East Asian Math. J.
Vol. 32 (2016), No. 3, pp. 355–364
http://dx.doi.org/10.7858/eamj.2016.026

ENERGY DECAY RATE FOR THE KELVIN-VOIGT TYPE WAVE EQUATION WITH BALAKRISHNAN-TAYLOR DAMPING AND ACOUSTIC BOUNDARY

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Abstract. In this paper, we study exponential stabilization of the vibrations of the Kelvin-Voigt type wave equation with Balakrishnan-Taylor damping and acoustic boundary in a bounded domain in R^n. To stabilize the systems, we incorporate separately, the internal material damping in the model as like Kang [3]. Energy decay rate are obtained by the exponential stability of solutions by using multiplier technique.

1. Introduction

In this paper, we consider the uniform stability of a mathematical problems governed by the following a nonlinear wave equations of the Kelvin-Voigt type with Balakrishnan-Taylor damping and acoustic boundary conditions:

$$|u'|^\rho u'' = (a^2 + b\int_\Omega |\nabla u|^2 dx + \sigma \int_\Omega \nabla u \cdot \nabla u' dx)\triangle u + 2\lambda \triangle u' \text{ in } \Omega \times R^+, \tag{1.1}$$

$$u = 0 \text{ on } \Gamma_0 \times R^+, \tag{1.2}$$

$$(a^2 + b\int_\Omega |\nabla u|^2 dx + \sigma \int_\Omega \nabla u \cdot \nabla u' dx)\frac{\partial u}{\partial \nu} + 2\lambda \frac{\partial u'}{\partial \nu} = y' \text{ on } \Gamma_1 \times R^+, \tag{1.3}$$

$$u' + p(x)y' + q(x)y = 0 \text{ on } \Gamma_1 \times R^+, \tag{1.4}$$

$$u(0) = u_0, u'(0) = u_1 \text{ in } \Omega, \tag{1.5}$$

$$y(0) = y_0 \text{ on } \Gamma_1 \tag{1.6}$$

where Ω is a bounded, connected set in $R^n (n \geq 1)$ having a smooth boundary $\Gamma = \partial\Omega$, consisting of two parts Γ_0 and Γ_1 such that $\overline{\Gamma_0} \cup \overline{\Gamma_1} = \Gamma$. Primes denote the time derivative, Δ the Laplacian in R^n taken in space variables, ν the unit normal of Γ pointing towards exterior of Ω and $R^+ := (0, \infty)$. The parameters $\lambda > 0$ is a small internal material damping coefficient, and $a > 0, b > 0, \sigma > 0$

Received February 23, 2016; Accepted March 17, 2016.
2010 *Mathematics Subject Classification.* 35L70; 35B40, 76Exx.
Key words and phrases. Kelvin-Voigt type, Energy decay, Balakrishnan-Taylor damping, Acoustic boundary, Stabilization, Lyapunov functional.

(pISSN 1226-6973, eISSN 2287-2833)

are constant real numbers. p and q are functions satisfying some conditions to be specified later. Physically, the integro-differential equations (1.1)-(1.6) occurs in the study of vibrations of damped flexible space structures in bounded domain in R^n. The nonlinear term $|u'|^\rho u''$, $\rho > 0$ is modeled materials whose density depends on the velocity u'. The term $2\lambda\Delta u'$ is the internal material damping of Kelvin-Voigt type of the structure. And also the model in hand, with Balakrishnan-Taylor damping($\sigma > 0$) and $\rho = 1$, was initially proposed by Balakrishnan and Taylor[16] in 1989 and Bass and Zes [17]. On the other hand, for the conditions $\rho = 1, \sigma = 0$, Kang[11] was worked in 2012. The boundary conditions considered here are of mixed Dirichlet and Neumann type and acoustic boundary. The analytical studies in the area of stabilization of distributed parameter system is currently of interest in view of application to vibration control of various structural elements. The phenomenon was first observed by Hunton as reported by Harrison [9]. The nonlinear model like (1.1) for transverse vibrations was originally derived by Kirchhoff [8]. Beale and Rosencrans[13] introduced acoustic boundary conditions of the general form

$$\begin{aligned}
&\frac{\partial u}{\partial \nu} = y' \text{ on } \Gamma_1 \times R^+ \\
&\gamma u' + m(x)y'' + p(x)y' + q(x)y = 0 \text{ on } \Gamma_1 \times R^+.
\end{aligned}$$

