ implies $\psi(t) = 0$, that is, $\bar{\rho}(t) \ge 0$ for $0 \le t \le \overline{T}(Y_0,U)$.

Let us verify that $\bar{w} \geq 0$. Similarily, if we use $H(\bar{w})$ and the third equation in (2.3), then we obtain from $u \geq 0$ that

$$\langle \bar{w}'(t), H'(\bar{w}(t))\rangle_{(H^1(I))',H^1(I)} \leq 2\|\bar{y}\|_{L^\infty(I)} \int_0^l H(\bar{w})dx \leq C\|\bar{y}\|_{H^1(I)} \int_0^l H(\bar{w})dx.$$

If we put $\phi(t) = \int_0^l H(\bar{w}(t))dx, 0 \leq t \leq \overline{T}(Y_0, U)$, then it is seen that

$$\frac{d}{dt}\phi(t) = \langle \bar{w}'(t), H'(\bar{w}(t))\rangle_{(H^1(I))',H^1(I)} \leq C\|\bar{y}\|_{H^1(I)}\phi(t).$$

Thus we obtain that

$$\begin{aligned}\phi(t) &\leq \phi(0)exp\big(C\int_0^t \|\bar{y}(s)\|_{H^1(I)}ds\big)\\ &\leq \phi(0)exp\big(C\overline{T}(Y_0,U)\int_0^t \|\bar{y}(s)\|^2_{H^1(I)}ds\big), \quad 0 \leq t \leq \overline{T}(Y_0,U).\end{aligned}$$

Since $\bar{y} \in L^2(0, \overline{T}(Y_0, U); H^1(I))$, we obtain $\phi(t) = 0$, that is, $\bar{w} \geq 0$ for $0 \leq t \leq \overline{T}(Y_0, U)$.

Finally, we show that $\bar{y} \geq 0$. Similarily, we obtain

$$\begin{aligned}\langle \bar{y}'(t), H'(\bar{y}(t))\rangle_{(H^1(I))',H^1(I)} &\leq 2\big(\|\bar{y}\|_{L^\infty(I)} + \|\bar{w}\|_{L^\infty(I)}\big)\int_0^l H(\bar{y})dx\\ &\leq C\big(\|\bar{y}\|_{H^1(I)} + \|\bar{w}\|_{H^1(I)}\big)\int_0^l H(\bar{y})dx.\end{aligned}$$

If we put $\eta(t) = \int_0^l H(\bar{y}(t))dx, 0 \leq t \leq \overline{T}(Y_0, U)$, then it is seen that

$$\frac{d}{dt}\eta(t) = \langle \bar{y}'(t), H'(\bar{y}(t))\rangle_{(H^1(I))',H^1(I)} \leq C\big(\|\bar{y}\|_{H^1(I)} + \|\bar{w}\|_{H^1(I)}\big)\eta(t).$$

Thus we obtain that

$$\begin{aligned}\eta(t) &\leq \eta(0)exp\Big(C\int_0^t \big(\|\bar{y}(s)\|_{H^1(I)} + \|\bar{w}(s)\|_{H^1(I)}\big)ds\Big)\\ &\leq \eta(0)exp\Big(C\overline{T}(Y_0,U)\int_0^t \big(\|\bar{y}(s)\|^2_{H^1(I)} + \|\bar{w}(s)\|^2_{H^1(I)}\big)ds\Big), \quad 0 \leq t \leq \overline{T}(Y_0,U).\end{aligned}$$

Since $\bar{y}, \bar{w} \in L^2(0, \overline{T}(Y_0, U); H^1(I))$, we obtain $\eta(t) = 0$, that is, $\bar{y} \geq 0$ for $0 \leq t \leq \overline{T}(Y_0, U)$.

