

Bayesian testing for the homogeneity of the shape parameters of several inverse Gaussian distributions

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Abstract

We develop the testing procedures about the homogeneity of the shape parameters of several inverse Gaussian distributions in our paper. We propose default Bayesian testing procedures for the shape parameters under the reference priors. The Bayes factor based on the proper priors gives the successful results for Bayesian hypothesis testing. For the case of the lack of information, the noninformative priors such as Jeffreys' prior or the reference prior can be used. Jeffreys' prior or the reference prior involves the undefined constants in the computation of the Bayes factors. Therefore under the reference priors, we develop the Bayesian testing procedures with the intrinsic Bayes factors and the fractional Bayes factor. Simulation study for the performance of the developed testing procedures is given, and an example for illustration is given.

Keywords: Fractional Bayes factor, intrinsic Bayes factor, reference prior, shape parameter.

1. Introduction

Suppose that the observations are inverse Gaussian distributed with μ and λ . Then the probability density function is given by

$$f(y) = \sqrt{\frac{\lambda}{2\pi}} y^{-\frac{3}{2}} \exp\left\{-\frac{\lambda(y-\mu)^2}{2\mu^2 y}\right\}, y > 0. \quad (1.1)$$

Here the mean parameter is μ and the scale parameter is λ . The shape parameter for the inverse Gaussian distribution is defined as η , where $\eta = \lambda/\mu$. The shape parameter has an important meaning in inverse Gaussian distribution.

The inverse Gaussian distribution for the analysis of right skewed data has been used in the wide areas, that is, life testing, reliability, economics and biology as discussed in Chhikara and Folks (1989) and Seshadri (1999).

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Chhikara and Folks (1989) derived an exact method of seeking a confidence interval for the ratio of means when the shape parameters of these two populations are the same. Consequently, the use of this exact method requires a test to check whether the shape parameters of the two populations are equal or not. The skewness and kurtosis of inverse Gaussian distribution involve the shape parameter. Specially, the skewness and the kurtosis are $3\sqrt{\mu/\lambda}$ and $3 + 15\mu/\lambda$, respectively. And the coefficient of variation is $\sqrt{\mu/\lambda}$. Obviously, the skewness, the kurtosis and the coefficient of variation (CV) are all directly related to the shape parameter. The CV test statistic or its reciprocal is very important in medical sciences, physical sciences, social sciences and finance, see Bai *et al.* (2011). Thus, it is important to test the equality for inverse Gaussian shape parameters. Niu *et al.* (2014) developed three tests for the equality of shape parameters in inverse Gaussian distributions.

In the present paper, we consider testing for the homogeneity of the shape parameters in several inverse Gaussian distributions. The Bayes factor based on the proper priors give the successful results for Bayesian hypothesis testing. But for the case of the lack of information, the noninformative priors such as the reference prior (Berger and Bernardo, 1989, 1992) or Jeffreys' prior can be used. Jeffreys' prior or the reference prior involves the undefined constants in the computation of the Bayes factors. The studies of Spiegelhalter and Smith (1982), O'Hagan (1995) and Berger and Pericchi (1996) have been done on this problem.

Spiegelhalter and Smith (1982) proposed the using of imaginary training sample to remove the undefined constants. However for multiple model comparisons, the Bayes factor of Spiegelhalter and Smith (1982) does not guarantee the coherency. Berger and Pericchi (1996) developed the intrinsic Bayes factor to eliminate the undefined constants using the portion of data. O'Hagan (1995) introduced the fractional Bayes factor to remove the undefined constant using the likelihood with the fraction. In many statistical applications (Kang *et al.*, 2013, 2014), these methods have been useful. Berger and Pericchi (2001) introduces the useful exposition for the objective Bayesian hypothesis testing.

We develop the default Bayesian testing about the homogeneity of the shape parameters in several inverse Gaussian distributions based on the Bayes factors. The remaining sections in this paper is given as follows. We provide the Bayes factors of hypothesis testing in Section 2. In Section 3, we develop the Bayesian testing using the intrinsic Bayes factors and the fractional Bayes factor with reference prior. Simulation study for the performance of the developed Bayesian testing is given, and an example for testing of homogeneity is given in Section 4.

