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GLOBAL ATTRACTOR FOR SOME BEAM EQUATION WITH NONLINEAR SOURCE AND DAMPING TERMS

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Abstract. Global attractor is a basic concept to study the long-time behavior of solutions of the various equations. This paper is investigated with the existence of a global attractor for the beam equation

$$u_{tt} + \Delta^2 u - \nabla \cdot \{\sigma(|\nabla u|^2)\nabla u\} + f(u) + a(x)g(u_t) = h,$$

using multipliers technique and Nakao's Lemma.

1. Introduction

In this paper, we consider the existence of global attractor for the following beam equation with nonlinear damping and critical nonlinearity:

$$u_{tt} + \Delta^2 u - \nabla \cdot \{\sigma(|\nabla u|^2)\nabla u\} + f(u) + a(x)g(u_t) = h \text{ in } \Omega \times \mathbb{R}^+ \quad (1)$$

$$u = 0 \text{ on } \partial\Omega \times \mathbb{R}^+, \quad (2)$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \text{ in } \Omega, \quad (3)$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, $\mathbb{R}^+ = [0,\infty)$, $\sigma(v)$ is a function like $\sigma(v) = 1/\sqrt{1+v}$ and $a(x)$ is a nonnegative smooth function on $\bar{\Omega}$.

Many author studied global attractor for the various equations [6, 7, 16]. The existence of attractor for wave equation with critical exponent was obtained in [8, 9, 18, 19]. Nakao [12] dealt with the global attractor of the quasi-linear wave equation with a strong dissipation. In [17], the authors showed the existence of global attractor for plate equation with nonlinear damping. In [13, 14], the authors dealt with the global attractor for nonlinear parabolic equation of m-Laplace type in $\mathbb{R}^N$. Recently, the research of beam equations have attracted considerable attention(see [1, 2, 4, 5] and reference there in). For instance Ma and Narciso[10] proved the global attractors for a model of extensible beam with nonlinear damping and source terms.

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Global attractor is a basic concept to study the long-time behavior of solutions for nonlinear evolution equations with various dissipations [3, 7, 15]. Motivated these papers, we will consider the global attractor of the beam equation $(1)-(3)$ with nonlinear damping and critical nonlinearity. To our knowledge, attractors for the problem $(1)-(3)$ with nonlinear damping and critical nonlinearity were not previously considered. This paper organized as follows. In section 2, we introduce the assumption and some results about the theory of attractors and the existence of solution to the problems $(1)-(3)$ and we state our main results. Section 3 is devoted to the proof of our main results.

2. Preliminaries and the Main Result

Throughout this paper, $E(t)$ is the energy of the solution at time t to problem $(1)-(3)$ defined by

$$E(t)=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\Delta u(t)\|_2^2+\frac{1}{2}\int_\Omega F(|\nabla u|^2)dx+\int_\Omega \tilde{f}(u(t))dx-\int_\Omega hu(t)dx,$$

where $F(s)=\int_0^s\sigma(\eta)d\eta$ and $\tilde{f}(z)=\int_0^z f(s)ds$.

Let us begin with precise hypotheses on the functions F, f and g. Suppose that

$$\sup_{0\le t\le T}\|\nabla u(t)\|_\infty^2<L.$$

(**H1**) $F(|\nabla u|^2)\ge k_0|\nabla u|^2$ if $|\nabla u|^2<L$, where $k_0=\sup_t\min_{0\le\theta\le|\nabla u|^2}\sigma(\theta)$.
(**H2**) There exists a constant $\tilde{k}>0$ such that $\sigma(|\nabla u|^2)|\nabla u|^2\ge\tilde{k}F(|\nabla u|^2)$ if $|\nabla u|^2<L$, where $\tilde{k}=\sigma(|\nabla u|^2)/\sup\sigma(\theta), 0<\theta<|\nabla u|^2$.
Since $\sigma(v)$ belongs to $C^{m+2}([0,L])$ for some $L>0$, there exist $M=\sup_t\max_{0\le\theta\le|\nabla u|^2}\sigma(\theta)$.
By (**H1**) and (**H2**), we get the following inequality

$$\tilde{k}F(|\nabla u|^2)\le\sigma(|\nabla u|^2)|\nabla u|^2\le M|\nabla u|^2\le\frac{M}{k_0}F(|\nabla u|^2). \tag{4}$$