Therefore, we conclude that $\overline{F}(\overline{Y}) = F(\overline{Y})$. Thus we see that $\overline{Y}$ is a local solution of (2.1). By the uniqueness, we see that $\overline{Y} = Y$ for $0 \leq t \leq \overline{T}(Y_0, U)$. Therefore, $Y(t) \geq 0$ for $0 \leq t \leq \overline{T}(Y_0, U)$. Finally, we can show that $\overline{T}(Y_0, U) = T(Y_0, U)$ by a contradiction as in [9]. □

Theorem 2.3. *For any $Y_0 \in \mathcal{K}$ and $U \in \mathcal{U}_{ad}$,* (2.1) *has a unique global weak solution*

$$0 \leq Y \in H^1(0, T; \mathcal{V}') \cap \mathcal{C}([0, T]; \mathcal{H}) \cap L^2(0, T; \mathcal{V}).$$

Proof. Let $Y = \begin{pmatrix} y \\ \rho \\ w \end{pmatrix}$ be any local weak solution as in Theorem 2.1 on an interval $[0, S]$. Then, if we use the method as in [9], we obtain the following estimates

$$\frac{d}{dt}\int_0^l (y^2 + \xi\rho^2 + w^2)dx + \mu \int_0^l (y^2 + \xi\rho^2 + w^2)dx \\ + \nu \int_0^l \Big(\Big|\frac{\partial y}{\partial x}\Big|^2 + \xi\Big|\frac{\partial \rho}{\partial x}\Big|^2 + \Big|\frac{\partial w}{\partial x}\Big|^2\Big)dx \le C(1 + \int_0^l u^2 dx), \quad (2.4)$$

where $\xi = 4c^2q^{-1}\delta^{-1}$, $\mu = \min\{2\epsilon^{-1}, 1, \delta^{-1}q\}$ and $\nu = \min\{a, b, d\}$.

If we solve the following differential inequality

$$\frac{d}{dt}\int_0^l (y^2 + \xi\rho^2 + w^2)dx + \mu \int_0^l (y^2 + \xi\rho^2 + w^2)dx \le C(1 + \|u\|^2_{L^2(I)}),$$

we have

$$\min\{1, \xi\}\|Y\|^2_{L^\infty(0,S;\mathcal{H})} \le \|y(t)\|^2_{L^2(I)} + \xi\|\rho(t)\|^2_{L^2(I)} + \|w(t)\|^2_{L^2(I)} \\ \le e^{-\mu t}\big(\|y_0\|^2_{L^2(I)} + \xi\|\rho_0\|^2_{L^2(I)} + \|w_0\|^2_{L^2(I)}\big) + C\big(1 + \|u\|^2_{L^2(0,T;L^2(I))}\big). \quad (2.5)$$

If we use (2.4), we obtain

$$k\int_0^t (\|y(s)\|^2_{H^1(I)} + \xi\|\rho(s)\|^2_{H^1(I)} + \|w(s)\|^2_{H^1(I)})ds \\ \le (\|y_0\|^2_{L^2(I)} + \xi\|\rho_0\|^2_{L^2(I)} + \|w_0\|^2_{L^2(I)}) + C\int_0^T (1 + \|u\|^2_{L^2(I)})ds, \quad 0 \le t \le S, \quad (2.6)$$

where $k = \min\{\mu, \nu\}$. Thus, we take $t_1 \in (0, S)$ so that $y(t_1), \rho(t_1), w(t_1) \in L^2(I)$. By (2.5) and (2.6), we see $\|y\|_{L^2(t_1,S;H^1(I))\cap L^\infty(t_1,S;L^2(I))}$, $\|\rho\|_{L^2(t_1,S;H^1(I))\cap L^\infty(t_1,S;L^2(I))}$, and $\|w\|_{L^2(t_1,S;H^1(I))\cap L^\infty(t_1,S;L^2(I))}$ do not depend on S. As a consequence, $\|y\|_{H^1(t_1,S;(H^1(I))')}$, $\|\rho\|_{H^1(t_1,S;(H^1(I))')}$, $\|w\|_{H^1(t_1,S;(H^1(I))')}$, and $\|y\|_{C([t_1,S];L^2(I))}$, $\|\rho\|_{C([t_1,S];L^2(I))}$, $\|w\|_{C([t_1,S];L^2(I))}$ do not depend on S. This shows that y, ρ, w can be extended as a weak solution beyond the S. By the standard argument on the extension of the weak solutions, we can then prove the desired result. □

Moreover, we also obtain the stability result with respect to the control.