2. Intrinsic and fractional Bayes factors

We consider hypotheses H_1, H_2, \dots, H_q . Under hypothesis H_i , the probability distribution is $f_i(\mathbf{x}|\theta_i)$. The $\pi_i(\theta_i)$ is the prior distribution for hypothesis H_i , and the p_i is the prior probability for hypothesis $H_i, i = 1, 2, \dots, q$. Then under the hypothesis H_i , the posterior probability is given by

$$P(H_i|\mathbf{x}) = 1 / \left(\sum_{j=1}^q \frac{p_j}{p_i} \cdot B_{ji} \right), \quad (2.1)$$

where B_{ji} is the Bayes factor for hypothesis H_j and hypothesis H_i , and is given as follow:

$$B_{ji} = \frac{m_j(\mathbf{x})}{m_i(\mathbf{x})} = \frac{\int f_j(\mathbf{x}|\theta_j)\pi_j(\theta_j)d\theta_j}{\int f_i(\mathbf{x}|\theta_i)\pi_i(\theta_i)d\theta_i}. \tag{2.2}$$

The B_{ji} provide the relative support for H_j and H_i . In the calculation of B_{ji} , the density form of prior distributions $\pi_i(\theta_i)$ and $\pi_j(\theta_j)$ required. In general, we can use the noninformative priors π_i^N such as Jeffreys' prior and the reference prior. But the noninformative prior π_i^N is not a density. Therefore the noninformative prior π_i^N in (2.2) lead to undefined constants in the computation of B_{ji} . To overcome this problem, O'Hagan (1995) developed the fractional Bayes factor, and Berger and Pericchi (1996) suggested the intrinsic Bayes factor.

One solution for this undefined constant is using the training sample. We denote $\mathbf{x}(l)$ as the training sample and denote $\mathbf{x}(-l)$ as the remainder of the data with

$$0 < m_i^N(\mathbf{x}(l)) < \infty, i = 1, \dots, q. \tag{2.3}$$

Then the posteriors $\pi_i^N(\theta_i|\mathbf{x}(l))$ are proper. Thus the Bayes factor $B_{ji}(l)$ is given by

$$B_{ji}(l) = \frac{\int f(\mathbf{x}(-l)|\theta_j, \mathbf{x}(l))\pi_j^N(\theta_j|\mathbf{x}(l))d\theta_j}{\int f(\mathbf{x}(-l)|\theta_i, \mathbf{x}(l))\pi_i^N(\theta_i|\mathbf{x}(l))d\theta_i} = B_{ji}^N \cdot B_{ij}^N(\mathbf{x}(l)), \tag{2.4}$$

where

$$B_{ji}^N = B_{ji}^N(\mathbf{x}) = \frac{m_j^N(\mathbf{x})}{m_i^N(\mathbf{x})} \text{ and } B_{ij}^N(\mathbf{x}(l)) = \frac{m_i^N(\mathbf{x}(l))}{m_j^N(\mathbf{x}(l))}$$

represent the Bayes factors for the full data \mathbf{x} and training samples $\mathbf{x}(l)$, respectively.

In the computation $B_{ij}^N(\mathbf{x}(l))$, Berger and Pericchi (1996) suggested the use of a minimal training sample. Since $B_{ji}(l)$ depends on the choice of the minimal training sample, Berger and Pericchi (1996) consider the average over all the possible minimal training samples. Let L denote the number of all possible minimal training samples. The arithmetic intrinsic Bayes factor (AIBF) for H_j and H_i is defined as

$$B_{ji}^{AI} = B_{ji}^N \times \frac{1}{L} \sum_{l=1}^L B_{ij}^N(\mathbf{x}(l)), \tag{2.5}$$

And the median intrinsic Bayes factor (MIBF) for H_j and H_i is defined as

$$B_{ji}^{MI} = B_{ji}^N \times ME[B_{ij}^N(\mathbf{x}(l))], \tag{2.6}$$

where the median for all the training sample Bayes factors is ME (Berger and Pericchi, 1998). Hence we can compute the posterior probability of H_i using B_{ji}^{AI} or B_{ji}^{MI} in (2.1)

The fractional Bayes factor (O'Hagan, 1995) have the similar concept as the insights of the intrinsic Bayes factor. To make proper prior, the fraction b used in each likelihood function. Therefore the fractional Bayes factor (FBF) for hypothesis H_j and hypothesis H_i is defined as

$$B_{ji}^F = B_{ji}^N \times \frac{\int L^b(\theta_i)\pi_i^N(\theta_i)d\theta_i}{\int L^b(\theta_j)\pi_j^N(\theta_j)d\theta_j} = B_{ji}^N \times \frac{m_i^b(\mathbf{x})}{m_j^b(\mathbf{x})}. \tag{2.7}$$

O'Hagan (1995) suggest three methods for using the fraction b . Let m is the size of the minimal training sample. The popular method for b is $b = m/n$. For this choice, we can refer to O'Hagan (1995, 1997) and the suggestion of Berger and Mortera in O'Hagan (1995).

