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EXISTENCE OF COINCIDENCE POINT UNDER GENERALIZED NONLINEAR CONTRACTION WITH APPLICATIONS

Bhavana Deshpande, Amrish Handa, and Shamim Ahmad Thoker

Abstract. We present coincidence point theorem for g-non-decreasing mappings satisfying generalized nonlinear contraction on partially ordered metric spaces. We show how multidimensional results can be seen as simple consequences of our unidimensional coincidence point theorem. We also obtain the coupled coincidence point theorem for generalized compatible pair of mappings F, $G : X^2 \to X$ by using obtained coincidence point results. Furthermore, an example and an application to integral equation are also given to show the usability of obtained results. Our results generalize, modify, improve and sharpen several well-known results.

1. Introduction and Preliminaries

In order to fix the framework needed to state our main results, we recall the following notions. For simplicity, we denote from now on $X \times X \times ... \times X$ (n times) by X^n, where $n \in \mathbb{N}$ with $n \geq 2$ and X is a non-empty set. Let $\{A, B\}$ be a partition of the set $\Lambda_n = \{1, 2, ..., n\}$, that is, A and B are nonempty subsets of Λ_n such that $A \cup B = \Lambda_n$ and $A \cap B = \emptyset$. We will denote $\Omega_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq A, \sigma(B) \subseteq B\}$ and $\Omega^{'}_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq B, \sigma(B) \subseteq A\}$. Henceforth, let $\sigma_1, \sigma_2, ..., \sigma_n$ be n mappings from Λ_n into itself and let Υ be the $n-$tuple $(\sigma_1, \sigma_2, ..., \sigma_n)$. Let $F : X^n \to X$ and $g : X \to X$ be two mappings. For brevity, we denote $g(x)$ by gx where $x \in X$.

In [10], Guo and Lakshmikantham introduced the notion of coupled fixed point for single-valued mappings.

Definition 1. [10]. Let $F : X^2 \to X$ be a given mapping. An element $(x, y) \in X^2$ is called a coupled fixed point of F if

$$F(x,\ y) = x \text{ and } F(y,\ x) = y.$$

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Gnana-Bhaskar and Lakshmikantham [2] obtained some coupled fixed point theorems for single-valued mappings by defining the notion of mixed monotone property on partially ordered metric spaces.

Definition 2. [2]. Let $(X, \preceq)$ be a partially ordered set. Suppose $F : X^2 \to X$ be a given mapping. We say that F has the mixed monotone property if for all $x,\ y \in X$, we have

$$x_1,\ x_2 \in X,\ x_1 \preceq x_2 \implies F(x_1,\ y) \preceq F(x_2,\ y),$$

and

$$y_1,\ y_2 \in X,\ y_1 \preceq y_2 \implies F(x,\ y_1) \succeq F(x,\ y_2).$$

After that, Lakshmikantham and Ciric [18] presented the existence of coupled coincidence point by defining the notion of mixed g-monotone property.

Definition 3. [18]. Let $F : X^2 \to X$ and $g : X \to X$ be given mappings. An element $(x,\ y) \in X^2$ is called a coupled coincidence point of the mappings F and g if

$$F(x,\ y) = gx \text{ and } F(y,\ x) = gy.$$

Definition 4. [18]. Let $F : X^2 \to X$ and $g : X \to X$ be given mappings. An element $(x,\ y) \in X^2$ is called a common coupled fixed point of the mappings F and g if

$$x = F(x,\ y) = gx \text{ and } y = F(y,\ x) = gy.$$

Definition 5. [18]. The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be commutative if

$$gF(x,\ y) = F(gx,\ gy), \text{ for all } (x,\ y) \in X^2.$$

Definition 6. [18]. Let $(X, \preceq)$ be a partially ordered set. Suppose $F : X^2 \to X$ and $g : X \to X$ are given mappings. We say that F has the mixed g-monotone property if for all $x,\ y \in X$, we have

$$x_1,\ x_2 \in X,\ gx_1 \preceq gx_2 \implies F(x_1,\ y) \preceq F(x_2,\ y),$$

and

$$y_1,\ y_2 \in X,\ gy_1 \preceq gy_2 \implies F(x,\ y_1) \succeq F(x,\ y_2).$$

If g is the identity mapping on X, then F satisfies the mixed monotone property.

Subsequently, Choudhury and Kundu [4] modify the results of Lakshmikantham and Ciric [18] by introducing the notion of compatibility in coupled coincidence point context.

Definition 7. [4]. The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be compatible if

$$\begin{aligned} \lim_{n\to\infty} d(gF(x_n,\ y_n),\ F(gx_n,\ gy_n)) &= 0, \\ \lim_{n\to\infty} d(gF(y_n,\ x_n),\ F(gy_n,\ gx_n)) &= 0, \end{aligned}$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\begin{aligned}\lim_{n\to\infty} F(x_n,\ y_n) &= \lim_{n\to\infty} gx_n = x,\\ \lim_{n\to\infty} F(y_n,\ x_n) &= \lim_{n\to\infty} gy_n = y,\ \text{for some } x,\ y \in X.\end{aligned}$$

A great deal of these studies investigate contractions on partially ordered metric spaces because of their applicability to initial value problems defined by differential or integral equations.

Hussain et al. [12] proved some coupled coincidence point results by introducing a new concept of generalized compatibility of a pair of mappings F, $G : X^2 \to X$.

Definition 8. [12]**.** Suppose that F, $G : X^2 \to X$ are two mappings. F is said to be $G-$increasing with respect to $\preceq$ if for all x, y, u, $v \in X$, with $G(x, y) \preceq G(u, v)$ we have $F(x, y) \preceq F(u, v)$.

Definition 9. [12]**.** Let F, $G : X^2 \to X$ be two mappings. We say that the pair $\{F, G\}$ is commuting if

$$F(G(x,\ y),\ G(y,\ x)) = G(F(x,\ y),\ F(y,\ x)),\ \text{for all } x,\ y \in X.$$

Definition 10. [12]**.** Suppose that F, $G : X^2 \to X$ are two mappings. An element $(x, y) \in X^2$ is called a coupled coincidence point of mappings F and G if

$$F(x,\ y) = G(x,\ y) \text{ and } F(y,\ x) = G(y,\ x).$$

Definition 11. [12]**.** Let $(X, \preceq)$ be a partially ordered set, $F : X^2 \to X$ and $g : X \to X$ are two mappings. We say that F is g-increasing with respect to $\preceq$ if for any x, $y \in X$,

$$gx_1 \preceq gx_2 \text{ implies } F(x_1,\ y) \preceq F(x_2,\ y),$$

and

$$gy_1 \preceq gy_2 \text{ implies } F(x,\ y_1) \preceq F(x,\ y_2).$$

Definition 12. [12]**.** Let $(X, \preceq)$ be a partially ordered set and $F : X^2 \to X$ be a mapping. We say that F is increasing with respect to $\preceq$ if for any x, $y \in X$,

$$x_1 \preceq x_2 \text{ implies } F(x_1,\ y) \preceq F(x_2,\ y),$$

and

$$y_1 \preceq y_2 \text{ implies } F(x,\ y_1) \preceq F(x,\ y_2).$$

Definition 13. [12]**.** Let F, $G : X^2 \to X$ are two mappings. We say that the pair $\{F, G\}$ is generalized compatible if

$$\begin{aligned}\lim_{n\to\infty} d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(F(x_n,\ y_n),\ F(y_n,\ x_n))) &= 0,\\ \lim_{n\to\infty} d(F(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(F(y_n,\ x_n),\ F(x_n,\ y_n))) &= 0,\end{aligned}$$

whenever (x_n) and (y_n) are sequences in X such that

$$\begin{aligned}\lim_{n\to\infty} G(x_n,\ y_n) &= \lim_{n\to\infty} F(x_n,\ y_n) = x,\\ \lim_{n\to\infty} G(y_n,\ x_n) &= \lim_{n\to\infty} F(y_n,\ x_n) = y,\ \text{for some } x,\ y \in X.\end{aligned}$$

Obviously, a commuting pair is a generalized compatible but not conversely in general.

Erhan et al. [8], indicate that the results obtained in Hussain et al. [12] can be deduce from the coincidence point results in the existing literature.

In [8] Erhan et al. used the following definitions:

Definition 14. [1, 9]**.** A coincidence point of two mappings $T,\ g : X \to X$ is a point $x \in X$ such that $Tx = gx$.