Theorem 2.4. *For any $Y_0 \in \mathcal{K}$, let $Y_1 = \begin{pmatrix} y_1 \\ \rho_1 \\ w_1 \end{pmatrix}$ and $Y_2 = \begin{pmatrix} y_2 \\ \rho_2 \\ w_2 \end{pmatrix}$ be the solutions with respect to $U_1 = \begin{pmatrix} 0 \\ 0 \\ \delta^{-1}u_1 \end{pmatrix}, U_2 = \begin{pmatrix} 0 \\ 0 \\ \delta^{-1}u_2 \end{pmatrix} \in \mathcal{U}_{ad}$. Then, we* *have the following estimate*

$$\|Y_1(t) - Y_2(t)\|^2_{L^\infty(0,T;\mathcal{H})} \le C\|u_1(t) - u_2(t)\|^2_{L^2(0,T;L^2(I))} \tag{2.7}$$

for all $t \in [0, T]$.

Proof. Let $\tilde{u} = u_1 - u_2$, $\tilde{y} = y_1 - y_2$, $\tilde{\rho} = \rho_1 - \rho_2$ and $\tilde{w} = w_1 - w_2$. Then $\tilde{y}, \tilde{\rho}$ and $\tilde{w}$ satisfy

$$\begin{aligned}
&\frac{\partial \tilde{y}}{\partial t} - a\frac{\partial^2 \tilde{y}}{\partial x^2} + \frac{1}{\epsilon}\tilde{y} = \frac{1}{\epsilon}\Big(2\tilde{y} + q\tilde{w} - w_1\tilde{y} - y_2\tilde{w} - (y_1 + y_2)\tilde{y}\Big) \quad \text{in } I \times (0,T],\\
&\frac{\partial \tilde{\rho}}{\partial t} - b\frac{\partial^2 \tilde{\rho}}{\partial x^2} + \tilde{\rho} = \tilde{y} \quad \text{in } I \times (0,T],\\
&\frac{\partial \tilde{w}}{\partial t} - d\frac{\partial^2 \tilde{w}}{\partial x^2} + \frac{q}{\delta}\tilde{w} = \frac{1}{\delta}\big(- w_1\tilde{y} - y_2\tilde{w} + c\tilde{\rho} + \tilde{u}\big) \quad \text{in } I \times (0,T],\\
&\frac{\partial \tilde{y}}{\partial x}(0,t) = \frac{\partial \tilde{y}}{\partial x}(l,t) = 0 \quad \text{on } (0,T],\\
&\frac{\partial \tilde{\rho}}{\partial x}(0,t) = \frac{\partial \tilde{\rho}}{\partial x}(l,t) = 0 \quad \text{on } (0,T],\\
&\frac{\partial \tilde{w}}{\partial x}(0,t) = \frac{\partial \tilde{w}}{\partial x}(l,t) = 0 \quad \text{on } (0,T],\\
&\tilde{y}(x,0) = 0, \quad \tilde{\rho}(x,0) = 0, \quad \tilde{w}(x,0) = 0 \quad \text{in } I.
\end{aligned} \tag{2.8}$$

Taking the scalar product with $\tilde{y}$ to the first equation of (2.8), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\tilde{y}\|^2_{L^2(I))} + \frac{m_1}{2}\|\tilde{y}\|^2_{H^1(I)} &\\
\le C\big(\|y_1\|^2_{H^1(I)} + \|y_2\|^2_{H^1(I)} + \|w_1\|^2_{H^1(I)} + 1\big)&\\
\times \big(\|\tilde{y}\|^2_{L^2(I))} + \|\tilde{w}\|^2_{L^2(I)}\big),&
\end{aligned} \tag{2.9}$$