Recently, wave equations with acoustic boundary conditions have been treated by many authors([11],[13],[5],[1],[6],[10],[12],[3]). In [11], the authors studied the nonlinear wave equations

$$\begin{aligned}
&u'' - M\left(\int_\Omega |u|^2 dx\right)\Delta u + |u'|^\alpha u' = 0 \text{ in } \Omega \times R^+, \\
&u = 0 \text{ on } \Gamma_0 \times R^+, \\
&\frac{\partial u}{\partial \nu} = y' \text{ on } \Gamma_1 \times R^+, \\
&\gamma u' + m(x)y'' + p(x)y' + q(x)y = 0 \text{ on } \Gamma_1 \times R^+.
\end{aligned}$$

They proved the existence of solutions, but did not give decay rate for solutions. As regards uniform decay rates for solutions to problems with acoustic boundary conditions, there are not much literature ([14],[11],[6],[10],[12]). Frota and Larkin[6] established global solvability and decay estimates for a linear wave equation with boundary conditions

$$\begin{aligned}
&\frac{\partial u}{\partial \nu} = h(x)y' \text{ on } \Gamma_1 \times R^+ \\
&\gamma u' + p(x)y' + q(x)y = 0 \text{ on } \Gamma_1 \times R^+.
\end{aligned}$$

In this paper we are motivated by boundary conditions of Park[11] and results of Gorain[7] and Kang[16][17]. The aim of this paper is to study stabilization of the generalized nonlinear Kirchhoff type wave equations governed by (1.1)-(1.6) with the mixed boundary conditions. To our knowledge, this problem has not been considered by predecessors and is studied first, as a Kirchhoff of kelvin-Voigt type model, in this paper. The plan of this paper as follows. In section 2, we give some notation, some conditions and material needed for our work. In section 3, we drive the stability on account of internal material damping of Kelvin-Voigt type with Balakrishnan-Taylor damping and acoustic boundary. The notation used in this paper is standard and can be found in Gorain[7] and Kang [17].

2. Preliminaries and some notations

In this section, we present some notations and some material in the proof of our result. Throughout this paper, we use the notation $V = \{u \in H^1(\Omega) : u = 0 \ on \ \Gamma_0\}$ the subspace of the classical Sobolev space $H^1(\Omega)$ of real valued functions of order one. Let k be the smallest positive constant independent of t (depends only on Ω) satisfying the Poincare inequality

$$\int_\Omega u^2 dx \leq k \int_\Omega |\nabla u|^2 dx \text{ for every } u \in V. \tag{2.1}$$

And also let $\overline{k}$ be the smallest positive constant independent of t (depends only on Γ_1) satisfying the embedding inequality

$$\int_{\Gamma_1} u^2 d\Gamma \leq \overline{k} \int_\Omega |\nabla u|^2 dx \text{ for every } u \in V. \tag{2.2}$$

We assume that

$$\rho \text{ satisfies } 0 < \rho \leq \tfrac{n}{n-2}, \text{ if } n \geq 3 \text{ or } \rho > 0, \text{ if } n = 1, 2. \tag{2.3}$$

and since $V \hookrightarrow L^{\rho+2}(\Omega)$,

$$\text{there exist a positive constant } K \text{ such that } ||u||_{\rho+2} \leq K||\nabla u||_2. \tag{2.4}$$