Definition 15. [8]**.** A partially ordered metric space $(X,\ d,\ \preceq)$ is a metric space $(X,\ d)$ provided with a partial order $\preceq$.

Definition 16. [2, 12]**.** A partially ordered metric space $(X,\ d,\ \preceq)$ is said to be non-decreasing-regular (respectively, non-increasing-regular) if for every sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \to x$ and $x_n \preceq x_{n+1}$ (respectively, $x_n \succeq x_{n+1}$) for all $n \geq 0$, we have that $x_n \preceq x$ (respectively, $x_n \succeq x$) for all $n \geq 0$. $(X,\ d,\ \preceq)$ is said to be regular if it is both non-decreasing-regular and non-increasing-regular.

Definition 17. [8]**.** Let $(X,\ \preceq)$ be a partially ordered set and let $T,\ g : X \to X$ be two mappings. We say that T is $(g,\ \preceq)$-non-decreasing if $Tx \preceq Ty$ for all x, $y \in X$ such that $gx \preceq gy$. If g is the identity mapping on X, we say that T is $\preceq$-non-decreasing.

Very recently, the concept of multidimensional fixed/coincidence point introduced by Roldan et al. in [22], which is an extension of Berzig and Samet's notion given in [3], which extended and generalized the mentioned fixed point results to higher dimensions. However, they used permutations of variables and distinguished between the first and the last variables.

A partial order $\preceq$ on X can be extended to a partial order $\sqsubseteq$ on X^n in the following way. If $(X,\ \preceq)$ be a partially ordered space, $x,\ y \in X$ and $i \in \Lambda_n$, we will use the following notations:

$$x \preceq_i y \Rightarrow \begin{cases} x \preceq y, & \text{if } i \in A,\\ x \succeq y, & \text{if } i \in B.\end{cases}$$

Consider on the product space X^n the following partial order: for $Y = (y_1,\ y_2,\ ...,\ y_i,\ ...,\ y_n)$, $V = (v_1,\ v_2,\ ...,\ v_i,\ ...,\ v_n) \in X^n$,

$$Y \sqsubseteq V \Leftrightarrow y_i \preceq_i v_i. \tag{1}$$

Notice that $\sqsubseteq$ depends on A and B. We say that two points Y and V are comparable, if $Y \sqsubseteq V$ or $V \sqsubseteq Y$. Obviously, $(X^n,\ \sqsubseteq)$ is a partially ordered set.

Definition 18. [15, 22, 23]**.** A point $(x_1,\ x_2,\ ...,\ x_n) \in X^n$ is called a $\Upsilon-$coincidence point of the mappings $F : X^n \to X$ and $g : X \to X$ if

$$F(x_{\sigma_i(1)},\ x_{\sigma_i(2)},\ ...,\ x_{\sigma_i(n)}) = gx_i, \text{ for all } i \in \Lambda_n.$$

If g is the identity mapping on X, then $(x_1,\ x_2,\ ...,\ x_n) \in X^n$ is called a $\Upsilon-$fixed point of the mapping F.

It is clear that the previous definition extends the notions of coupled, tripled, and quadruple fixed points. In fact, if we represent a mapping $\sigma : \Lambda_n \to \Lambda_n$ throughout its ordered image, that is, $\sigma = (\sigma(1),\ \sigma\ (2),\ ...,\ \sigma\ (n))$, then

(i) Gnana-Bhaskar and Lakshmikantham's coupled fixed points occur when $n = 2$, $\sigma_1 = (1,\ 2)$ and $\sigma_2 = (2,\ 1)$,

(ii) Berinde and Borcut's tripled fixed points are associated with $n = 3$, $\sigma_1 = (1,\ 2,\ 3)$, $\sigma_2 = (2,\ 1,\ 2)$ and $\sigma_3 = (3,\ 2,\ 1)$,

(iii) Karapinar's quadruple fixed points are considered when $n = 4$, $\sigma_1 = (1,\ 2,\ 3,\ 4)$, $\sigma_2 = (2,\ 3,\ 4,\ 1)$, $\sigma_3 = (3,\ 4,\ 1,\ 2)$ and $\sigma_4 = (4,\ 1,\ 2,\ 3)$.

These cases consider A as the odd numbers in $\{1,\ 2,\ ...,\ n\}$ and B as its even numbers. However, Berzig and Samet [3] use $A = \{1,\ 2,\ ...,\ m\}$, $B = \{m + 1,\ ...,\ n\}$ and arbitrary mappings.

Definition 19. [22]**.** Let $(X,\ \preceq)$ be a partially ordered space. We say that F has the mixed $(g,\ \preceq)$-monotone property if F is g-monotone non-decreasing in arguments of A and g-monotone non-increasing in arguments of B, that is, for all $x_1,\ x_2,\ ...,\ x_n,\ y,\ z \in X$ and all i,

$$gy \preceq gz \Rightarrow F(x_1, ..., x_{i-1}, y, x_{i+1}, ..., x_n) \preceq_i F(x_1, ..., x_{i-1}, z, x_{i+1}, ..., x_n).$$

Remark 1. [14]**.** In order to ensure the existence of Υ-coincidence/fixed points, it is very important to assume that the mixed g-monotone property is compatible with the permutation of the variables, that is, the mappings of $\Upsilon = (\sigma_1,\ \sigma_2,\ ...,\ \sigma_n)$ should verify $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$.

Definition 20. [23, 26]**.** Let $(X,\ d)$ be a metric space and define $\Delta_n,\ \rho_n : X^n \times X^n \to [0,\ +\infty)$, for $Y = (y_1,\ y_2,\ ...,\ y_n)$, $V = (v_1,\ v_2,\ ...,\ v_n) \in X^n$, by

$$\Delta_n(Y,\ V) = \frac{1}{n}\sum_{i=1}^{n} d(y_i,\ v_i) \text{ and } \rho_n(Y,\ V) = \max_{1\leq i\leq n} d(y_i,\ v_i).$$

Then Δ_n and ρ_n are metric on X^n, $(X,\ d)$ is complete if and only if $(X^n,\ \Delta_n)$ is complete. Similarly, $(X,\ d)$ is complete if and only if $(X^n,\ \rho_n)$ is complete. It is easy to see that

$$\begin{aligned} \Delta_n(Y^k,\ Y) \quad &\to \quad 0 \text{ (as } k \to \infty) \Leftrightarrow d(y_i^k,\ y_i) \to 0 \text{ (as } k \to \infty), \\ \text{and } \rho_n(Y^k,\ Y) \quad &\to \quad 0 \text{ (as } k \to \infty) \Leftrightarrow d(y_i^k,\ y_i) \to 0 \text{ (as } k \to \infty),\ i \in \Lambda_n, \end{aligned}$$

where $Y^k = (y_1^k,\ y_2^k,\ ...,\ y_n^k)$ and $Y = (y_1,\ y_2,\ ...,\ y_n) \in X^n$.

Lemma 1.1. [23, 26, 27]**.** *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and let $F : X^n \to X$ and $g : X \to X$ be two mappings. Let $\Upsilon = (\sigma_1,\ \sigma_2,\ ...,\ \sigma_n)$*

be an $n-$tuple of mappings from Λ_n into itself verifying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Define F_Υ, $G : X^n \to X^n$, for all $y_1, y_2, ..., y_n \in X$, by

$$\begin{aligned} F_\Upsilon(y_1, y_2, ..., y_n) &= \begin{pmatrix} F(y_{\sigma_1(1)}, y_{\sigma_1(2)}, ..., y_{\sigma_1(n)}), \\ F(y_{\sigma_2(1)}, y_{\sigma_2(2)}, ..., y_{\sigma_2(n)}), \\ ..., F(y_{\sigma_n(1)}, y_{\sigma_n(2)}, ..., y_{\sigma_n(n)}) \end{pmatrix}, \\ \text{and } G(y_1, y_2, ..., y_n) &= (gy_1, gy_2, ..., gy_n). \end{aligned}$$

(1) *If F has the mixed $(g, \preceq)-$monotone property, then F_Υ is monotone $(G, \sqsubseteq)-$non-decreasing.*

(2) *If F is $d-$continuous, then F_Υ is Δ_n-continuous and ρ_n-continuous.*

(3) *If g is $d-$continuous, then G is Δ_n-continuous and ρ_n-continuous.*

(4) *A point $(y_1, y_2, ..., y_n) \in X^n$ is a $\Upsilon-$fixed point of F if and only if $(y_1, y_2, ..., y_n)$ is a fixed point of F_Υ.*

(5) *A point $(y_1, y_2, ..., y_n) \in X^n$ is a $\Upsilon-$coincidence point of F and g if and only if $(y_1, y_2, ..., y_n)$ is a coincidence point of F_Υ and G.*

(6) *If $(X, d, \preceq)$ is regular, then $(X^n, \Delta_n, \sqsubseteq)$ and $(X^n, \rho_n, \sqsubseteq)$ are also regular.*

The commutativity and compatibility of the mappings are defined as follows.