(**H3**) The function $f:\mathbb{R}\to\mathbb{R}$ belongs to C^1 and $f(0)=0$. Also, there exist constants $k_0,k_1,A_0,A_1>0$ such that

(i) $|f(u)-f(v)|\le k_1(1+|u|^\rho+|v|^\rho)|u-v|,\quad\forall u,v\in\mathbb{R}$.

(ii) $|f'(u)|\le k_0(1+|u|^\rho),\quad\forall u\in\mathbb{R}$, where $0<\rho\le\frac{2}{N-2}$ if $N\ge3$ and $\rho>0$ if $N=1,2$.

(iii) $-A_0\le\tilde{f}(u)\le\frac{1}{2}f(u)u+A_1,\quad\forall u\in\mathbb{R}$, where $\tilde{f}(z)=\int_0^z f(s)ds$.

(**H4**) $a(x)\in L^\infty(\Omega)$, $a(x)\ge\alpha_0>0$ in Ω.
(**H5**) The function $g:\mathbb{R}\to\mathbb{R}$ belongs to C^1 and $g(0)=0$. There exist constants $k_2,k_3>0$ such that

(i) $|g(u)-g(v)|\le k_2(1+|u|^r+|v|^r)|u-v|,\quad\forall u,v\in\mathbb{R}$.

(ii) $(g(u)-g(v))(u-v)\ge k_3|u-v|^{r+2},\quad\forall u,v\in\mathbb{R}$,

where $0<r\le\frac{2}{N-2}$ if $N\ge3$ and $r>0$ if $N=1,2$.

As basic space we use

$$\mathcal{H}=H_0^2(\Omega)\times L^2(\Omega)$$

equipped with the norm

$$\|(u,v)\|_{\mathcal{H}}^2 = \|\triangle u\|_2^2 + \|v\|_2^2,$$

where $\|\cdot\|_p$ denote L^p norm.

We also need the following lemma of Nakao[11].

Lemma 2.1. *Let $\phi(t)$ be a nonnegative function on $\mathbb{R}^+$ satisfying*

$$\sup_{t\le s\le t+T} \phi(s)^{1+\gamma} \le C\{\phi(t) - \phi(t+T)\}$$

with $T > 0, \gamma > 0$ and C is some positive constant. Then $\phi(t)$ has the decay property

$$\phi(t) \le \left\{\phi(0)^{-\gamma} + \frac{\gamma}{C}(t-T)\right\}^{-\frac{1}{\gamma}}$$

for $t \ge T$.

Definition 1. Let $S(t)$ be a C_0-semigroup defined on a complete metric space H. Then $S(t)$ has a global attractor in H if and only if it satisfied following two conditions.

(i) A bounded set $\mathbf{B} \subset H$ is an **absorbing set** for $S(t)$: For any bounded set $\mathbf{B} \subset H$ is an absorbing set for $S(t)$ if for any bounded set $B \subset H$, there exists $t_\tau = t_\tau(B) \ge 0$ such that

$$S(t)B \subset \mathbf{B}, \quad \forall t \ge t_\tau,$$

which defines $(H, S(t))$ as a dissipative dynamical system.