For the functions p and q, we assume that $p, q \in C(\Gamma_1)$ and $p(x) > 0$ and $q(x) > 0$ for all $x \in \Gamma_1$. This assumption implies that there exist positive constants $p_i, q_i (i = 0, 1)$ such that

$$p_0 \leq p(x) \leq p_1, \ q_0 \leq q(x) \leq q_1 \ \text{ for all } x \in \Gamma_1. \tag{2.5}$$

By using Gälerkin's approximation and the methods of Gorain[7] and Park[12], we can obtain the following existence result for the solution subject to (1.1)-(1.6) under the conditions on p and q as above. For the initial data $(u_0, u_1, y_0) \in (V \cap H^2(\Omega)) \times V \times L^2(\Gamma_1)$, there exists a unique pair of functions (u, y), which is a solution to the problem (1.1)-(1.6) in the class

$$u \in L^\infty(0, T; V \times H^2(\Omega)), u' \in L^\infty(0, T; V),$$
$$u'' \in L^\infty(0, T; L^2(\Omega)), y, y' \in L^2(0, \infty; L^2(\Gamma_1)).$$

In the order to state our main results, we define the energy of problem (1.1)-(1.6) by

$$E(t) = \frac{1}{\rho+2}\int_\Omega |u'|^{\rho+2}dx + \frac{a^2}{2}\int_\Omega |\nabla u|^2 dx + \frac{b}{4}(\int_\Omega |\nabla u|^2 dx)^2 + \frac{1}{2}\int_{\Gamma_1} q(x)(y)^2 d\Gamma. \tag{2.6}$$

3. Stability on account of internal damping of Kelvin-Voigt type

If we differentiate (2.6) with respect to t and use the governing Eq.(1.1) we get

$$\begin{aligned}
E'(t) &= \int_\Omega |u'|^\rho u'' u' dx + a^2 \int_\Omega \nabla u \cdot \nabla u' dx + b\int_\Omega ||\nabla u||^2 \nabla u \cdot \nabla u' dx \\
&\quad + \int_{\Gamma_1} q(x) y y' d\Gamma \\
&= \int_\Omega \{(a^2 + b\int_\Omega |\nabla u|^2 dx + \sigma \int_\Omega \nabla u \cdot \nabla u' dx)\triangle u + 2\lambda \triangle u'\} u' dx \\
&\quad + a^2 \int_\Omega \nabla u \cdot \nabla u' dx + b\int_\Omega ||\nabla u||^2 \nabla u \cdot \nabla u' dx + \int_{\Gamma_1} q(x) y y' d\Gamma.
\end{aligned}$$

Application of Green's formula and using the boundary conditions (1.2)-(1.4) and then a simplification, we get

$$\begin{aligned}
E'(t) &= -2\lambda \int_\Omega |\nabla u'|^2 dx - \sigma(\int_\Omega \nabla u \cdot \nabla u' dx)^2 + \int_{\Gamma_1} (u' + q(x)y) y' d\Gamma \\
&= -2\delta \int_\Omega |u'|^2 dx - \sigma(\int_\Omega \nabla u \cdot \nabla u' dx)^2 - \int_{\Gamma_1} p(x)(y')^2 d\Gamma < 0, \quad \forall t \in R^+.
\end{aligned} \tag{3.1}$$

We see from (3.1) that the energy E is a decreasing function of time and hence

$$E(t) \le E(0) \quad \forall t \ge 0, \tag{3.2}$$

where

$$\begin{aligned}
E(0) = \frac{1}{\rho+2}\int_\Omega |u_1|^{\rho+2} dx + \frac{a^2}{2}\int_\Omega |\nabla u_0|^2 dx + \frac{b}{4}(\int_\Omega |\nabla u_0|^2 dx)^2 \\
+ \frac{1}{2}\int_{\Gamma_1} q(x)(y(x,0))^2 d\Gamma.
\end{aligned}$$

Under what conditions does this energy E decay with time uniformly? An affirmative answer is contained in the following theorem.