Definition 21. [22]**.** We will say that two mappings T, $g : X \to X$ are commuting if $gTx = Tgx$ for all $x \in X$. We will say that $F : X^n \to X$ and $g : X \to X$ are commuting if $gF(x_1, x_2, ..., x_n) = F(gx_1, gx_2, ..., gx_n)$ for all $x_1, x_2, ..., x_n \in X$.

The following notion was introduced in order to avoid the necessity of commutativity.

Definition 22. [4, 11, 20, 21]**.** Let $(X, d, \preceq)$ be an ordered metric space. Two mappings T, $g : X \to X$ are said to be O-compatible if

$$\lim_{n\to\infty} d(gTx_n, Tgx_n) = 0,$$

provided that $\{x_n\}$ is a sequence in X such that $\{gx_n\}$ is $\preceq$-monotone, that is, it is either non-increasing or non-decreasing with respect to $\preceq$ and

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_n \in X.$$

The natural extension to an arbitrary number of variables is the following one.

Definition 23. [21]**.** Let $(X, d, \preceq)$ be an ordered metric space and let $F : X^n \to X$ and $g : X \to X$ be two mappings. Let $\Upsilon = (\sigma_1, \sigma_2, ..., \sigma_n)$ be an $n-$tuple of mappings Λ_n into itself verifying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. We will say that (F, g) is a $(O, \Upsilon)-$compatible pair if, for all $i \in \Lambda_n$,

$$\lim_{m\to\infty} d(gF(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, ..., x_m^{\sigma_i(n)}), F(gx_m^{\sigma_i(1)}, gx_m^{\sigma_i(2)}, ..., gx_m^{\sigma_i(n)})) = 0,$$

whenever $\{x_m^1\}$, $\{x_m^2\}$, ..., $\{x_m^n\}$ are sequences in X such that $\{gx_m^1\}$, $\{gx_m^2\}$, ..., $\{gx_m^n\}$ are $\preceq$ −monotone and

$$\lim_{m\to\infty} F(x_m^{\sigma_i(1)},\ x_m^{\sigma_i(2)},\ ...,\ x_m^{\sigma_i(n)}) = \lim_{n\to\infty} gx_m^i \in X \text{ for all } i \in \Lambda_n.$$

Lemma 1.2. [21]. *If F and g are $(O,\ \Upsilon)$-compatible, then F_Υ and G are O-compatible.*

In [7], Ding et al. obtained coupled coincidence and common coupled fixed point theorems for generalized nonlinear contraction on partially ordered metric spaces which generalize the results of Lakshmikantham and Ciric [18]. Our basic references are[5,6,7,8,12,13,16,17,24,25,26,27].

In this paper, we present coincidence point theorem for g-non-decreasing mappings satisfying generalized nonlinear contraction on partially ordered metric spaces. We show how multidimensional results can be seen as simple consequences of our unidimensional coincidence point theorem. We also obtain the coupled coincidence point theorem for generalized compatible pair of mappings F, $G : X^2 \to X$ by using obtained coincidence point results. Furthermore, an example and an application to integral equation are also given to show the usability of obtained results. Our results generalize, modify, improve and sharpen the results of Gnana-Bhaskar and Lakshmikantham [2], Ding et al. [7] and Lakshmikantham and Ciric [18].

2. Main results

Let Φ denote the set of all functions $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ satisfying

(i_φ) φ is non-decreasing,

(ii_φ) $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, where $\varphi^{n+1}(t) = \varphi^n(\varphi(t))$.

It is clear that $\varphi(t) < t$ for each $t > 0$. In fact, if $\varphi(t_0) \geq t_0$ for some $t_0 > 0$, then, since φ is non-decreasing, $\varphi^n(t_0) \geq t_0$ for all $n \in \mathbb{N}$, which contradicts with $\lim_{n\to\infty} \varphi^n(t_0) = 0$. In addition, it is easy to see that $\varphi(0) = 0$.

Theorem 2.1. *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and let T, $g : X \to X$ be two mappings such that the following properties are fulfilled:*

(i) $T(X) \subseteq g(X)$,

(ii) T is $(g,\ \preceq)$-non-decreasing,

(iii) there exists $x_0 \in X$ such that $gx_0 \preceq Tx_0$,

(iv) there exists $\varphi \in \Phi$ such that

$$d(Tx,\ Ty) \leq \varphi\left(M(x,\ y)\right),$$

where

$$M(x,\ y) = \max\left\{ \begin{array}{c} d(gx,\ gy),\ d(gx,\ Tx),\ d(gy,\ Ty), \\ \frac{d(gx,\ Ty)+d(gy,\ Tx)}{2} \end{array} \right\},$$

for all x, $y \in X$ such that $gx \preceq gy$. Also assume that, at least, one of the following conditions holds:

(a) $(X,\ d)$ is complete, T and g are continuous and the pair $(T,\ g)$ is O-compatible,

(b) $(X,\ d)$ *is complete,* T *and* g *are continuous and commuting,*

(c) $(g(X),\ d)$ *is complete and* $(X,\ d,\ \preceq)$ *is non-decreasing-regular,*

(d) $(X,\ d)$ *is complete,* $g(X)$ *is closed and* $(X,\ d,\ \preceq)$ *is non-decreasing-regular,*

(e) $(X,\ d)$ *is complete,* g *is continuous, and monotone-non-decreasing, the pair* $(T,\ g)$ *is O-compatible and* $(X,\ d,\ \preceq)$ *is non-decreasing-regular.*

Then T *and* g *have, at least, a coincidence point.*

Proof. We divide the proof into three steps.

Step 1. We claim that there exists a sequence $\{x_n\} \subseteq X$ such that $\{gx_n\}$ is $\preceq$-non-decreasing and $gx_{n+1} = Tx_n$, for all $n \geq 0$. Let $x_0 \in X$ be arbitrary. Since $Tx_0 \in T(X) \subseteq g(X)$, therefore there exists $x_1 \in X$ such that $Tx_0 = gx_1$. Then $gx_0 \preceq Tx_0 = gx_1$. Since T is $(g,\ \preceq)$-non-decreasing, therefore $Tx_0 \preceq Tx_1$. Again, since $Tx_1 \in T(X) \subseteq g(X)$, therefore there exists $x_2 \in X$ such that $Tx_1 = gx_2$. Then $gx_1 = Tx_0 \preceq Tx_1 = gx_2$. Since T is $(g,\ \preceq)$-non-decreasing, therefore $Tx_1 \preceq Tx_2$. Repeating this argument, there exists a sequence $\{x_n\}_{n\geq 0}$ such that $\{gx_n\}$ is $\preceq$-non-decreasing, $gx_{n+1} = Tx_n \preceq Tx_{n+1} = gx_{n+2}$ and

$$gx_{n+1} = Tx_n \text{ for all } n \geq 0. \tag{2}$$

Step 2. We claim that $\{gx_n\}_{n\geq 0}$ is a Cauchy sequence in X. Now, by contractive condition (iv), we have

$$d(gx_{n+1},\ gx_{n+2}) = d(Tx_n,\ Tx_{n+1}) \leq \varphi\left(M(x_n,\ x_{n+1})\right), \tag{3}$$

where

$$\begin{aligned}
& M(x_n,\ x_{n+1}) \\
= \ & \max\left\{ \begin{array}{c} d(gx_n,\ gx_{n+1}),\ d(gx_n,\ Tx_n),\ d(gx_{n+1},\ Tx_{n+1}), \\ \frac{d(gx_n,\ Tx_{n+1})+d(gx_{n+1},\ Tx_n)}{2} \end{array} \right\} \\
= \ & \max\left\{ \begin{array}{c} d(gx_n,\ gx_{n+1}),\ d(gx_n,\ gx_{n+1}),\ d(gx_{n+1},\ gx_{n+2}), \\ \frac{d(gx_n,\ gx_{n+2})+d(gx_{n+1},\ gx_{n+1})}{2} \end{array} \right\} \\
\leq \ & \max\left\{ \begin{array}{c} d(gx_n,\ gx_{n+1}),\ d(gx_{n+1},\ gx_{n+2}), \\ \frac{d(gx_n,\ gx_{n+1})+d(gx_{n+1},\ gx_{n+2})}{2} \end{array} \right\} \\
\leq \ & \max\left\{d(gx_n,\ gx_{n+1}),\ d(gx_{n+1},\ gx_{n+2})\right\}.
\end{aligned}$$