(ii) $S(t)$ is **asymptotically smooth** in H : If for any bounded positive invariant set $B \subset H$, there exists a compact set $K \subset \bar{B}$ such that

$$\text{dist } (S(t)B, K) \to 0 \;\text{ as }\; t \to \infty.$$

Theorem 2.2. [10] *Let $S(t)$ be a dissipative C_0-semigroup defined on a metric space H. Then $S(t)$ has a compact global attractor in H if and only if it is asymptotically smooth in H.*

Lemma 2.3. [10] *Assume that for any bounded positive invariant set $B \subset H$, and for any $\varepsilon > 0$, there exists $T = T(\varepsilon, B)$ such that*

$$d(S(T)\alpha, S(T)\beta) \le \varepsilon + \varphi_T(\alpha,\beta), \quad \forall \alpha,\beta \in B.$$

Here $\varphi_T : H \times H \to \mathbb{R}$ satisfies $\liminf_{m\to\infty}\liminf_{n\to\infty}\varphi_T(z_n, z_m) = 0$, where $\{z_n\} \subset B$ is any sequence. Then $S(t)$ is asymptotically smooth.

The existence and the regularity of solutions u to the problem $(1)-(3)$ are given by the following standard well-known result (see [19]):

Theorem 2.4. *Assume that $(H3)-(H4)$ hold and $h \in L^2(\Omega)$. Also $a(\cdot) \in C^{m-1}(\overline{\Omega})$ and $(u_0, u_1) \in \mathcal{H}$. Then the problem $(1)-(3)$ admits a unique solution $u(t) \in C(\mathbb{R}^+; H_0^2(\Omega)) \cap C^1(\mathbb{R}^+; L^2(\Omega))$ and (u, u_t) depends continuously on initial data in $\mathcal{H}$.*

The main result of this paper reads as follows:

Theorem 2.5. *Under the hypotheses* $(H1)-(H3)$, *the associate semigroup* $S(t)$ *of problem* $(1)-(3)$ *has a global attractor* $\mathbb{A}$ *in* $\mathcal{H} = H_0^2(\Omega) \times L^2(\Omega)$. *Further, there exists a constant* $C_h > 0$ *such that*

$$\mathbb{A} \subset \mathbf{B} = \{(u,v) \in \mathcal{H} | \|\Delta u\|_2^2 + \|v\|_2^2 \leq C_h(\|h\|_2^2 + A_0 + A_1)\}$$

and for each bounded subset $B \subset \mathcal{H}$, *there exists a constant* $C(B)$ *such that*

$$dist(S(t)B, \mathbf{B}) \leq C(B)(1+t)^{-\frac{1}{r}}.$$

3. Proof of Theorem 2.5

(I) : Absorbing set

Lemma 3.1. *Under the hypotheses* $(H1)-(H3)$, $S(t)$ *has an absorbing set* $\mathbf{B} \subset \mathcal{H}$.

Proof. We note that $B \subset \mathcal{H}$ is an arbitrary fixed bounded set and $(u(t), u_t(t)) = S(t)(u_0, u_1), (u_0, u_1) \in B$ is solutions of problem $(1)-(3)$. We set the modified energy functional

$$E_1(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}\int_\Omega F(|\nabla u|^2)dx + \int_\Omega \tilde{f}(u(t))dx - \int_\Omega hu(t)dx$$
$$+A_0|\Omega| + \frac{1}{\lambda_1}\|h\|_2^2 = E(t) + A_0|\Omega| + \frac{1}{\lambda_1}\|h\|_2^2,$$

where $\lambda_1 > 0$ is the first eigenvalue of the bi-harmonic operator Δ^2 in $H_0^2(\Omega)$.

We claim that

$$\mathbf{B} = \{(u,v) \in \mathcal{H} | \|\Delta u\|_2^2 + \|v\|_2^2 \leq C_h(\|h\|_2^2 + A_0 + A_1)\}$$

is an absorbing set for $S(t)$.