3. Bayesian hypothesis testing procedures

Let $X_{ij}, i = 1, \dots, k, j = 1, \dots, n_i$ denote observations from inverse Gaussian distribution with the shape parameter η_i and the mean parameter μ_i . Then likelihood function is

$$f(\mathbf{x}|\eta_1, \dots, \eta_k, \mu_1, \dots, \mu_k) = (2\pi)^{-\frac{n}{2}} \left(\prod_{i=1}^k \prod_{j=1}^{n_i} x_{ij}^{-\frac{3}{2}} \right) \left(\prod_{i=1}^k (\mu_i \eta_i)^{\frac{n_i}{2}} \right) \exp\left\{-\sum_{i=1}^k \sum_{j=1}^{n_i} \frac{\eta_i (x_{ij} - \mu_i)^2}{2\mu_i x_{ij}}\right\}, \quad (3.1)$$

where $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_k)$ and $\mathbf{x}_i = (x_{i1}, \dots, x_{in_i})$ and $n = n_1 + \dots + n_k$. We want to test the hypotheses $H_1 : \eta_1 = \dots = \eta_k$ versus $H_2 : \eta_1 \neq \dots \neq \eta_k$ using the intrinsic Bayes factors and the fractional Bayes factor.

3.1. Bayesian testing procedure using the fractional Bayes factor

From (3.1) the likelihood function under the hypothesis $H_1 : \eta_1 = \dots = \eta_k \equiv \eta$ is

$$L_1(\eta, \mu_1, \dots, \mu_k | \mathbf{x}) = (2\pi)^{-\frac{n}{2}} \left(\prod_{i=1}^k \prod_{j=1}^{n_i} x_{ij}^{-\frac{3}{2}} \right) \eta^{\frac{n}{2}} \left(\prod_{i=1}^k \mu_i^{\frac{n_i}{2}} \right) \exp\left\{-\sum_{i=1}^k \sum_{j=1}^{n_i} \frac{\eta (x_{ij} - \mu_i)^2}{2\mu_i x_{ij}}\right\}. \quad (3.2)$$

The reference prior of $(\eta, \mu_1, \dots, \mu_k)$ in the hypothesis H_1 , developed by Kang *et al.* (2015), and is given by

$$\pi_1^N(\eta, \mu_1, \dots, \mu_k) \propto \eta^{-\frac{1}{2}} (1 + 2\eta)^{-\frac{1}{2}} \mu_1^{-1} \dots \mu_k^{-1}. \quad (3.3)$$

Then the component $m_1^b(\mathbf{x})$ of the FBF under H_1 using the likelihood (3.2) and the reference prior (3.3) is given as follow:

$$\begin{aligned} m_1^b(\mathbf{x}) &= \int_0^\infty \int_0^\infty \dots \int_0^\infty L_1^b(\eta, \mu_1, \dots, \mu_k | \mathbf{x}) \pi_1^N(\eta, \mu_1, \dots, \mu_k) d\eta d\mu_1 \dots d\mu_k \\ &= \int_0^\infty \int_0^\infty \dots \int_0^\infty (2\pi)^{-\frac{bn}{2}} \left(\prod_{i=1}^k \prod_{j=1}^{n_i} x_{ij}^{-\frac{3b}{2}} \right) \eta^{\frac{bn-1}{2}} (1 + 2\eta)^{-\frac{1}{2}} \left(\prod_{i=1}^k \mu_i^{\frac{bn_i}{2}-1} \right) \\ &\quad \times \exp\left\{-\frac{b\eta}{2} \sum_{i=1}^k \left[s_i \mu_i + \frac{n_i (\bar{x}_i - \mu_i)^2}{\mu_i \bar{x}_i} \right]\right\} d\eta d\mu_1 \dots d\mu_k, \end{aligned} \quad (3.4)$$

where $\bar{x}_i = \sum_{j=1}^{n_i} x_{ij}/n_i$ and $s_i = \sum_{j=1}^{n_i} (\frac{1}{x_{ij}} - \frac{1}{\bar{x}_i})$. For the hypothesis H_2 , the reference prior for $(\eta_1, \dots, \eta_k, \mu_1, \dots, \mu_k)$ is

$$\pi^N(\eta_1, \dots, \eta_k, \mu_1, \dots, \mu_k) \propto \prod_{i=1}^k \eta_i^{-\frac{1}{2}} (1 + 2\eta_i)^{-\frac{1}{2}} \mu_i^{-1}. \quad (3.5)$$