If $d(gx_{n+1},\ gx_{n+2}) \geq d(gx_n,\ gx_{n+1})$. Then

$$M(x_n,\ x_{n+1}) \leq d(gx_{n+1},\ gx_{n+2}). \tag{4}$$

From (3), (4) and by the fact that $\varphi(t) < t$ for all $t > 0$, we get

$$d(gx_{n+1},\ gx_{n+2}) \leq \varphi\left(d(gx_{n+1},\ gx_{n+2})\right) < d(gx_{n+1},\ gx_{n+2}),$$

which is a contradiction. Hence, $d(gx_n,\ gx_{n+1}) \geq d(gx_{n+1},\ gx_{n+2})$. Then

$$M(x_n,\ x_{n+1}) \leq d(gx_n,\ gx_{n+1}). \tag{5}$$

Thus, by (3) and (5), we have for all $n \in \mathbb{N}$,

$$d(gx_{n+1},\ gx_{n+2}) \leq \varphi\left(d(gx_n,\ gx_{n+1})\right) \leq \varphi^n\left(d(gx_0,\ gx_1)\right) \leq \varphi^n(\delta), \tag{6}$$

where

$$\delta = d(gx_0,\ gx_1).$$

Without loss of generality, we can assume that $d(gx_0,\ gx_1) \neq 0$. In fact, if this is not true, then $gx_0 = gx_1 = Tx_0$, that is, x_0 is a coincidence point of g and T. Thus, for $m,\ n \in \mathbb{N}$ with $m > n$, by triangle inequality and (6), we get

$$\begin{aligned}
& d(gx_n,\ gx_{m+n}) \\
\leq\ & d(gx_n,\ gx_{n+1}) + d(gx_{n+1},\ gx_{n+2}) + ... + d(gx_{n+m-1},\ gx_{m+n}) \\
\leq\ & \varphi^n(\delta) + \varphi^{n+1}(\delta) + ... + \varphi^{n+m-1}(\delta) \\
\leq\ & \sum_{i=n}^{n+m-1} \varphi^i(\delta),
\end{aligned}$$

which implies, by (ii_φ), that $\{gx_n\}_{n\geq 0}$ is a Cauchy sequence in X.

Step 3. We claim that T and g have a coincidence point distinguishing between cases $(a) - (e)$.

Suppose now that (a) holds, that is, $(X,\ d)$ is complete, T and g are continuous and the pair $(T,\ g)$ is O-compatible. Since $(X,\ d)$ is complete, therefore there exists $z \in X$ such that $\{gx_n\} \to z$ and $\{Tx_n\} \to z$. Since T and g are continuous, therefore $\{Tgx_n\} \to Tz$ and $\{ggx_n\} \to gz$. Since the pair $(T,\ g)$ is O-compatible, therefore $\lim_{n\to\infty} d(gTx_n,\ Tgx_n) = 0$. Thus, we conclude that

$$d(gz,\ Tz) = \lim_{n\to\infty} d(ggx_{n+1},\ Tgx_n) = \lim_{n\to\infty} d(gTx_n,\ Tgx_n) = 0,$$

that is, z is a coincidence point of T and g.

Suppose now that (b) holds, that is, $(X,\ d)$ is complete, T and g are continuous and commuting. It is evident that (b) implies (a).

Suppose now that (c) holds, that is, $(g(X),\ d)$ is complete and $(X,\ d,\ \preceq)$ is non-decreasing-regular. As $\{gx_n\}_{n\geq 0}$ is a Cauchy sequence in the complete space $(g(X),\ d)$, so there exist $y \in g(X)$ such that $\{gx_n\} \to y$. Let $z \in X$ be any point such that $y = gz$, then $\{gx_n\} \to gz$. Indeed, as $(X,\ d,\ \preceq)$ is non-decreasing-regular and $\{gx_n\}$ is $\preceq$-non-decreasing and converging to gz, we deduce that $gx_n \preceq gz$ for all $n \geq 0$. Applying the contractive condition (iv), we get

$$d(gx_{n+1},\ Tz) = d(Tx_n,\ Tz) \leq \varphi\left(M(x_n,\ z)\right), \tag{7}$$

where

$$\begin{aligned}
M(x_n,\ z) &= \max\left\{ \begin{array}{c} d(gx_n,\ gz),\ d(gx_n,\ Tx_n),\ d(gz,\ Tz), \\ \frac{d(gx_n,\ Tz)+d(gz,\ Tx_n)}{2} \end{array} \right\} \\
&= \max\left\{ \begin{array}{c} d(gx_n,\ gz),\ d(gx_n,\ gx_{n+1}),\ d(gz,\ Tz), \\ \frac{d(gx_n,\ Tz)+d(gz,\ gx_{n+1})}{2} \end{array} \right\}.
\end{aligned}$$

Since $\lim_{n\to\infty} gx_n = gz$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$M(x_n,\ z) = d(gz,\ Tz). \tag{8}$$

By (7) and (8), we get

$$d(gx_{n+1},\ Tz) \leq \varphi\left(d(gz,\ Tz)\right).$$

Now, we claim that $d(gz, Tz) = 0$. If this is not true, then $d(gz, Tz) > 0$, which, by the fact that $\varphi(t) < t$ for all $t > 0$, implies

$$d(gx_{n+1},\ Tz) < d(gz,\ Tz).$$

Letting $n \to \infty$ in the above inequality and using $\lim_{n\to\infty} gx_n = gz$, we get

$$d(gz,\ Tz) < d(gz,\ Tz),$$

which is a contradiction. Hence we must have $d(gz,\ Tz) = 0$, that is, z is a coincidence point of T and g.

Suppose now that (d) holds, that is, $(X,\ d)$ is complete, $g(X)$ is closed and $(X,\ d,\ \preceq)$ is non-decreasing-regular. It follows from the fact that a closed subset of a complete metric space is also complete. Then, $(g(X),\ d)$ is complete and $(X,\ d,\ \preceq)$ is non-decreasing-regular. Thus (c) can apply here.

Suppose now that (e) holds, that is, $(X,\ d)$ is complete, g is continuous, the pair $(T,\ g)$ is O-compatible and $(X,\ d,\ \preceq)$ is non-decreasing-regular. As $(X,\ d)$ is complete, so there exists $z \in X$ such that $\{gx_n\} \to z$. Since $Tx_n = gx_{n+1}$ for all $n \geq 0$, we also have that $\{Tx_n\} \to z$. As g is continuous, then $\{ggx_n\} \to gz$. Furthermore, since the pair $(T,\ g)$ is O-compatible, we have $\lim_{n\to\infty} d(ggx_{n+1}, Tgx_n) = \lim_{n\to\infty} d(gTx_n,\ Tgx_n) = 0$. As $\{ggx_n\} \to gz$ the previous property means that $\{Tgx_n\} \to gz$.