Theorem 3.1. *If $u = u(x,t)$ is a regular solution of the system (1.1)-(1.6) with initial values $(u_0, u_1, y_0) \in V \times L^2(\Omega) \times L^2(\Gamma_1)$, then the energy $E(t)$ of the system defined by (2.6) satisfies*

$$E(t) < Me^{-\mu t}E(0), t \in (0, \infty)$$

for some real constants $M > 1$(3.25) and $\mu > 0$(3.22).

Firstly, we need to prove the following lemma.

Lemma 3.2. *For every solution $u = u(x,t)$ of the system (1.1)-(1.6), the time derivative of the functional Ψ defined by*

$$\Psi(t) = \frac{1}{\rho+1}\int_\Omega |u'|^\rho u' u dx + \lambda \int_\Omega |\nabla u|^2 dx + \frac{\sigma}{4}(\int_\Omega |\nabla u|^2 dx)^2 + \int_{\Gamma_1} uy d\Gamma + \frac{1}{2}\int_{\Gamma_1} p(x) y^2 d\Gamma \tag{3.3}$$

satisfies

$$\Psi'(t) \le \frac{1}{\rho+1}\int_\Omega |u'|^{\rho+2} dx + 2\int_{\Gamma_1} uy' d\Gamma - \int_{\Gamma_1} q(x)(y)^2 d\Gamma - a^2\int_\Omega |\nabla u|^2 dx - b(\int_\Omega |\nabla u|^2 dx)^2 + \frac{\sigma}{4}(\int_\Omega \nabla u \cdot u' dx)^2, t \in R^+. \tag{3.4}$$

Proof. If we differentiate (3.3) with respect to t and replace u'' by the relation (1.1), then we get

$$\begin{aligned}
\Psi'(t) &= \int_\Omega |u'|^\rho u'' u dx + \frac{1}{\rho+1}\int_\Omega |u'|^{\rho+2} dx + 2\lambda\int_\Omega \nabla u \cdot \nabla u' dx \\
&\quad + \sigma(\int_\Omega |\nabla u|^2 dx)\int_\Omega \nabla u \cdot \nabla u' dx + \int_{\Gamma_1}(u'y + uy')d\Gamma + \int_{\Gamma_1} p(x) y y' d\Gamma \\
&= \int_\Omega \{(a^2 + b\int_\Omega |\nabla u|^2 dx + \sigma\int_\Omega |\nabla u|^2 dx)\triangle u + 2\lambda \triangle u'\} u dx \\
&\quad + \frac{1}{\rho+1}\int_\Omega |u'|^{\rho+2} dx + 2\lambda\int_\Omega \nabla u \cdot \nabla u' dx \\
&\quad + \sigma(\int_\Omega |\nabla u|^2 dx)\int_\Omega \nabla u \cdot \nabla u' dx + \int_{\Gamma_1} uy' d\Gamma + \int_{\Gamma_1} y(u' + p(x) y')d\Gamma.
\end{aligned}$$

Applying Green's formula and boundary conditions, we have

$$\begin{aligned}
\Psi'(t) &= \int_{\Gamma_1}\{(a^2 + b\int_\Omega |\nabla u|^2 dx + \sigma\int_\Omega |\nabla u|^2 dx)\frac{\partial u}{\partial \nu} + 2\lambda\frac{\partial u'}{\partial \nu}\} u d\Gamma \\
&\quad - (a^2 + b\int_\Omega |\nabla u|^2 dx)\int_\Omega |\nabla u|^2 dx \\
&\quad - \sigma(\int_\Omega |\nabla u|^2 dx)^2 + \sigma(\int_\Omega |\nabla u|^2 dx)(\int_\Omega \nabla u \cdot \nabla u' dx) \\
&\quad + \frac{1}{\rho+1}\int_\Omega |u'|^{\rho+2} dx + \int_{\Gamma_1} uy' d\Gamma + \int_{\Gamma_1} y(u' + p(x) y')d\Gamma.
\end{aligned} \tag{3.5}$$