The above prior developed by Kang *et al.* (2004) and also see Kang *et al.* (2015). For the hypothesis H_2 , the likelihood function is

$$\begin{aligned}
 &L_2(\eta_1, \dots, \eta_k, \mu_1, \dots, \mu_k | \mathbf{x}) \\
 &= (2\pi)^{-\frac{n}{2}} \left(\prod_{i=1}^k \prod_{j=1}^{n_i} x_{ij}^{-\frac{3}{2}} \right) \left(\prod_{i=1}^k (\eta_i \mu_i)^{\frac{n_i}{2}} \right) \exp \left\{ - \sum_{i=1}^k \sum_{j=1}^{n_i} \frac{\eta_i (x_{ij} - \mu_i)^2}{2\mu_i x_{ij}} \right\}. \quad (3.6)
 \end{aligned}$$

Hence in H_2 , the component $m_2^b(\mathbf{x})$ of FBF using the reference prior (3.5) and the likelihood (3.6) is given by

$$\begin{aligned}
 m_2^b(\mathbf{x}) &= \int_0^\infty \dots \int_0^\infty \int_0^\infty \dots \int_0^\infty L_2^b(\eta_1, \dots, \eta_k, \mu_1, \dots, \mu_k | \mathbf{x}) \\
 &\times \pi_2^N(\eta_1, \dots, \eta_k, \mu_1, \dots, \mu_k) d\eta_1 \dots d\eta_k d\mu_1 \dots d\mu_k \\
 &= \int_0^\infty \dots \int_0^\infty \int_0^\infty \dots \int_0^\infty (2\pi)^{-\frac{bn}{2}} \left(\prod_{i=1}^k \prod_{j=1}^{n_i} x_{ij}^{-\frac{3b}{2}} \right) \left(\prod_{i=1}^k \eta_i^{\frac{bn_i-1}{2}} (1+2\eta_i)^{-\frac{1}{2}} \mu_i^{\frac{bn_i}{2}-1} \right) \\
 &\times \exp \left\{ - \sum_{i=1}^k \frac{b\eta_i}{2} \left[s_i \mu_i + \frac{n_i(\bar{x}_i - \mu_i)^2}{\mu_i \bar{x}_i} \right] \right\} d\eta_1 \dots d\eta_k d\mu_1 \dots d\mu_k. \quad (3.7)
 \end{aligned}$$

Therefore the element B_{21}^N of FBF is given by

$$B_{21}^N = \frac{S_2(\mathbf{x})}{S_1(\mathbf{x})}, \quad (3.8)$$

where

$$\begin{aligned}
 S_1(\mathbf{x}) &= \int_0^\infty \int_0^\infty \dots \int_0^\infty \eta^{\frac{n-1}{2}} (1+2\eta)^{-\frac{1}{2}} \left(\prod_{i=1}^k \mu_i^{\frac{n_i}{2}-1} \right) \\
 &\times \exp \left\{ - \frac{\eta}{2} \sum_{i=1}^k \left[s_i \mu_i + \frac{n_i(\bar{x}_i - \mu_i)^2}{\mu_i \bar{x}_i} \right] \right\} d\eta d\mu_1 \dots d\mu_k
 \end{aligned}$$

and

$$\begin{aligned}
 S_2(\mathbf{x}) &= \int_0^\infty \dots \int_0^\infty \int_0^\infty \dots \int_0^\infty \left(\prod_{i=1}^k \eta_i^{\frac{bn_i-1}{2}} (1+2\eta_i)^{-\frac{1}{2}} \mu_i^{\frac{bn_i}{2}-1} \right) \\
 &\times \exp \left\{ - \sum_{i=1}^k \frac{\eta_i}{2} \left[s_i \mu_i + \frac{n_i(\bar{x}_i - \mu_i)^2}{\mu_i \bar{x}_i} \right] \right\} d\eta_1 \dots d\eta_k d\mu_1 \dots d\mu_k.
 \end{aligned}$$

And the ratio of marginal densities with fraction b is

$$\frac{m_1^b(\mathbf{x})}{m_2^b(\mathbf{x})} = \frac{S_1(\mathbf{x}; b)}{S_2(\mathbf{x}; b)}, \quad (3.9)$$

where

$$\begin{aligned}
 S_1(\mathbf{x}; b) &= \int_0^\infty \int_0^\infty \dots \int_0^\infty \eta^{\frac{bn-1}{2}} (1+2\eta)^{-\frac{1}{2}} \left(\prod_{i=1}^k \mu_i^{\frac{bn_i}{2}-1} \right) \\
 &\times \exp \left\{ - \frac{b\eta}{2} \sum_{i=1}^k \left[s_i \mu_i + \frac{n_i(\bar{x}_i - \mu_i)^2}{\mu_i \bar{x}_i} \right] \right\} d\eta d\mu_1 \dots d\mu_k
 \end{aligned}$$