Indeed, as $(X,\ d,\ \preceq)$ is non-decreasing-regular and $\{gx_n\}$ is $\preceq$-non-decreasing and converging to z, we deduce that $gx_n \preceq z$ for all $n \geq 0$. Applying the contractive condition (iv), we get

$$d(Tgx_n,\ Tz) \leq \varphi\left(M(gx_n,\ z)\right), \tag{9}$$

where

$$M(gx_n,\ z) = \max\left\{ \begin{array}{c} d(ggx_n,\ gz),\ d(ggx_n,\ Tgx_n),\ d(gz,\ Tz), \\ \frac{d(ggx_n,\ Tz)+d(gz,\ Tgx_n)}{2} \end{array} \right\}.$$

Since $\{ggx_n\} \to gz$, therefore there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$M(gx_n,\ z) = d(gz,\ Tz). \tag{10}$$

By (9) and (10), we get

$$d(Tgx_n,\ Tz) \leq \varphi\left(d(gz,\ Tz)\right).$$

Now, we claim that $d(gz, Tz) = 0$. If this is not true, then $d(gz, Tz) > 0$, which, by the fact that $\varphi(t) < t$ for all $t > 0$, implies

$$d(Tgx_n,\ Tz) < d(gz,\ Tz).$$

Letting $n \to \infty$ in the above inequality and using $\{Tgx_n\} \to gz$, we get

$$d(gz,\ Tz) < d(gz,\ Tz),$$

which is a contradiction. Hence we must have $d(gz,\ Tz) = 0$, that is, z is a coincidence point of T and g. ∎

Next we give an $n-$dimensional coincidence point theorem for mixed monotone mappings. For brevity, $(y_1,\ y_2,\ ...,\ y_n)$, $(v_1,\ v_2,\ ...,\ v_n)$ and $(y_0^1,\ y_0^2,\ ...,\ y_0^n)$ will be denoted by Y, V and Y_0 respectively. Consider the mappings F_Υ, $G : X^n \to X^n$ defined by

$$\begin{aligned} F_\Upsilon(Y) &= \begin{pmatrix} F(y_{\sigma_1(1)},\ y_{\sigma_1(2)},\ ...,\ y_{\sigma_1(n)}), \\ F(y_{\sigma_2(1)},\ y_{\sigma_2(2)},\ ...,\ y_{\sigma_2(n)}), \\ ...,\ F(y_{\sigma_n(1)},\ y_{\sigma_n(2)},\ ...,\ y_{\sigma_n(n)}) \end{pmatrix}, \quad (11) \\ \text{and } G(Y) &= (gy_1,\ gy_2,\ ...,\ gy_n), \text{ for } Y \in X^n. \end{aligned}$$

Under these conditions, the following properties hold:

Lemma 2.2. *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and let $F : X^n \to X$ and $g : X \to X$ be two mappings. Then*

(1) If there exist $y_0^1,\ y_0^2,\ ...,\ y_0^n \in X$ verifying $gy_0^i \preceq_i F(y_0^{\sigma_i(1)},\ y_0^{\sigma_i(2)},\ ...,\ y_0^{\sigma_i(n)})$, for all $i \in \Lambda_n$, then there exists $Y_0 \in X^n$ such that $G(Y_0) \sqsubseteq F_\Upsilon(Y_0)$.

(2) If there exists $\varphi \in \Phi$ such that

$$d(F(y_1,\ y_2,\ ...,\ y_n),\ F(v_1,\ v_2,\ ...,\ v_n)) \leq \varphi\left(M(y_1,\ ...,\ y_n,\ v_1,\ ...,\ v_n)\right), \quad (12)$$

for all $y_1,\ y_2,\ ...,\ y_n,\ v_1,\ v_2,\ ...,\ v_n \in X$ with $y_i \preceq_i v_i$, for $i \in \Lambda_n$, where

$$\begin{aligned} &M(y_1,\ ...,\ y_n,\ v_1,\ ...,\ v_n) \quad (13) \\ = &\max_{1\leq i\leq n} \left\{ \begin{array}{c} d(gy_i,\ gv_i),\ d(gy_i,\ F(y_{\sigma_i(1)},\ ...,\ y_{\sigma_i(n)})), \\ d(gv_i,\ F(v_{\sigma_i(1)},\ ...,\ v_{\sigma_i(n)})), \\ \frac{d(gy_i,\ F(v_{\sigma_i(1)},\ ...,\ v_{\sigma_i(n)}))+d(gv_i,\ F(y_{\sigma_i(1)},\ ...,\ y_{\sigma_i(n)}))}{2} \end{array} \right\}, \end{aligned}$$

then

$$\rho_n(F_\Upsilon(Y),\ F_\Upsilon(V)) \leq \varphi\left(M_{\rho_n}(Y,\ V)\right), \quad (14)$$

where

$$M_{\rho_n}(Y,\ V) = \max\left\{ \begin{array}{c} \rho_n(G(Y),\ G(V)),\ \rho_n(G(Y),\ F_\Upsilon(Y)),\ \rho_n(G(V),\ F_\Upsilon(V)), \\ \frac{\rho_n(G(Y),\ F_\Upsilon(V))+\rho_n(G(V),\ F_\Upsilon(Y))}{2} \end{array} \right\}, \quad (15)$$

for all $Y,\ V \in X^n$ with $G(Y) \sqsubseteq G(V)$.

Proof. (1) is obvious.

(2) Suppose that $G(Y) \sqsubseteq G(V)$ for $Y,\ V \in X^n$. For fixed $i \in A$, we have $y_{\sigma_i(t)} \preceq_t v_{\sigma_i(t)}$ for $t \in \Lambda_n$. From (12), we have

$$\begin{aligned} &d(F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ ...,\ y_{\sigma_i(n)}),\ F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ ...,\ v_{\sigma_i(n)})) \quad (16) \\ \leq\ &\varphi\left(M(y_1,\ y_2,\ ...,\ y_n,\ v_1,\ v_2,\ ...,\ v_n)\right), \end{aligned}$$

for all $i \in A$. Similarly, for fixed $i \in B$, we have $y_{\sigma_i(t)} \succeq_t v_{\sigma_i(t)}$ for $t \in \Lambda_n$. It follows from (12) that

$$\begin{aligned} & d(F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ ...,\ y_{\sigma_i(n)}),\ F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ ...,\ v_{\sigma_i(n)})) \\ \leq\ & d(F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ ...,\ v_{\sigma_i(n)}),\ F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ ...,\ y_{\sigma_i(n)})) \\ \leq\ & \varphi\left(M(y_1,\ y_2,\ ...,\ y_n,\ v_1,\ v_2,\ ...,\ v_n)\right), \end{aligned} \tag{17}$$

for all $i \in B$. By (1), (11), (16), (17) and φ is non-decreasing, we have

$$\rho_n(F_\Upsilon(Y),\ F_\Upsilon(V)) \leq \varphi\left(M_{\rho_n}(G(Y),\ G(V))\right),$$

for all $Y,\, V \in X^n$ with $G(Y) \sqsubseteq G(V)$. ∎

Theorem 2.3. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $F : X^n \to X$ and $g : X \to X$ be two mappings and $\Upsilon = (\sigma_1,\ \sigma_2,\ ...,\ \sigma_n)$ be an $n-$tuple of mappings from Λ_n into itself verifying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Suppose that the following properties are fulfilled:*

(i) $F(X^n) \subseteq g(X)$,

(ii) F has the mixed g-monotone property,

(iii) there exist $y_0^1,\ y_0^2,\ ...,\ y_0^n \in X$ verifying $gy_0^i \preceq_i F(y_0^{\sigma_i(1)},\ y_0^{\sigma_i(2)},\ ...,\ y_0^{\sigma_i(n)})$, for all $i \in \Lambda_n$,

(iv) there exist $\varphi \in \Phi$ satisfying (12). Also assume that at least one of the following conditions holds:

(a) $(X,\ d)$ is complete, F and g are continuous and the pair $(F,\ g)$ is $(O,\ \Upsilon)$-compatible,

(b) $(X,\ d)$ is complete, F and g are continuous and commuting,

(c) $(g(X),\ d)$ is complete and $(X,\ d,\ \preceq)$ is non-decreasing-regular,

(d) $(X,\ d)$ is complete, $g(X)$ is closed and $(X,\ d,\ \preceq)$ is regular,

(e) $(X,\ d)$ is complete, g is continuous, the pair $(F,\ g)$ is $(O,\ \Upsilon)$-compatible and $(X,\ d,\ \preceq)$ is non-decreasing-regular.

Then F and g have, at least, a $\Upsilon-$coincidence point.

Proof. It is only necessary to apply Theorem 2.1 to the mappings $T = F_\Upsilon$ and $g = G$ in the ordered metric space $(X^n,\ \rho_n,\ \sqsubseteq)$ taking into account all items of Lemma 1.1 and Lemma 2.2. ∎

Next, we derive the two dimensional version of Theorem 2.1. Define the mappings $T_F,\, T_G : X^2 \to X^2$, for all $(x,\ y) \in X^2$, by

$$T_F(x,\ y) = (F(x,\ y),\ F(y,\ x)) \text{ and } T_G(x,\ y) = (G(x,\ y),\ G(y,\ x)). \tag{18}$$

Lemma 2.4. *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and let $F,\, G : X^2 \to X$ be two mappings. Then*

(1) If F is d-continuous, then T_F is ρ_2-continuous.