The proof is given by similar argument as in [10]. We will sketch it briefly. Using $(H3)$ and the fact $\|u\|_2^2 \leq \lambda_1^{-1}\|\Delta u\|_2^2, \forall u \in H_0^2(\Omega)$, we get

$$E_1(t) \geq \frac{1}{4}(\|\Delta u\|_2^2 + \|u_t\|_2^2) + \frac{1}{2}\int_\Omega F(|\nabla u|^2)dx \geq \frac{1}{4}(\|\Delta u\|_2^2 + \|u_t\|_2^2). \tag{5}$$

Now we shall derive the inequality

$$E_1(t) \leq C_1(\|h\|_2^2 + A_0 + A_1), \forall t > t_B > 0.$$

By multiplying the equation (1) by u_t and integrating over Ω, we obtain

$$\frac{d}{dt}E(t) = -\int_\Omega a(x)g(u_t)u_t dx.$$

Therefore,

$$E(t) - E(t+1) = \int_t^{t+1}\int_\Omega a(x)g(u_t)u_t dxds \geq 0.$$

Setting

$$D(t)^2 = E_1(t) - E_1(t+1) = E(t) - E(t+1).$$

Using condition (**H4**) and Höder inequality we get

$$\int_t^{t+1} \|u_t\|_2^2 ds = \int_t^{t+1} \int_\Omega |u_t|^2 dxds \le \frac{1}{k_3}|\Omega|^{\frac{r}{r+2}} D(t)^{\frac{4}{r+2}}.$$

Then by Mean Value Theorem for integrals there exist two numbers $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u_t(t_i)\|_2^2 \le \frac{4}{k_3}|\Omega|^{\frac{r}{r+2}} D(t)^{\frac{4}{r+2}}, \quad i = 1, 2.$$

By multiplying the equation (1) by u and integration over Ω, we obtain that

$$\|\Delta u\|_2^2 = -\sigma(|\nabla u|^2)\|\nabla u\|_2^2 - \int_\Omega f(u)udx + \|u_t\|_2^2 - \frac{d}{dt}\int_\Omega u_t u dx - \int_\Omega a(x)g(u_t)udx + \int_\Omega hudx. \tag{6}$$

By definition of $E_1(t)$ and (6)

$$E_1(t) = \|u_t\|_2^2 + \frac{1}{2}\Big(\int_\Omega F(|\nabla u|^2)dx - \sigma(|\nabla u|^2)\|\nabla u\|_2^2\Big) + \int_\Omega (\tilde{f}(u) - \frac{1}{2}f(u)u)dx - \frac{1}{2}\frac{d}{dt}\int_\Omega u_t u dx - \frac{1}{2}\int_\Omega a(x)g(u_t)udx - \frac{1}{2}\int_\Omega hudx + A_0|\Omega| + \frac{1}{\lambda_1}\|h\|_2^2. \tag{7}$$

Using (4), we obtain

$$\int_\Omega F(|\nabla u|^2)dx - \sigma(|\nabla u|^2)\|\nabla u\|_2^2 \le \int_\Omega (1-\tilde{k})F(|\nabla u|^2)dx \le 2(1-\tilde{k})E_1 \tag{8}$$

From (7) and (8), we have

$$\tilde{k}\int_{t_1}^{t_2} E_1(s)ds \le \int_{t_1}^{t_2} \|u_t\|_2^2 ds - \frac{1}{2}\Big(\int_\Omega u_t(t_2)u(t_2)dx - \int_\Omega u_t(t_1)u(t_1)dx\Big) - \frac{1}{2}\int_{t_1}^{t_2}\int_\Omega hudxds + \frac{1}{2}\int_{t_1}^{t_2}\int_\Omega |a(x)g(u_t)u|dxds + \frac{1}{\lambda_1}\|h\|_2^2 + (A_0 + A_1)|\Omega|.$$

Using Mean value theorem, Young's inequality and definition of $D(t)^2$ and after some calculation, we derive

$$E_1(t) \le C_B D(t)^{\frac{4}{r+2}} + C(\|h\|_2^2 + A_0 + A_1).$$

This inequality implies that

$$E_1(t)^{1+\frac{r}{2}} \le C_B(E_1(t) - E_1(t+1)) + C(\|h\|_2^2 + A_0 + A_1)^{\frac{r+2}{2}}.$$