where $m_1 = \min\{a, \epsilon^{-1}\}$. Taking the scalar product with $\tilde{\rho}$ to the second equation of (2.8), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\rho}\|^2_{L^2(I))} + \frac{m_2}{2}\|\tilde{\rho}\|^2_{H^1(I)} \le \|\tilde{y}\|^2_{L^2(I)}, \tag{2.10}$$

where $m_2 = \min\{b, 1\}$. Finally, we take the scalar product with $\tilde{w}$ to the third equation of (2.8), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\tilde{w}\|^2_{L^2(I))} + \frac{m_3}{2}\|\tilde{w}\|^2_{H^1(I)} &\\
\le C\big(\|w_1\|^2_{H^1(I)} + \|y_2\|^2_{H^1(I)} + 1\big)&\\
\times \big(\|\tilde{y}\|^2_{L^2(I))} + \|\tilde{w}\|^2_{L^2(I))} + \|\tilde{\rho}\|^2_{L^2(I)}\big) + C\|\tilde{u}\|^2_{L^2(I)},&
\end{aligned} \tag{2.11}$$

where $m_3 = \min\{d, q\delta^{-1}\}$. From (2.9), (2.10) and (2.11), we have

$$\frac{d}{dt}\|\widetilde{Y}\|^2_{\mathcal{H}} + m\|\widetilde{Y}\|^2_{\mathcal{V}} \le C\big(\|Y_1\|^2_{\mathcal{V}} + \|Y_2\|^2_{\mathcal{V}} + 1\big)\|\widetilde{Y}\|^2_{\mathcal{H}} + C\|\tilde{u}\|^2_{L^2(I)},$$

where $m = \min\{m_1, m_2, m_3\}$. Using Gronwall's inequality, we obtain that

$$\|\widetilde{Y}(t)\|_{\mathcal{H}}^2 + m\int_0^t \|\widetilde{Y}(t)\|_{\mathcal{V}}^2 ds$$
$$\leq C\|\tilde{u}\|_{L^2(0,T;L^2(I))}^2 e^{\int_0^T C(\|Y_1(s)\|_{\mathcal{V}}^2 + \|Y_2(s)\|_{\mathcal{V}}^2 + 1)ds} \leq C\|u_1 - u_2\|_{L^2(0,T;L^2(I)}^2$$

for all $t \in [0, T]$. Hence, we otain (2.7). □

3. Existence of the optimal control

The problem $(\mathbf{P})$ is obviously formulated as follows:

$$(\overline{\mathbf{P}}) \qquad \text{minimize } J(U),$$

where

$$J(U) = \int_0^T \|Y(U) - Y_d\|_{\mathcal{V}}^2 dt + \gamma \int_0^T \|U\|_{\mathcal{H}}^2 dt, \quad U \in \mathcal{U}_{ad}.$$

Here, $Y_d = \begin{pmatrix} y_d \\ \rho_d \\ w_d \end{pmatrix}$ is a fixed element of $L^2(0,T;\mathcal{V})$ with y_d, ρ_d , $w_d \in L^2(0,T;H^1(I))$. γ is a positive constant.

Theorem 3.1. *There exists an optimal control* $\overline{U} \in \mathcal{U}_{ad}$ *for* $(\overline{\mathbf{P}})$ *such that* $J(\overline{U}) = \min_{U\in\mathcal{U}_{ad}} J(U)$.

Proof. Let $\{U_n\} \subset \mathcal{U}_{ad}$ be a minimizing sequence such that

$$\lim_{n\to\infty} J(U_n) = \min_{U\in\mathcal{U}_{ad}} J(U).$$

Since $\{U_n\}$ is bounded in $L^2(0,T;\mathcal{H})$, we can assume that $U_n \to \overline{U}$ weakly in $L^2(0,T;\mathcal{H})$. For simplicity, we will write Y_n instead of the solution $Y(U_n)$ of (2.1) corresponding to U_n. Using the boundedness of Y_n, we infer that $Y_n = \begin{pmatrix} y_n \\ \rho_n \\ w_n \end{pmatrix} \to \overline{Y} = \begin{pmatrix} \bar{y} \\ \bar{\rho} \\ \bar{w} \end{pmatrix}$ weakly in $L^2(0,T;\mathcal{V}) \cap H^1(0,T;\mathcal{V}')$. Since $\mathcal{V}$ is compactly embedded in $\mathcal{H}$, we have