(2) If F is G-increasing with respect to $\preceq$, then T_F is $(T_G,\ \sqsubseteq)$-non-decreasing.

(3) If there exist two elements $x_0,\ y_0 \in X$ with $G(x_0,\ y_0) \preceq F(x_0,\ y_0)$ and $G(y_0,\ x_0) \succeq F(y_0,\ x_0)$, then there exists a point $(x_0,\ y_0) \in X^2$ such that $T_G(x_0,\ y_0) \sqsubseteq T_F(x_0,\ y_0)$.

(4) For any $x,\ y \in X$, there exist $u,\ v \in X$ such that $F(x,\ y) = G(u,\ v)$ and $F(y,\ x) = G(v,\ u)$, then $T_F(X^2) \subseteq T_G(X^2)$.

(5) Assume there exists $\varphi \in \Phi$ such that

$$d(F(x,\ y),\ F(u,\ v)) \leq \varphi\left(M(x,\ y,\ u,\ v)\right), \tag{19}$$

where

$$M(x,\ y,\ u,\ v) = \max\left\{\begin{array}{c} d(G(x,\ y),\ G(u,\ v)),\ d(G(x,\ y),\ F(x,\ y)), \\ d(G(u,\ v),\ F(u,\ v)),\ d(G(y,\ x),\ G(v,\ u)), \\ d(G(y,\ x),\ F(y,\ x)),\ d(G(v,\ u),\ F(v,\ u)), \\ \frac{d(G(x,\ y),\ F(u,\ v))+d(G(u,\ v),\ F(x,\ y))}{2}, \\ \frac{d(G(y,\ x),\ F(v,\ u))+d(G(v,\ u),\ F(y,\ x))}{2} \end{array}\right\}, \tag{20}$$

for all $x,\ y,\ u,\ v \in X$, where $G(x,\ y) \preceq G(u,\ v)$ and $G(y,\ x) \succeq G(v,\ u)$, then

$$\rho_2(T_F(x,\ y),\ T_F(u,\ v)) \leq \varphi\left(M_{\rho_2}((x,\ y),\ (u,\ v))\right), \tag{21}$$

where

$$M_{\rho_2}((x,\ y),\ (u,\ v)) = \max\left\{\begin{array}{c} \rho_2(T_G(x,\ y),\ T_G(u,\ v)), \\ \rho_2(T_G(x,\ y),\ T_F(x,\ y)), \\ \rho_2(T_G(u,\ v),\ T_F(u,\ v)), \\ \frac{\rho_2(T_G(x,\ y),\ T_F(u,\ v))+\rho_2(T_G(u,\ v),\ T_F(x,\ y))}{2} \end{array}\right\}, \tag{22}$$

for all $(x,\ y),\ (u,\ v) \in X^2$, where $T_G(x,\ y) \sqsubseteq T_G(u,\ v)$.

(6) If the pair $\{F,\ G\}$ is generalized compatible, then the mappings T_F and T_G are O-compatible in $(X^2,\ \rho_2,\ \sqsubseteq)$.

(7) A point $(x,\ y) \in X^2$ is a coupled coincidence point of F and G if and only if it is a coincidence point of T_F and T_G.

Proof. Statement (1), (3), (4), and (7) are obvious.

(2) Assume that F is G-increasing with respect to $\preceq$ and let $(x,\ y),\ (u,\ v) \in X^2$ be such that $T_G(x,\ y) \sqsubseteq T_G(u,\ v)$. Then $G(x,\ y) \preceq G(u,\ v)$ and $G(y,\ x) \succeq G(v,\ u)$. Since F is G-increasing with respect to $\preceq$, we have that $F(x,\ y) \preceq F(u,\ v)$ and $F(y,\ x) \succeq F(v,\ u)$. Therefore $T_F(x,\ y) \sqsubseteq T_F(u,\ v)$ which shows that T_F is $(T_G,\ \sqsubseteq)$-non-decreasing.

(5) Suppose that there exists $\varphi \in \Phi$ such that

$$d(F(x,\ y),\ F(u,\ v)) \leq \varphi\left(M(x,\ y,\ u,\ v)\right),$$

for all $x,\ y,\ u,\ v \in X$, where $G(x,\ y) \preceq G(u,\ v)$ and $G(y,\ x) \succeq G(v,\ u)$ and let $(x,\ y),\ (u,\ v) \in X^2$ be such that $T_G(x,\ y) \sqsubseteq T_G(u,\ v)$. Therefore $G(x,\ y) \preceq G(u,\ v)$ and $G(y,\ x) \succeq G(v,\ u)$. From (19), we have

$$d(F(x,\ y),\ F(u,\ v)) \leq \varphi\left(M(x,\ y,\ u,\ v)\right). \tag{23}$$

Furthermore $G(y,x) \succeq G(v,u)$ and $G(x,y) \preceq G(u,v)$, the contractive condition (19), implies that

$$d(F(y,\ x),\ F(v,\ u)) \leq \varphi\left(M(x,\ y,\ u,\ v)\right). \tag{24}$$

Combining (23) and (24), we get

$$\max\left\{d(F(x,\ y),\ F(u,\ v)),\ d(F(y,\ x),\ F(v,\ u))\right\} \leq \varphi\left(M(x,\ y,\ u,\ v)\right). \tag{25}$$

It follows from (25) that

$$\begin{aligned} & \rho_2(T_F(x,\ y),\ T_F(u,\ v)) \\ = & \ \rho_2\left((F(x,\ y),\ F(y,\ x)),\ (F(u,\ v),\ F(v,\ u))\right) \\ = & \ \max\left\{d(F(x,\ y),\ F(u,\ v)),\ d(F(y,\ x),\ F(v,\ u))\right\} \\ \leq & \ \varphi\left(M(x,\ y,\ u,\ v)\right) \\ \leq & \ \varphi\left(M_{\rho_2}((x,\ y),\ (u,\ v))\right). \end{aligned}$$

(6) Let $\{(x_n,\ y_n)\} \subseteq X^2$ be any sequence such that $T_F(x_n,\ y_n) \overset{\rho_2}{\to} (x,\ y)$ and $T_G(x_n,\ y_n) \overset{\rho_2}{\to} (x,\ y)$ (note that it is not require to suppose that $\{T_G(x_n,\ y_n)\}$ is $\sqsubseteq$-monotone). Thus

$$\begin{aligned} & (F(x_n,\ y_n),\ F(y_n,\ x_n)) \overset{\rho_2}{\to} (x,\ y) \\ \Rightarrow & \ F(x_n,\ y_n) \overset{d}{\to} x \text{ and } F(y_n,\ x_n) \overset{d}{\to} y, \end{aligned}$$

and

$$\begin{aligned} & (G(x_n,\ y_n),\ G(y_n,\ x_n)) \overset{\rho_2}{\to} (x,\ y) \\ \Rightarrow & \ G(x_n,\ y_n) \overset{d}{\to} x \text{ and } G(y_n,\ x_n) \overset{d}{\to} y. \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{n\to\infty} F(x_n,\ y_n) &= \lim_{n\to\infty} G(x_n,\ y_n) = x \in X, \\ \lim_{n\to\infty} F(y_n,\ x_n) &= \lim_{n\to\infty} G(y_n,\ x_n) = y \in X. \end{aligned}$$

Since the pair $\{F,\ G\}$ is generalized compatible, therefore

$$\begin{aligned} \lim_{n\to\infty} d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(F(x_n,\ y_n),\ F(y_n,\ x_n))) &= 0, \\ \lim_{n\to\infty} d(F(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(F(y_n,\ x_n),\ F(x_n,\ y_n))) &= 0. \end{aligned}$$

In particular,

$$\begin{aligned} & \lim_{n\to\infty} \rho_2(T_G T_F(x_n,\ y_n),\ T_F T_G(x_n,\ y_n)) \\ = & \lim_{n\to\infty} \rho_2(T_G(F(x_n,\ y_n),\ F(y_n,\ x_n)),\ T_F(G(x_n,\ y_n),\ G(y_n,\ x_n))) \\ = & \lim_{n\to\infty} \rho_2 \begin{pmatrix} (G(F(x_n,\ y_n),\ F(y_n,\ x_n)),\ G(F(y_n,\ x_n),\ F(x_n,\ y_n))), \\ (F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ F(G(y_n,\ x_n),\ G(x_n,\ y_n))) \end{pmatrix} \\ = & \lim_{n\to\infty} \max \begin{Bmatrix} d(G(F(x_n,\ y_n),\ F(y_n,\ x_n)),\ F(G(x_n,\ y_n),\ G(y_n,\ x_n))), \\ d(G(F(y_n,\ x_n),\ F(x_n,\ y_n)),\ F(G(y_n,\ x_n),\ G(x_n,\ y_n))) \end{Bmatrix} \\ = & \ 0. \end{aligned}$$