and

$$S_2(\mathbf{x}; b) = \int_0^\infty \cdots \int_0^\infty \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^k \eta_i^{\frac{bn_i-1}{2}} (1+2\eta_i)^{-\frac{1}{2}} \mu_i^{\frac{bn_i}{2}-1} \right) \\ \times \exp \left\{ - \sum_{i=1}^k \frac{b\eta_i}{2} \left[s_i \mu_i + \frac{n_i(\bar{x}_i - \mu_i)^2}{\mu_i \bar{x}_i} \right] \right\} d\eta_1 \cdots d\eta_k d\mu_1 \cdots d\mu_k.$$

Therefore the FBF of H_2 versus H_1 is

$$B_{21}^F = \frac{S_1(\mathbf{x}; b) S_2(\mathbf{x})}{S_1(\mathbf{x}) S_2(\mathbf{x}; b)}. \quad (3.10)$$

In the computation of the Bayes factor (3.10), we are needed two dimensional integration.

3.2. Bayesian testing procedure using the intrinsic Bayes factor

In the fractional Bayes factor, the component B_{21}^N of the intrinsic Bayes factor is already calculated. Thus we only compute the marginal densities of the hypotheses H_1 and H_2 for the minimal training sample. The marginal densities of $(X_{11_1}, X_{11_2}, \dots, X_{kk_1}, X_{kk_2})$ are finite for all $1 \leq i_1 < i_2 \leq n_i, i = 1, \dots, k$ under each hypothesis (Kang *et al.*, 2004, 2015). Hence we think that the size of a minimal training sample is $2k$.

The marginal density $m_1^N(x_{11_1}, x_{11_2}, \dots, x_{kk_1}, x_{kk_2})$ under H_1 is given by

$$m_1^N(x_{11_1}, x_{11_2}, \dots, x_{kk_1}, x_{kk_2}) \\ = \int_0^\infty \int_0^\infty \cdots \int_0^\infty f(x_{11_1}, x_{11_2}, \dots, x_{kk_1}, x_{kk_2} | \eta, \mu_1, \dots, \mu_k) \\ \times \pi_1^N(\eta, \mu_1, \dots, \mu_k) d\eta d\mu_1 \cdots d\mu_k \\ = \int_0^\infty \int_0^\infty \cdots \int_0^\infty (2\pi)^{-k} \left(\prod_{i=1}^k \prod_{j=1}^2 x_{ii_j}^{-\frac{3}{2}} \right) \eta^{\frac{2k-1}{2}} (1+2\eta)^{-\frac{1}{2}} \\ \times \exp \left\{ - \frac{\eta}{2} \sum_{i=1}^k \left[w_i \mu_i + \frac{(x_{ii_1} + x_{ii_2} - 2\mu_i)^2}{\mu_i(x_{ii_1} + x_{ii_2})} \right] \right\} d\eta d\mu_1 \cdots d\mu_k,$$

where $w_i = (x_{ii_1} + x_{ii_2})/2$. And in H_2 , the marginal density $m_2^N(x_{11_1}, x_{11_2}, \dots, x_{kk_1}, x_{kk_2})$ is as follow:

$$m_2^N(x_{11_1}, x_{11_2}, \dots, x_{kk_1}, x_{kk_2}) \\ = \int_0^\infty \cdots \int_0^\infty \int_0^\infty \cdots \int_0^\infty f(x_{11_1}, x_{11_2}, \dots, x_{kk_1}, x_{kk_2} | \eta_1, \dots, \eta_k, \mu_1, \dots, \mu_k) \\ \times \pi_2^N(\eta_1, \dots, \eta_k, \mu_1, \dots, \mu_k) d\eta_1 \cdots d\eta_k d\mu_1 \cdots d\mu_k \\ = \int_0^\infty \cdots \int_0^\infty \int_0^\infty \cdots \int_0^\infty (2\pi)^{-k} \left(\prod_{i=1}^k \prod_{j=1}^2 x_{ii_j}^{-\frac{3}{2}} \right) \left(\prod_{i=1}^k \eta_i^{\frac{1}{2}} (1+2\eta_i)^{-\frac{1}{2}} \right) \\ \times \exp \left\{ - \sum_{i=1}^k \frac{\eta_i}{2} \left[w_i \mu_i + \frac{(x_{ii_1} + x_{ii_2} - 2\mu_i)^2}{\mu_i(x_{ii_1} + x_{ii_2})} \right] \right\} d\eta_1 \cdots d\eta_k d\mu_1 \cdots d\mu_k.$$