Applying Lemma 2.1 to above inequality and $t \to \infty$, then

$$E_1(t) \le C(\|h\|_2^2 + A_0 + A_1), \quad \forall t > t_B, \tag{9}$$

where $t_B > 0$ and depending on B. Combining (5) and (9), we completed the proof of Lemma 3.1. □

(II): Asymptotic Smoothness

Now we prove that $S(t)$ is asymptotically smooth. To prove the asymptotic smoothness, we will use Lemma 2.3.

Lemma 3.2. *Under the hypotheses* $(H1)-(H3)$*,* $S(t)$ *is asymptotically smooth in* $\mathcal{H}$*.*

Proof. Let u, v be two solutions of problem (1) with the initial data $(u_0, u_1), (v_0, v_1) \in B$, respectively. Here $B \subset \mathcal{H}$ is a bounded positive invariant set for $S(t)$. Putting $w = u - v$, we have

$$\begin{aligned} & w_{tt} + \Delta^2 w + a(x)(g(u_t) - g(v_t)) \\ & = \nabla \cdot \{\sigma(|\nabla u|^2)\nabla u\} - \nabla \cdot \{\sigma(|\nabla v|^2)\nabla v\} - (f(u) - f(v)) \qquad (10) \\ & \quad \text{in} \quad \Omega \times [0, \infty), \\ & \frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \times [0, \infty), \qquad (11) \\ & w(x, 0) = u_0 - v_0, \quad w_t(x, 0) = u_1 - v_1 \quad \text{in} \quad \Omega. \qquad (12) \end{aligned}$$

Now we define the functional $E_w(t)$ such as

$$E_w(t) = \|w_t(t)\|_2^2 + \|\Delta w(t)\|_2^2 + \sigma(|\nabla u|^2)\|\nabla w\|_2^2.$$

We will claim that

$$E_w(t) \le C_B(1+t)^{-\frac{2}{r}} + C_T\left(\sup_{0\le\alpha\le T}\int_\alpha^{\alpha+1}\|\nabla w\|_2 ds\right)^{\frac{2}{r+2}}, \quad 0 \le t \le T. \qquad (13)$$

If it is done, we will apply Lemma 2.3. By (13) and definition of $E_w(t)$, there exist constants $\tilde{C}_B, \tilde{C}_T$ such that

$$\|w(t), w_t(t)\|_{\mathcal{H}} \le \tilde{C}_B(1+t)^{-\frac{1}{r}} + \tilde{C}_T \sup_{0\le\alpha\le T}\left(\int_\alpha^{\alpha+1}\|\nabla w(s)\|_2 ds\right)^{\frac{1}{r+2}}, \quad 0 \le t \le T. \qquad (14)$$

Given $\varepsilon > 0$, we fix a sufficiently large T so that,

$$\tilde{C}_B(1+T)^{-\frac{1}{r}} < \varepsilon.$$

Then we define $\psi_T : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ by

$$\psi_T((u_0, u_1), (v_0, v_1)) = \tilde{C}_T \sup_{0\le\alpha\le T}\left(\int_\alpha^{\alpha+1}\|\nabla w(s)\|_2 ds\right)^{\frac{1}{r+2}}.$$

From (14) and fixed a sufficiently large T, we obtain

$$\|S(T)(u_0, u_1) - S(T)(v_0, v_1)\|_{\mathcal{H}} \le \varepsilon + \psi_T((u_0, u_1), (v_0, v_1)),$$

for all $(u_0, u_1), (v_0, v_1) \in B$.