Using the boundary conditions(1.5),(1.6) and Young's inequality, relation (3.5) can be written as

$$\begin{aligned}\Psi'(t) =&\frac{1}{\rho+1}\int_\Omega |u'|^{\rho+2}dx - a^2\int_\Omega |\nabla u|^2 dx - b(\int_\Omega |\nabla u|^2 dx)^2\\&-\sigma(\int_\Omega |\nabla u|^2dx)^2 + \sigma(\int_\Omega |\nabla u|^2 dx)(\int_\Omega \nabla u\cdot\nabla u' dx)\\&+2\int_{\Gamma_1} uy' d\Gamma + \int_{\Gamma_1} y(u'+p(x)y')d\Gamma.\\ \le&\frac{1}{\rho+1}\int_\Omega |u'|^{\rho+2}dx + 2\int_{\Gamma_1} uy'd\Gamma - \int_{\Gamma_1} q(x)(y)^2 d\Gamma\\&-a^2\int_\Omega|\nabla u|^2dx - b(\int_\Omega |\nabla u|^2dx)^2 + \frac{\sigma}{4}(\int_\Omega \nabla u\cdot\nabla u'dx)^2, \quad \forall t\in R^+.\end{aligned}$$

Hence the proof of lemma complete. □

Proof of Theorem 1. We introduce a modified energy like Lyapunov functional V by

$$V(t) = E(t) + \epsilon\Psi(t) \quad \text{for } t\ge 0. \tag{3.6}$$

Now, using the Cauchy-Schwarz's inequality, the Hölder inequality, Young's inequality, the Poincare inequality (2.1)-(2.4) and the defined of energy (2.6), we obtain estimate as follow

$$\begin{aligned}|\frac{1}{\rho+1}\int_\Omega |u'|^\rho u'u dx| \le&\frac{1}{\rho+1}\int_\Omega |u'|^{\rho+1}|u|dx\\ \le&\frac{1}{\rho+1}(\int_\Omega |u'|^{\rho+2}dx)^{\frac{\rho+1}{\rho+2}}(\int_\Omega |u|^{\rho+2}dx)^{\frac{1}{\rho+2}}\\ \le&\frac{1}{\rho+2}\int_\Omega |u'|^{\rho+2}dx + \frac{1}{(\rho+1)(\rho+2)}\int_\Omega |u|^{\rho+2}dx\\ \le&\frac{1}{\rho+2}||u'||^{\rho+2}_{\rho+2} + \frac{K^{\rho+2}}{(\rho+1)(\rho+2)}||\nabla u||^{\rho+2}\\ \le&\frac{1}{\rho+2}||u'||^{\rho+2}_{\rho+2} + \frac{K^{\rho+2}}{(\rho+1)(\rho+2)}(\frac{2E(0)}{a^2})^{\frac{\rho}{2}}||\nabla u||^2\\ \le&\{1+\frac{K^{\rho+2}}{(\rho+1)(\rho+2)}(\frac{2E(0)}{a^2})^{\frac{\rho}{2}}\frac{2}{a^2}\}E(t),\end{aligned} \tag{3.7}$$

$$0\le \lambda\int_\Omega |\nabla u|^2 dx \le \frac{2\lambda}{a^2}E(t), \tag{3.8}$$

$$\begin{aligned}|\int_{\Gamma_1} uyd\Gamma| \le&\int_{\Gamma_1}\frac{1}{2q(x)}u^2d\Gamma + \frac{1}{2}\int_{\Gamma_1} q(x)y^2d\Gamma\\ \le&\frac{\bar{k}}{2q_0}\int_\Omega |\nabla u|^2dx + \frac{1}{2}\int_{\Gamma_1} q(x)y^2 d\Gamma \le (\frac{\bar{k}}{a^2q_0}+1)E(t),\end{aligned} \tag{3.9}$$

and

$$\frac{1}{2}\int_{\Gamma_1} p(x)y^2 d\Gamma \le \frac{p_1}{2q_0}\int_{\Gamma_1} q(x)y^2 d\Gamma \le \frac{p_1}{q_0}E(t). \tag{3.10}$$

and

$$\frac{\sigma}{4}(\int_\Omega |\nabla u|^2 dx)^2 dx \le \frac{\sigma}{b}E(t). \tag{3.11}$$