Hence, the mappings T_F and T_G are O-compatible in $(X^2,\ \rho_2,\ \sqsubseteq)$. ∎

Theorem 2.5. *Let $(X,\ \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F,\, G : X^2 \to X$ be two generalized compatible mappings such that F is G-increasing with respect to $\preceq$, G is continuous and there exist two elements $x_0,\, y_0 \in X$ with*

$$G(x_0,\ y_0) \preceq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \succeq F(y_0,\ x_0).$$

Suppose that there exists $\varphi \in \Phi$ satisfying (19) and for any $x,\, y \in X$, there exist $u,\, v \in X$ such that

$$F(x,\ y) = G(u,\ v) \text{ and } F(y,\ x) = G(v,\ u). \tag{26}$$

Also suppose that either
(a) F is continuous or
(b) $(X,\ d,\ \preceq)$ is regular.
Then F and G have a coupled coincidence point.

Proof. It is only require to use Theorem 2.1 to the mappings $T = T_F$ and $g = T_G$ in the partially ordered metric space $(X^2,\ \rho_2,\ \sqsubseteq)$ by applying Lemma 2.4. ∎

Corollary 2.6. *Let $(X,\ \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F,\, G : X^2 \to X$ be two commuting mappings satisfying (19) and (26) such that F is G-increasing with respect to $\preceq$, G is continuous and there exist two elements $x_0,\, y_0 \in X$ with*

$$G(x_0,\ y_0) \preceq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \succeq F(y_0,\ x_0).$$

Also suppose that either
(a) F is continuous or
(b) $(X,\ d,\ \preceq)$ is regular.
Then F and G have a coupled coincidence point.

Next, we deduce results without g-mixed monotone property of F.

Corollary 2.7. *Let $(X,\ \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F : X^2 \to X$ and $g : X \to X$ be two mappings such that F is g-increasing with respect to $\preceq$ and there exist $\varphi \in \Phi$ such that*

$$d(F(x,\ y),\ F(u,\ v)) \leq \varphi\left(M_g(x,\ y,\ u,\ v)\right), \tag{27}$$

where

$$M_g(x,\ y,\ u,\ v) = \max\left\{\begin{array}{c} d(gx,\ gu),\ D(gx,\ F(x,\ y)), D\ (gu,\ F(u,\ v)), \\ \frac{D(gx,\ F(u,\ v))+D(gu,\ F(x,\ y))}{2}, \\ d(gy,\ gv),\ D(gy,\ F(y,\ x)),\ D(gv,\ F(v,\ u)), \\ \frac{D(gy,\ F(v,\ u))+D(gv,\ F(y,\ x))}{2} \end{array}\right\},$$

for all $x,\, y,\, u,\, v \in X$, where $gx \preceq gu$ and $gy \succeq gv$. Suppose that $F(X^2) \subseteq g(X)$, g is continuous and the pair $\{F,\, g\}$ is compatible. Also suppose that either
(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

If there exist two elements $x_0,\ y_0 \in X$ with

$$gx_0 \preceq F(x_0,\ y_0) \text{ and } gy_0 \succeq F(y_0,\ x_0).$$

Then F and g have a coupled coincidence point.

Corollary 2.8. *Let $(X,\ \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F : X^2 \to X$ and $g : X \to X$ be two mappings satisfying (27) such that F is g-increasing with respect to $\preceq$. Suppose that $F(X^2) \subseteq g(X)$, g is continuous and the pair $\{F,\ g\}$ is commuting. Also suppose that either*

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

If there exist two elements $x_0,\ y_0 \in X$ with

$$gx_0 \preceq F(x_0,\ y_0) \text{ and } gy_0 \succeq F(y_0,\ x_0).$$

Then F and g have a coupled coincidence point.

Now, we deduce result without mixed monotone property of F.

Corollary 2.9. *Let $(X,\ \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F : X^2 \to X$ be an increasing mapping with respect to $\preceq$ and there exists $\varphi \in \Phi$ such that*

$$d(F(x,\ y),\ F(u,\ v)) \leq \varphi\left(m(x,\ y,\ u,\ v)\right), \tag{28}$$

where

$$m(x,\ y,\ u,\ v) = \max\left\{\begin{array}{c} d(x,\ u),\ D(x,\ F(x,\ y)),\ D(u,\ F(u,\ v)), \\ \frac{D(x,\ F(u,\ v))+D(u,\ F(x,\ y))}{2}, \\ d(y,\ v),\ D(y,\ F(y,\ x)),\ D(v,\ F(v,\ u)), \\ \frac{D(y,\ F(v,\ u))+D(v,\ F(y,\ x))}{2} \end{array}\right\},$$

for all $x,\ y,\ u,\ v \in X$, where $x \preceq u$ and $y \succeq v$. Also suppose that either

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

If there exist two elements $x_0,\ y_0 \in X$ with

$$x_0 \preceq F(x_0,\ y_0) \text{ and } y_0 \succeq F(y_0,\ x_0).$$

Then F has a coupled fixed point.

Example 1. Suppose that $X = [0,\ 1]$, equipped with the usual metric $d : X^2 \to [0,\ +\infty)$ with the natural ordering of real numbers $\leq$. Let $F,\ G : X^2 \to X$ be defined as

$$\begin{aligned} F(x,\ y) &= \left\{\begin{array}{c} \frac{x^2-y^2}{3},\ \text{if } x \geq y, \\ 0,\ \text{if } x < y, \end{array}\right. \\ \text{and } G(x,\ y) &= \left\{\begin{array}{c} x^2-y^2,\ \text{if } x \geq y, \\ 0,\ \text{if } x < y. \end{array}\right. \end{aligned}$$

Define $\varphi : [0, +\infty) \to [0, +\infty)$ as follows

$$\varphi(t) = \begin{cases} \frac{t}{3}, & \text{for } t \neq 1, \\ 1, & \text{for } t = 1. \end{cases}$$

First, we shall show that the contractive condition (19) holds for the mappings F and G. Let $x, y, u, v \in X$ such that $G(x, y) \preceq G(u, v)$ and $G(y, x) \succeq G(v, u)$, we have

$$\begin{aligned} d(F(x, y), F(u, v)) &= \left| \frac{x^2 - y^2}{3} - \frac{u^2 - v^2}{3} \right| \\ &= \frac{1}{3} \left| G(x, y) - G(u, v) \right| \\ &= \frac{1}{3} d(G(x, y), G(u, v)) \\ &\leq \frac{1}{3} M(x, y, u, v) \\ &\leq \varphi\left(M(x, y, u, v)\right). \end{aligned}$$

Thus the contractive condition (19) holds for all $x, y, u, v \in X$. In addition, like in [12], all the other conditions of Theorem 2.5 are satisfied and $z = (0, 0)$ is a coupled coincidence point of F and G.

3. Application to integral equations

We study the existence of the solution to a Fredholm nonlinear integral equation, as an application of the results established in previous section. We shall consider the following integral equation, for all $p \in I = [a, b]$.

$$x(p) = \int_a^b (K_1(p, q) + K_2(p, q)) \left[f(q, x(q)) + g(q, x(q)) \right] dq + h(p). \tag{29}$$

Let Θ denote the set of all functions $\theta : [0, +\infty) \to [0, +\infty)$ satisfying

(i_θ) θ is non-decreasing,

(ii_θ) $\theta(p) \leq p$.

Condition 1. We assume that the functions K_1, K_2, f, g fulfill the following conditions:

(i) $K_1(p, q) \geq 0$ and $K_2(p, q) \leq 0$ for all $p, q \in I$.