Since $\{u^n\}$ is bounded in $C([0, \infty); H_0^2(\Omega)) \cap C^1([0, \infty); L^2(\Omega))$ and $H_0^2(\Omega) \hookrightarrow H_0^1(\Omega)$ compactly, there exists a subsequence $\{u^{n_i}\}$ which converges strongly in $C([0, T+1]; H_0^1(\Omega))$. Then

$$\lim_{i\to\infty}\lim_{j\to\infty}\psi((u_0^{n_i}, u_1^{n_i}), (u_0^{n_j}, u_1^{n_j})) = 0.$$

Therefore, by Lemma 2.3, $S(t)$ is asymptotically smooth.

From now on we claim (13). Indeed the proof is similar to Lemma 3.4 in [10]. So we will sketch it briefly. By multiplying the equation (10) – (12) by w_t and integrating over Ω, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}E_w(t) + \int_\Omega a(x)(g(u_t) - g(v_t))w_t dx \\
&= \sigma'(|\nabla u|^2)\nabla u \nabla u_t \|\nabla w\|_2^2 - (\sigma(|\nabla u|^2) - \sigma(|\nabla v|^2))\nabla v \nabla w_t \\
&\quad - \int_\Omega (f(u) - f(v))w_t dx \\
&= -\sigma'(|\nabla u|^2)\|\nabla w\|_2^2 \int_\Omega \Delta u u_t dx + (\sigma(|\nabla u|^2) - \sigma(|\nabla v|^2)) \int_\Omega \Delta v w_t dx \\
&\quad - \int_\Omega (f(u) - f(v))w_t dx \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Now we will estimate I_1, I_2, I_3. It is obvious that

$$|I_1| \le C_1\|\nabla w(t)\|_2^2,$$

here we use continuity of σ'.

$$|I_2| \le C_2\|\nabla w\|_2^{\frac{r+2}{r+1}} + \frac{\alpha k_3}{4}\|w_t\|_{r+2}^{r+2},$$

here we use the fact $\sigma(|\nabla u|^2) - \sigma(|\nabla v|^2) \le \sigma'(\sup\{\nabla u\|_2^2, \|\nabla v\|_2^2\})\|\nabla w\|(\|\nabla u\| + \|\nabla v\|)$. Using Hölder inequality with $\frac{\rho}{2(\rho+1)} + \frac{1}{2(\rho+1)} + \frac{1}{2} = 1$, we obtain that

$$\begin{aligned}
|I_3| &\le k_1\left(\int_\Omega (1 + |u|^\rho + |v|^\rho)^{\frac{2(\rho+1)}{\rho}} dx\right)^{\frac{\rho}{2(\rho+1)}} \|w\|_{2(\rho+1)}\|w_t\|_2 \\
&\le C_3\|\nabla w\|_2\|w_t\|_{r+2} \le C_4\|\nabla w\|_2^{\frac{r+2}{r+1}} + \frac{\alpha k_3}{4}\|w_t\|_{r+2}^{r+2}.
\end{aligned}$$

On the other hand,

$$\int_\Omega a(x)(g(u_t) - g(v_t))(u_t - v_t)dx \ge \alpha k_3\|w_t\|_{r+2}^{r+2}.$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}E_w(t) + \frac{\alpha k}{2}\|w\|_{r+2}^{r+2} \le C\|\nabla w\|_2(\|\nabla w\|_2 + \|\nabla w\|_2^{\frac{1}{r+1}}).$$

So, we derive that

$$\int_t^{t+1} \|w_t\|_{r+2}^{r+2} ds \le E_w(t) - E_w(t+1) + C_5\int_t^{t+1} \|\nabla w(s)\|_2 ds \equiv G(t)^2. \quad (15)$$