Thus the AIBF of hypotheses H_2 and H_1 is as follow:

$$B_{21}^{AI} = \frac{S_2(\mathbf{x})}{S_1(\mathbf{x})} \left[\frac{1}{L} \sum_{1_1 < 1_2}^{n_1} \cdots \sum_{k_1 < k_2}^{n_k} \frac{T_1(x_{11_1}, x_{11_2}, \dots, x_{kk_1}, x_{kk_2})}{T_2(x_{11_1}, x_{11_2}, \dots, x_{kk_1}, x_{kk_2})} \right], \tag{3.11}$$

where $L = \prod_{i=1}^k n_i(n_i - 1)/2$,

$$T_1(x_{11_1}, x_{11_2}, \dots, x_{kk_1}, x_{kk_2}) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \eta^{\frac{2k-1}{2}} (1 + 2\eta)^{-\frac{1}{2}} \times \exp \left\{ -\frac{\eta}{2} \sum_{i=1}^k \left[w_i \mu_i + \frac{(x_{ii_1} + x_{ii_2} - 2\mu_i)^2}{\mu_i(x_{ii_1} + x_{ii_2})} \right] \right\} d\eta d\mu_1 \cdots d\mu_k.$$

and

$$T_2(x_{11_1}, x_{11_2}, \dots, x_{kk_1}, x_{kk_2}) = \int_0^\infty \cdots \int_0^\infty \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^k \eta_i^{\frac{1}{2}} (1 + 2\eta_i)^{-\frac{1}{2}} \right) \times \exp \left\{ -\sum_{i=1}^k \frac{\eta_i}{2} \left[w_i \mu_i + \frac{(x_{ii_1} + x_{ii_2} - 2\mu_i)^2}{\mu_i(x_{ii_1} + x_{ii_2})} \right] \right\} d\eta_1 \cdots d\eta_k d\mu_1 \cdots d\mu_k.$$

Lastly, the MIBF of hypotheses H_2 and H_1 is as follow:

$$B_{21}^{MI} = \frac{S_2(\mathbf{x})}{S_1(\mathbf{x})} ME \left[\frac{T_1(x_{11_1}, x_{11_2}, \dots, x_{kk_1}, x_{kk_2})}{T_2(x_{11_1}, x_{11_2}, \dots, x_{kk_1}, x_{kk_2})} \right]. \tag{3.12}$$

In the AIBF and the MIBF of hypotheses H_2 and H_1 , the computations of the Bayes factors are needed two dimensional integration.

4. Numerical studies

For the comparison of the Bayesian testing procedures, we compute the posterior probabilities for several configurations of $(\eta_1, \mu_1), \dots, (\eta_k, \mu_k)$ and (n_1, \dots, n_k) . In particular, for fixed $(\eta_i, \mu_i), i = 1, \dots, k$, we take 200 independent random samples from the inverse Gaussian distributions with (η_i, μ_i) and sample sizes n_i , respectively. We want to test the hypotheses $H_1 : \eta_1 = \dots = \eta_k$ versus $H_2 : \eta_1 \neq \dots \neq \eta_k$. Under the assumption of the equal prior probability, the posterior probabilities of H_1 are calculated. In general the decision in hypothesis testing based on the Bayes factor is to choose the most probable hypothesis, the one for which the posterior probability of the hypothesis largest.

Tables 4.1 provides the average of the posterior probabilities and the standard deviations of posterior probabilities. And $P^F(\cdot)$ is the posterior probabilities of the hypothesis H_1 using FBF, $P^{AI}(\cdot)$ is the posterior probabilities of the hypothesis H_1 using AIBF, and $P^{MI}(\cdot)$ is the posterior probabilities of the hypothesis H_1 using MIBF. From the results of Table 4.1, the FBF, the AIBF and the MIBF conclude that the hypothesis H_1 is true when the values of η_2 are close to values of η_1 , and the hypothesis H_2 is true when the values of η_2 are far

from values of η_1 . In particular, the AIBF and the MIBF represent a similar values for all configurations. We can know that the posterior probabilities of the hypothesis H_1 based on all Bayes factors are increasing or decreasing when H_1 is true or H_2 is true as the sample sizes are increasing, respectively. Generally, the AIBF and the MIBF prefer the hypothesis H_1 than the FBF.