Thus the inequality (3.7)-(3.11) and (3.3) yield for Ψ that estimates

$$\begin{aligned}
&-[2+\frac{2K^{\rho+2}}{a^2(\rho+1)(\rho+2)}(\frac{2E(0)}{a^2})^{\rho/2}+\frac{\bar{k}}{a^2q_0}]E(t)\le G(t)\\
&\le [2+\frac{2K^{\rho+2}}{a^2(\rho+1)(\rho+2)}(\frac{2E(0)}{a^2})^{\rho/2}+\frac{2\lambda}{a^2}+\frac{\bar{k}}{a^2q_0}+\frac{p_1}{q_0}+\frac{\sigma}{b}]E(t).
\end{aligned} \tag{3.12}$$

Then it follows from (3.12) that

$$\{1-\epsilon M_1)\}E(t)\le V(t)\le\{1+\epsilon M_2\}E(t) \quad \forall t\ge 0, \tag{3.13}$$

where we assume that

$$0<\epsilon<\frac{1}{M_1},$$

so that left hand side of (3.13) is positive. Here

$$M_1 := 2+\frac{2K^{\rho+2}}{a^2(\rho+1)(\rho+2)}(\frac{2E(0)}{a^2})^{\frac{\rho}{2}}+\frac{\bar{k}}{a^2q_0}>0$$

and

$$M_2 := 2+\frac{2K^{\rho+2}}{a^2(\rho+1)(\rho+2)}(\frac{2E(0)}{a^2})^{\frac{\rho}{2}}+\frac{2\lambda}{a^2}+\frac{\bar{k}}{a^2q_0}+\frac{p_1}{q_0}+\frac{\sigma}{b}>0.$$

Next, differentiating $V(t)$(defined by (3.6)) with respect to t using expression $E'(t)$(defined by (3.1)) and Lemma 3.1, we have

$$\begin{aligned}
V'(t)\le &-\sigma(\int_\Omega \nabla u\cdot\nabla u' dx)^2-2\lambda\int_\Omega|\nabla u'|^2dx-\int_{\Gamma_1}p(x)(y')^2d\Gamma\\
&+\epsilon\{\frac{1}{\rho+1}\int_\Omega|u'|^{\rho+2}dx+2\int_{\Gamma_1}uy'd\Gamma-\int_{\Gamma_1}q(x)(y)^2d\Gamma\\
&-a^2\int_\Omega|\nabla u|^2dx-b(\int_\Omega|\nabla u|^2dx)^2+\frac{\sigma}{4}(\int_\Omega\nabla u\cdot\nabla u'dx)^2\}.
\end{aligned} \tag{3.14}$$

Now, using the Cauchy-Schwarz's inequality,the Poincare inequality, the conditions (2.2)-(2.3) and the definition of energy (2.6), we obtain estimate

$$\begin{aligned} |2\epsilon \int_{\Gamma_1} uy' dx| &\leq \int_{\Gamma_1} p(x)(y')^2 d\Gamma + \epsilon^2 \int_{\Gamma_1} \frac{1}{p(x)} u^2 d\Gamma \\ &\leq \int_{\Gamma_1} p(x)(y')^2 d\Gamma + \frac{2\bar{k}\epsilon^2}{a^2 p_0} \frac{a^2}{2} \int_\Omega |\nabla u|^2 dx \\ &\leq \int_{\Gamma_1} p(x)(y')^2 d\Gamma + \frac{2\bar{k}\epsilon^2}{a^2 p_0} E(t). \end{aligned} \tag{3.15}$$