(ii) There exist positive numbers λ, μ and $\theta \in \Theta$ such that for all $x, y \in \mathbb{R}$ with $x \succeq y$, the following conditions hold:

$$0 \leq f(q, x) - f(q, y) \leq \lambda \theta(x - y), \tag{30}$$

$$-\mu \theta(x - y) \leq g(q, x) - g(q, y) \leq 0. \tag{31}$$

(iii)

$$\max\{\lambda, \mu\} \sup_{p \in I} \int_a^b [K_1(p, q) - K_2(p, q)] dq \leq \frac{1}{4}. \tag{32}$$

Definition 24. [19]. A pair $(\alpha,\ \beta) \in X^2$ with $X = C(I,\ \mathbb{R})$, where $C(I,\ \mathbb{R})$ denote the set of all continuous functions from I to $\mathbb{R}$, is called a coupled lower-upper solution of (29) if, for all $p \in I$,

$$\begin{aligned} \alpha(p) &\leq \int_a^b K_1(p,\ q)\,[f(q,\ \alpha(q)) + g(q,\ \beta(q))]\,dq \\ &\quad + \int_a^b K_2(p,\ q)\,[f(q,\ \beta(q)) + g(q,\ \alpha(q))]\,dq + h(p), \\ \text{and } \beta(p) &\geq \int_a^b K_1(p,\ q)\,[f(q,\ \beta(q)) + g(q,\ \alpha(q))]\,dq \\ &\quad + \int_a^b K_2(p,\ q)\,[f(q,\ \alpha(q)) + g(q,\ \beta(q))]\,dq + h(p). \end{aligned}$$

Theorem 3.1. *Consider the integral equation (29) with $K_1,\ K_2 \in C(I \times I,\ \mathbb{R})$, $f,\ g \in C(I \times \mathbb{R},\ \mathbb{R})$ and $h \in C(I,\ \mathbb{R})$. Suppose that there exists a coupled lower-upper solution $(\alpha,\ \beta)$ of (29) and Condition 1 is satisfied. Then the integral equation (29) has a solution in $C(I,\ \mathbb{R})$.*

Proof. Consider $X = C(I,\ \mathbb{R})$, the natural partial order relation, that is, for x, $y \in C(I,\ \mathbb{R})$,

$$x \preceq y \Longleftrightarrow x(p) \leq y(p),\ \forall p \in I.$$

It is clear that X is a complete metric space with respect to the sup metric

$$d(x,\ y) = \sup_{p \in I} |x(p) - y(p)|\,.$$

Consider the following partial order on X^2 : for $(x,\ y),\ (u,\ v) \in X^2$,

$$(x,\ y) \preceq (u,\ v) \Longleftrightarrow x(p) \leq u(p) \text{ and } y(p) \geq v(p),\ \text{ for all } p \in I.$$

Define $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ as follows

$$\varphi(t) = \begin{cases} \frac{t}{2}, & \text{for } t \neq 1, \\ 1, & \text{for } t = 1, \end{cases}$$

and the mapping $F : X^2 \to X$ by

$$\begin{aligned} F(x,\ y)(p) &= \int_a^b K_1(p,\ q)\,[f(q,\ x(q)) + g(q,\ y(q))]\,dq \\ &\quad + \int_a^b K_2(p,\ q)\,[f(q,\ y(q)) + g(q,\ x(q))]\,dq + h(p), \end{aligned}$$

for all $p \in I$. It is easy to prove, like in [12], that F is increasing. Now for x, y, u, $v \in X$ with $x \succeq u$ and $y \preceq v$, we have

$$\begin{aligned} & F(x,y)(p) - F(u,v)(p) \\ = & \int_a^b K_1(p,q)[(f(q,x(q)) - f(q,u(q))) - (g(q,v(q)) - g(q,y(q)))]dq \\ & - \int_a^b K_2(p,q)[(f(q,v(q)) - f(q,y(q))) - (g(q,x(q)) - g(q,u(q)))]dq. \end{aligned}$$

Thus, by using (30) and (31), we get

$$\begin{aligned} & F(x,\ y)(p) - F(u,\ v)(p) \\ \leq & \int_a^b K_1(p,\ q)\left[\lambda\theta\left(x(q) - u(q)\right) + \mu\theta\left(v(q) - y(q)\right)\right]dq \\ & - \int_a^b K_2(p,\ q)\left[\lambda\theta\left(v(q) - y(q)\right) + \mu\theta\left(x(q) - u(q)\right)\right]dq. \end{aligned} \tag{33}$$

Since θ is non-decreasing and $x \succeq u$ and $y \preceq v$, we have

$$\begin{aligned} \theta\left(x(q) - u(q)\right) & \leq \theta\left(\sup_{q\in I}|x(q) - u(q)|\right) = \theta(d(x,\ u)), \\ \theta\left(v(q) - y(q)\right) & \leq \theta\left(\sup_{q\in I}|v(q) - y(q)|\right) = \theta(d(y,\ v)). \end{aligned}$$

Hence by (33), in fact that $K_2(p,\ q) \leq 0$, we obtain

$$\begin{aligned} & |F(x,\ y)(p) - F(u,\ v)(p)| \\ \leq & \int_a^b K_1(p,\ q)\left[\lambda\theta(d(x,\ u)) + \mu\theta(d(y,\ v))\right]dq \\ & - \int_a^b K_2(p,\ q)\left[\lambda\theta(d(y,\ v)) + \mu\theta(d(x,\ u))\right]dq \\ \leq & \int_a^b K_1(p,\ q)\left[\max\{\lambda,\ \mu\}\theta(d(x,\ u)) + \max\{\lambda,\ \mu\}\theta(d(y,\ v))\right]dq \\ & - \int_a^b K_2(p,\ q)\left[\max\{\lambda,\ \mu\}\theta(d(y,\ v)) + \max\{\lambda,\ \mu\}\theta(d(x,\ u))\right]dq. \end{aligned}$$

Since the objects on the right hand side of (33) are non-negative. Taking the supremum with respect to p, by using (32), we get

$$\begin{aligned} & d(F(x,\ y),\ F(u,\ v)) \\ \leq & \max\{\lambda,\ \mu\}\sup_{p\in I}\int_a^b (K_1(p,\ q) - K_2(p,\ q))\,dq.\left[\theta(d(x,\ u)) + \theta(d(y,\ v))\right] \\ \leq & \frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{4}. \end{aligned}$$

Thus

$$d(F(x,\ y),\ F(u,\ v)) \le \frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{4}. \tag{34}$$

Now, since θ is non-decreasing, we have

$$\begin{aligned} \theta(d(x,\ u)) &\le \theta\left(\max\left\{d(x,\ u),\ d(y,\ v)\right\}\right), \\ \theta(d(y,\ v)) &\le \theta\left(\max\left\{d(x,\ u),\ d(y,\ v)\right\}\right), \end{aligned}$$

which implies, by (ii_θ), that

$$\begin{aligned} \frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{2} &\le \theta\left(\max\left\{d(x,\ u),\ d(y,\ v)\right\}\right) \\ &\le \max\left\{d(x,\ u),\ d(y,\ v)\right\}. \end{aligned}$$

Hence

$$\frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{4} \le \frac{1}{2}\max\left\{d(x,\ u),\ d(y,\ v)\right\}. \tag{35}$$

Thus by (34) and (35), we have

$$\begin{aligned} d(F(x,\ y),\ F(u,\ v)) &\le \frac{1}{2}\max\left\{d(x,\ u),\ d(y,\ v)\right\} \\ &\le \frac{1}{2}m(x,\ y,\ u,\ v) \\ &\le \varphi\left(m(x,\ y,\ u,\ v)\right), \end{aligned}$$

which is the contractive condition of Corollary 2.9. Now, let $(\alpha,\ \beta) \in X^2$ be a coupled upper-lower solution of (29), then we have $\alpha(p) \le F(\alpha,\ \beta)(p)$ and $\beta(p) \ge F(\beta,\ \alpha)(p)$, for all $p \in I$, which shows that all hypothesis of Corollary 2.9 are satisfied. This proves that F has a coupled fixed point $(x,\ y) \in X^2$ which is the solution in $X = C(I,\ \mathbb{R})$ of the integral equation (29). ∎

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Bhavana Deshpande
Department of Mathematics, Govt. B.S. P.G. College, Jaora, Dist. Ratlam (M.P.), India
E-mail address: bhavnadeshpande@yahoo.com

Amrish Handa
Department of Mathematics Govt. P. G. Arts and Science College Ratlam (M.P.), India
E-mail address: amrishhanda83@gmail.com

Shamim Ahmad Thoker
Department of Mathematics Govt. P. G. Arts and Science College Ratlam (M.P.), India
E-mail address: shamimthoker@gmail.com