For practical application, we do not recommend the intrinsic Bayes factors because the number of the minimal training samples for this problem is very large despite the population size and sample sizes being small. Therefore the computation for intrinsic Bayes factors requires a lot of computing time. For example, if the population size is $k = 3$ and the sample sizes are $n_i = 10, i = 1, 2, 3$, then the number of the minimal training samples is 91,125.

Table 4.1 The Averages (standard deviations) for posterior probabilities

μ_1	μ_2	η_1	η_2	(n_1, n_2)	$P^F(H_1 \mathbf{x})$	$P^{AI}(H_1 \mathbf{x})$	$P^{MI}(H_1 \mathbf{x})$	
1.0	1.0	1.0	1.0	5,10	0.511 (0.112)	0.624 (0.126)	0.619 (0.128)	
				10,10	0.538 (0.127)	0.660 (0.132)	0.654 (0.134)	
				10,20	0.582 (0.133)	0.701 (0.135)	0.695 (0.137)	
				20,20	0.631 (0.132)	0.750 (0.128)	0.745 (0.128)	
		1.0	1.5	5,10	0.498 (0.135)	0.601 (0.150)	0.594 (0.150)	
				10,10	0.559 (0.135)	0.676 (0.142)	0.670 (0.142)	
				10,20	0.566 (0.157)	0.677 (0.163)	0.670 (0.163)	
				20,20	0.603 (0.163)	0.717 (0.160)	0.711 (0.161)	
		1.0	2.0	5,10	0.503 (0.150)	0.599 (0.166)	0.590 (0.166)	
				10,10	0.503 (0.174)	0.608 (0.191)	0.599 (0.190)	
				10,20	0.498 (0.215)	0.598 (0.231)	0.591 (0.230)	
				20,20	0.486 (0.237)	0.588 (0.253)	0.581 (0.253)	
	1.0	3.0	5,10	0.460 (0.180)	0.544 (0.203)	0.536 (0.201)		
			10,10	0.449 (0.206)	0.541 (0.229)	0.532 (0.228)		
			10,20	0.400 (0.246)	0.480 (0.276)	0.472 (0.273)		
			20,20	0.308 (0.245)	0.390 (0.280)	0.382 (0.277)		
	1.0	5.0	5,10	0.356 (0.224)	0.408 (0.255)	0.400 (0.250)		
			10,10	0.322 (0.218)	0.395 (0.248)	0.386 (0.245)		
			10,20	0.225 (0.244)	0.275 (0.277)	0.269 (0.272)		
			20,20	0.126 (0.179)	0.171 (0.219)	0.166 (0.215)		
	1.0	5.0	1.0	1.0	5,10	0.510 (0.117)	0.625 (0.129)	0.620 (0.131)
					10,10	0.544 (0.124)	0.668 (0.136)	0.663 (0.136)
					10,20	0.586 (0.146)	0.704 (0.155)	0.699 (0.155)
					20,20	0.627 (0.146)	0.746 (0.141)	0.741 (0.142)
1.0			1.5	5,10	0.501 (0.141)	0.606 (0.159)	0.599 (0.158)	
				10,10	0.526 (0.158)	0.642 (0.172)	0.636 (0.173)	
				10,20	0.561 (0.173)	0.670 (0.186)	0.664 (0.185)	
				20,20	0.588 (0.181)	0.702 (0.185)	0.696 (0.186)	
1.0			2.0	5,10	0.495 (0.150)	0.594 (0.167)	0.587 (0.166)	
				10,10	0.494 (0.183)	0.602 (0.203)	0.596 (0.203)	
				10,20	0.495 (0.205)	0.596 (0.220)	0.588 (0.219)	
				20,20	0.519 (0.215)	0.627 (0.230)	0.621 (0.230)	
1.0		3.0	5,10	0.445 (0.195)	0.527 (0.223)	0.521 (0.220)		
			10,10	0.417 (0.213)	0.510 (0.241)	0.502 (0.239)		
			10,20	0.395 (0.258)	0.469 (0.284)	0.462 (0.281)		
			20,20	0.277 (0.240)	0.352 (0.278)	0.346 (0.275)		
1.0		5.0	5,10	0.339 (0.221)	0.391 (0.252)	0.385 (0.247)		
			10,10	0.288 (0.224)	0.352 (0.261)	0.345 (0.256)		
			10,20	0.216 (0.236)	0.256 (0.270)	0.250 (0.265)		
			20,20	0.139 (0.186)	0.180 (0.224)	0.176 (0.220)		
1.0		10.0	1.0	1.0	5,10	0.496 (0.120)	0.612 (0.132)	0.607 (0.133)
					10,10	0.530 (0.152)	0.650 (0.167)	0.646 (0.168)
					10,20	0.592 (0.129)	0.712 (0.134)	0.708 (0.134)
					20,20	0.627 (0.149)	0.743 (0.152)	0.739 (0.153)
	1.0		1.5	5,10	0.504 (0.135)	0.609 (0.148)	0.602 (0.148)	
				10,10	0.539 (0.157)	0.651 (0.169)	0.645 (0.168)	
				10,20	0.559 (0.178)	0.667 (0.189)	0.661 (0.189)	
				20,20	0.584 (0.190)	0.695 (0.199)	0.689 (0.199)	
	1.0		2.0	5,10	0.474 (0.162)	0.570 (0.184)	0.563 (0.183)	
				10,10	0.508 (0.187)	0.613 (0.207)	0.606 (0.206)	
				10,20	0.475 (0.228)	0.570 (0.251)	0.563 (0.249)	
				20,20	0.493 (0.232)	0.598 (0.243)	0.591 (0.243)	
	1.0	3.0	5,10	0.422 (0.183)	0.503 (0.208)	0.496 (0.205)		
			10,10	0.420 (0.211)	0.513 (0.238)	0.504 (0.236)		
			10,20	0.431 (0.259)	0.510 (0.286)	0.502 (0.283)		
			20,20	0.321 (0.242)	0.405 (0.276)	0.398 (0.274)		
	1.0	5.0	5,10	0.372 (0.222)	0.432 (0.254)	0.426 (0.250)		
			10,10	0.272 (0.221)	0.335 (0.257)	0.328 (0.253)		
			10,20	0.237 (0.255)	0.278 (0.287)	0.272 (0.282)		
			20,20	0.141 (0.189)	0.183 (0.227)	0.179 (0.223)		