From (3.14)-(3.15), we have

$$\begin{aligned} V'(t) \leq &- \sigma(1 - \frac{\epsilon}{4})(\int_\Omega \nabla u \cdot \nabla u' dx)^2 - 2\lambda \int_\Omega |\nabla u'|^2 dx \\ &+ \frac{\epsilon}{\rho+1} \int_\Omega |u'|^{\rho+2} dx + \frac{2\bar{k}\epsilon^2}{a^2 p_0} E(t) - \epsilon \int_{\Gamma_1} p(x)(y')^2 d\Gamma \\ &- a^2\epsilon \int_\Omega |\nabla u|^2 dx - b\epsilon(\int_\Omega |\nabla u|^2 dx)^2 \\ = &- 2\lambda \int_\Omega |\nabla u'|^2 dx + \frac{\epsilon(3\rho+4)}{(\rho+1)(\rho+2)} \int_\Omega |u'|^{\rho+2} dx \\ &- \sigma(1 - \frac{\epsilon}{4})(\int_\Omega \nabla u \cdot \nabla u' dx)^2 - 2\epsilon(1 - \frac{\bar{k}\epsilon}{a^2 p_0}) E(t) - \frac{b\epsilon}{2}(\int_\Omega |\nabla u|^2 dx)^2. \end{aligned} \tag{3.16}$$

By the definition of energy (2.6), we note that

$$||u'||_{\rho+2}^{\rho+2} \leq (\rho+2)E(0). \tag{3.17}$$

By the inequality (3.17), we can take sufficiently small $\epsilon_0 > 0$ such that

$$0 < \epsilon_0 \leq \frac{2\lambda(\rho+1)||\nabla u'||^2}{(3\rho+4)E(0)}. \tag{3.18}$$

And also we can take sufficiently small ϵ satisfy

$$0 < \epsilon < 4, \quad 0 < \epsilon < \frac{a^2 p_0}{\bar{k}}, \quad 0 < \epsilon \leq \epsilon_0, \tag{3.19}$$

since

$$||u'||_{\rho+2}^{\rho+2} \leq (\rho+2)E(0), \quad -2\lambda||\nabla u'||^2 + \epsilon_0 \frac{3\rho+4}{(\rho+1)(\rho+2)} ||u'||_{\rho+2}^{\rho+2} \leq 0.$$

From(3.16)-(3.19), we obtain

$$\begin{aligned} V'(t) \leq & -2\epsilon(1-\frac{\bar{k}\epsilon}{a^2p_0})E(t) \\ & -\sigma(1-\frac{\epsilon}{4})(\int_\Omega \nabla u\cdot\nabla u' dx)^2 - \frac{b\epsilon}{2}(\int_\Omega |\nabla u|^2 dx)^2 \\ & -(2\lambda\int_\Omega |\nabla u'|^2 dx - \frac{\epsilon(3\rho+4)}{(\rho+1)(\rho+2)}\int_\Omega |u'|^{\rho+2} dx) \\ < & -2\epsilon(1-\frac{\bar{k}\epsilon}{a^2p_0})E(t), \quad \forall t>0, \end{aligned}$$

where we assume that

$$0<\epsilon<\min\{4, M_1^{-1}, \frac{a^2p_0}{\bar{k}}, \epsilon_0\}. \tag{3.20}$$

With the help of (3.13), the above yields the differential inequality

$$V'(t)+\mu V(t)<0 \quad \forall t\in R^+, \tag{3.21}$$

where

$$0<\mu=\frac{2\epsilon(a^2p_0-\bar{k}\epsilon)}{a^2p_0(1+\epsilon M_2)}. \tag{3.22}$$

Multiplying (3.21) by $e^{\mu t}$ and integrating over the time interval $[0,t]$, we get the estimate

$$V(t)<e^{-\mu t}V(0) \quad \forall t\in R^+. \tag{3.23}$$

Invoking the inequality (3.13) again in (3.23), we have

$$E(t)<Me^{-\mu t}E(0) \quad \forall t\in R^+, \tag{3.24}$$

where

$$M=\frac{1+\epsilon M_2}{1-\epsilon M_1}>1. \tag{3.25}$$

The finishes the proof of the theorem. □

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