$$Y_n \to \overline{Y} \text{ strongly in } L^2(0,T;\mathcal{H}). \tag{3.1}$$

Now, we will show that $\overline{Y}$ is a solution to (2.1) with the control $\overline{U}$. For any $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \in L^2(0,T;\mathcal{V})$, we consider

$$\int_0^T \langle Y_n'(t), \Phi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt + \int_0^T \langle AY_n(t), \Phi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt$$
$$= \int_0^T \langle F(Y_n(t)), \Phi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt + \int_0^T \langle U_n(t), \Phi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt.$$

We first observed that for any $\phi_1 \in L^2([0,T];H^1(I))$,

$$\int_0^T \langle y_n^2 - \bar{y}^2, \phi_1\rangle_{(H^1(I))',H^1(I)} dt$$
$$\leq C\Big(\|y_n\|_{L^\infty(0,T;L^2(I))} + \|y\|_{L^\infty(0,T;L^2(I))}\Big)\|y_n - y\|_{L^2(0,T;L^2(I))}\|\phi_1\|_{L^2(0,T;H^1(I))}.$$

From (3.1), we have

$$y_n^2 \to \bar{y}^2 \text{ weakly in } L^2(0,T;(H^1(I))').$$

Since

$$\begin{aligned}
&\langle y_n w_n - \bar{y}\bar{w}, \phi_1\rangle_{(H^1(I))',H^1(I)} \\
&\leq \Big(\int_0^l |y_n w_n - \bar{y}\bar{w}| dx\Big)\|\phi_1\|_{L^\infty(I)} \\
&\leq C\big(\|w_n\|_{L^2(I)} + \|\bar{y}\|_{L^2(I)}\big)\big(\|y_n - \bar{y}\|_{L^2(I)} + \|w_n - \bar{w}\|_{L^2(I)}\big)\|\phi_1\|_{H^1(I)}
\end{aligned}$$

we obtain

$$\begin{aligned}
&\int_0^T \langle y_n w_n - \bar{y}\bar{w}, \phi_1\rangle_{(H^1(I))',H^1(I)} dt \\
\leq & C\big(\|w_n\|_{L^\infty(0,T;L^2(I))} + \|\bar{y}\|_{L^\infty(0,T;L^2(I))}\big) \\
&\times \big(\|y_n - \bar{y}\|_{L^2(0,T;L^2(I))} + \|w_n - \bar{w}\|_{L^2(0,T;L^2(I))}\big)\|\phi_1\|_{L^2(0,T;H^1(I))}.
\end{aligned}$$

From (3.1), we have

$$y_n w_n \to \bar{y}\bar{w} \text{ weakly in } L^2(0,T;(H^1(I))').$$

For any $\Phi = \begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} \in L^2(0,T;\mathcal{V})$, we obtain

$$\begin{aligned}
\int_0^T \langle \overline{Y}'(t), \Phi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt &+ \int_0^T \langle A\overline{Y}(t), \Phi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt \\
&= \int_0^T \langle F(\overline{Y}(t)), \Phi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt + \int_0^T \langle \overline{U}(t), \Phi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt.
\end{aligned}$$

This then shows that $\overline{Y}(t)$ satisfies the equation of (2.1) for almost all $t \in (0,T)$. Therefore, by the uniqueness of the solution of (2.1), $\overline{Y} = Y(\overline{U})$.

Since $Y_n - Y_d$ is weakly convergent to $\overline{Y} - Y_d$ in $L^2(0,T;\mathcal{V})$, we have:

$$\min_{U\in\mathcal{U}_{ad}} J(U) \leq J(\overline{U}) \leq \liminf_{n\to\infty} J(U_n) = \min_{U\in\mathcal{U}_{ad}} J(U).$$

Hence, $J(\overline{U}) = \min_{U\in\mathcal{U}_{ad}} J(U)$. □

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