Example 4.1 This example presented in Niu *et al.* (2014). This data set can be obtained from <http://lib.stat.cmu.edu/DASL/Datafiles/Crash.html> originally provided by National Transportation Safety Administration. The data came from the trials in which stock automobiles were crashed into a wall at 35MPH, with dummies in the driver and front passenger seat. The information of injury variables on how the crash affected the dummies was collected, namely the extent of head injuries, chest deceleration, and left and right femur load. To illustrate our methods, here we only consider the problem of comparing the left femur load among three car makes, namely Dodge, Honda and Hyundai. Tian (2006) showed that the variable left femur load can be fitted by an inverse Gaussian distribution. Further Tian (2006) concluded that left femur loads were not different among these three car makes based on the proposed generalized test method. Ye *et al.* (2010) considered the hypothesis about whether the common mean of these populations are equal to 8.5.

The summary statistics of this data set are: $n_1 = 8, n_2 = 7, n_3 = 5, \bar{x}_1 = 8.578, \bar{x}_2 = 8.053, \bar{x}_3 = 15.968, s_1 = 0.203, s_2 = 0.150, s_3 = 0.082$. Niu *et al.* (2014) investigated whether the shape parameters of these three populations are equal or not. They showed that the null hypothesis of the common shape parameter can not be rejected with the p -value 0.854.

We want to test the hypotheses $H_1 : \eta_1 = \eta_2 = \eta_3$ versus $H_2 : \eta_1 \neq \eta_2 \neq \eta_3$ using the Bayesian tests. The Table 4.2 gives the results of the Bayes factors and the posterior probabilities. Therefore we know that all posterior probabilities using three Bayes factors select the same hypothesis. That is, all Bayes factor select the hypothesis H_1 because the posterior probability of the hypothesis H_1 is greater than 1/2. Therefore the test of Niu *et al.* (2014) and our Bayesian tests give the same results. Also the AIBF and the MIBF slightly seem to favor the simple hypothesis.

Table 4.2 Bayes factor and posterior probabilities of $H_1 : \eta_1 = \eta_2 = \eta_3$

B_{21}^F	$P^F(H_1 \mathbf{x})$	B_{21}^{AI}	$P^{AI}(H_1 \mathbf{x})$	B_{21}^{MI}	$P^{MI}(H_1 \mathbf{x})$
0.283	0.779	0.120	0.893	0.105	0.905

5. Concluding remarks

In our study, we proposed the objective Bayesian testing procedures using the intrinsic Bayes factors and the fractional Bayes factor for the homogeneity of the shape parameters of several inverse Gaussian distributions under the reference priors. We revealed that the proposed Bayes testing procedures give the reasonable decisions in all configuration of parameter. And the FBF trends to select the hypothesis H_2 than the MIBF and the AIBF. In the practical applications, we suggest using the FBF than the AIBF and MIBF because simplicity and implementation for the FBF.

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