For all $t_1 \in [t, t + \frac{1}{4}]$, $t_2 \in [t + \frac{3}{4}, t + 1]$, there exists

$$\|w_t(t_i)\|_2^2 \le C_6 G(t)^{\frac{4}{r+2}}.$$

Multiplying the equation (10) – (12) by w and integrating over Ω, then

$$\frac{d}{dt}\int_\Omega w_t w dx - \|w_t\|_2^2\|\Delta w(t)\|_2^2 + \int_\Omega a(x)(g(u_t)-g(v_t))wdx$$
$$= -\sigma(|\nabla u|^2)\|\nabla w\|_2^2 + \int_\Omega \Big(\sigma(|\nabla u|^2)-\sigma(|\nabla v|^2)\Big)\Delta v w dx$$
$$-\int_\Omega \Big(f(u(t))-f(v(t))\Big)wdx. \tag{16}$$

Integrating (16) from t_1 to t_2,

$$\int_{t_1}^{t_2}\Big(\|\Delta w\|_2^2+\sigma(|\nabla u|^2)\|\nabla w\|_2^2\Big)ds$$
$$\le \Big|\int_{t_1}^{t_2}\frac{d}{dt}\int_\Omega w_t(s)w(s)dxds\Big| + \Big|\int_{t_1}^{t_2}\|w_t(s)\|_2^2 ds\Big|$$
$$+\Big|\int_{t_1}^{t_2}\Big((\sigma(|\nabla u|^2)-\sigma(|\nabla v|^2))\int_\Omega \Delta v(s)w(s)dx\Big)ds\Big|$$
$$+\Big|\int_{t_1}^{t_2}\int_\Omega (f(u)-f(v))wdxds\Big| + \Big|\int_{t_1}^{t_2}\int_\Omega a(x)(g(u_t)-g(v_t))wdxds\Big|$$
$$\le C_6 G(t)^{\frac{4}{r+2}} + \frac{1}{4}\sup_{t\le\sigma\le t+1}E_w(\sigma) + C_7\int_t^{t+1}\|\nabla w(s)\|_2 ds. \tag{17}$$

Here we use the facts

$$\int_{t_1}^{t_2}\int_\Omega a(x)(g(u_t)-g(v_t))wdx \le C_6 G(t)^{\frac{4}{r+2}} + \frac{1}{8}\sup_{t\le\alpha\le t+1}E_w(\alpha),$$
$$\int_{t_1}^{t_2}\int_\Omega (f(u)-f(v))wdxds \le C_7\int_{t_1}^{t_2}\|\nabla w(s)\|_2 ds,$$

and

$$\int_{t_1}^{t_2}\Big\{\Big(\sigma(|\nabla u|^2)-\sigma(|\nabla v|^2)\Big)\int_\Omega \Delta v w dx\Big\}ds \le C_7\int_{t_1}^{t_2}\|\nabla w\|_2 ds.$$

Then by definition $E_w(t)$ and (17), we get

$$\int_{t_1}^{t_2}E_w(s)ds \le C_6 G(t)^{\frac{4}{r+2}} + \frac{1}{4}\sup_{t\le\sigma\le t+1}E_w(\sigma) + C_7\int_t^{t+1}\|\nabla w(s)\|_2 ds.$$

By Mean Value Theorem, there exists $t^* \in [t_1, t_2]$ such that

$$E_w(t^*) \le C_6 G(t)^{\frac{4}{r+2}} + \frac{1}{2}\sup_{t\le\sigma\le t+1}E_w(\sigma) + C_7\int_t^{t+1}\|\nabla w(s)\|_2 ds. \tag{18}$$

Also, using (15) we can derive that

$$\sup_{t\le\alpha\le t+1}E_w(\alpha) \le E_w(t^*) + G(t)^2 + 2C_5\int_t^{t+1}\|\nabla w(s)\|ds.$$

Using this fact, property of $G(t)$ and (18) then we obtain

$$\sup_{t\leq\alpha\leq t+1} E_w(\alpha)^{1+\frac{r}{2}} \leq C_8(E_w(t)-E_w(t+1)) + C_9 \sup_{0\leq\alpha\leq T}\int_\alpha^{\alpha+1} \|\nabla w(s)\|ds.$$

By applying Lemma 2.1 we can derive (13). So we prove that $S(t)$ is asymptotically smooth.

□

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