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Some Special Cases of a Continuous Time-Cost Tradeoff Problem with Multiple Milestones under a Chain Precedence Graph

Byung-Cheon Choi
Department of Business Administration, Chungnam National University

Jibok Chung*
Department of Retail Management, Kongju National University

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ABSTRACT

We consider a time-cost tradeoff problem with multiple milestones under a chain precedence graph. In the problem, some penalty occurs unless a milestone is completed before its appointed date. This can be avoided through compressing the processing time of the jobs with additional costs. We describe the compression cost as the convex or the concave function. The objective is to minimize the sum of the total penalty cost and the total compression cost. It has been known that the problems with the concave and the convex cost functions for the compression are NP-hard and polynomially solvable, respectively. Thus, we consider the special cases such that the cost functions or maximal compression amounts of each job are identical. When the cost functions are convex, we show that the problem with the identical costs functions can be solved in strongly polynomial time. When the cost functions are concave, we show that the problem remains NP-hard even if the cost functions are identical, and develop the strongly polynomial approach for the case with the identical maximal compression amounts.

Keywords: Project Scheduling, Time-Cost Tradeoff, Computational Complexity, Chain Precedence Graph

* Corresponding Author, E-mail: jbchung@kongju.ac.kr

1. INTRODUCTION

The time-cost tradeoff problem (TCTP) is the project scheduling such that the processing times can be decreased through the expenditure of additional resources such as labor and capital. Let the decrease in the processing time be referred to as *compression*. The TCTP was initiated from (Fulkerson, 1961; Ford and Fulkerson, 1962; Kelley, 1961). Afterwards, many models have been introduced (see (Artigues *et al*., 2008; Brucker *et al*., 1999; Demeulemeester and Herroelen, 2002; Weglarz, 1999; Weglarz *et al*., 2011) for a comprehensive review). The classical TCTP has a single milestone for the overall project, that is, the last job, and the cost function for compressing a job is linear. In reality, however,

- There exist multiple milestones throughout a project. For example, a venture capital firm invests small sum at first, and increases or decreases the investments depending on the progress of the project (Bell, 2000; Choi and Chung, 2014; Sahlman, 1994);
- The relation between the cost and the compression amount may be nonlinear (Moussourakis and Haksever, 2010), which may be described as the concave or the convex function. Note that the concave and convex functions mean the general laws of increasing

and diminishing marginal returns, respectively (Berman, 1964; Choi and Park, 2015; Falk and Horowitz, 1972; Lamberson and Hocking, 1970).

To reflect this situation, we consider a TCTP with multiple milestones such that the cost function for compressing the job is concave or convex. Furthermore, the structure of the precedence graph is described by the chain that can be found in a product development process following a sequential pattern (Roemer and Ahmadi, 2004).

Our problems can be formally stated as follows. The TCTP is described by a directed graph $G = (V, A)$, where $V = \{1, 2, \cdots, n\}$ is the set of jobs and A is the set of precedence relations. Relation $(i, j) \in A$ means that job j can be started after job i is completed. Let $D \subseteq V$ be the set of milestones. Note that without loss of generality, the problem with $D \subset V$ can be transformed into the one with $D = V$ through the simple argument in (Choi and Park, 2015). Thus, for simplicity of notation, we assume that $D = V$. Associated with job j is a initial processing time p_j and a maximal compression amount uj, $j = 1, 2, \cdots, n$. Let w_j and d_j be the penalty cost for tardiness and the due date of job j, respectively. Let $x = (x_1, x_2, \cdots, x_n)$ be a vector for which x_j is the compressed amount of job j and $0 \le x_j \le u_j$, $j = 1, 2, \cdots, n$. Let $f_j(x_j)$ be the cost arising from compressing x_j of job j, and a non-decreasing concave or convex function. Let $C_j(x)$ be the completion time of job j under x. Then, our problem is defined as

$$\begin{aligned} &\textit{minimize} \sum_{j\in T(x)} w_j + \sum_{j=1}^{n} f_j(x_j) \\ &\textit{subject to } 0 \le x_j \le u_j,\ j = 1, 2, \cdots, n, \end{aligned}$$

where $T(x) = \{j \mid C_j(x) > d_j\}$ is the set of tardy job sunder x. Let the TCTP with the convex and the concave compression functions be referred to as *convex-TCTP* and *concave-TCTP*, respectively. Let job j be referred to as a *just-in-time (JIT)* job in x if it is completed exactly on its due date, that is, $C_j(x) = d_j$.

It has been known from (Choi and Park, 2015) that the convex-TCTP is polynomially solvable while the concave-TCTP is NP-hard. In this paper, we consider their special cases such that

- The compression cost functions of each job are identical, that is $f_j(x) = f(x)$, $j = 1, 2, \cdots, n$. This implies that the level of each resource such as man or machine for compressing the jobs is similar or;
- The maximal compression amounts of each job are identical, that is, $u_j = u$, $j = 1, 2, \cdots, n$. This implies that each job has a similar attribute in the aspect of compression.

For simplicity of notation, let convex- and concave-TCTP with $f_j(x) = f(x)$, $j = 1, 2, \cdots, n$, be denoted *convex-* and *concave-TCTP-f*, respectively. Similarly, let convex- and concave-TCTP with $u_j = u$, $j = 1, 2, \cdots, n$ be denoted *convex-* and *concave-TCTP-u*, respectively. Now, we introduce the optimality properties in (Choi and Park, 2015) below.

- The TCTP has an optimal schedule satisfying at least one of the following conditions:
  i) All jobs are uncompressed;
  ii) There exists at least one JIT job
- The TCTP has an optimal schedule such that jobs processed after the last JIT job are uncompressed.

Throughout the paper, we consider only the schedules satisfying the properties above.

The rest of the paper is organized as follows. Section 2 presents a strongly polynomial-time approach for the convex-TCTP-f. Section 3 proves the NP-hardness of the concave-TCTP-f and the strong polynomiality of the concave-TCTP-u. Finally, Section 4 discussesour conclusions.

2. CONVEX-TCTP

In this section, we prove the strong polynomiality of the convex-TCTP-f. Note that the general convex-TCTP can be reduced to the convex-TCTP-u as follows. Let

$$\bar{f} = \begin{cases} f(x), & \textit{if } 0 \le x_j \le u_j \\ \dfrac{(M - f(u_j))(x - u_j)}{(u_{max} - u_i)} + f(u_j), & \textit{if } u_j \le x_j \le u_{max} \end{cases}$$

where $M > 0$ is a sufficiently large value and $u_{max} = \max\{u_j \mid j = 1, 2, \cdots, n\}$. Thus, the strong polynomiality of the convex-TCTP-u depends on the computational complexity of $CON(k, l)$ defined in Proposition 1 below. Thus, for the convex-TCTP, we consider only the convex-TCTP-f. If $f(x)$ is a linear function, then it is known from (Choi and Chung, 2014) that the convex-TCTP is solved in strongly polynomial time, though each compression cost function is different. Thus, assume that $f(x)$ consists of linear and non-linear parts whose total number is m. Let the i-th part of $f(x)$ be denoted $f^1(x)$, and the domain of the $f^1(x)$ be $\{x_j \mid v_{i-1} \le x_j \le v_i\}$, $i = 1, 2, \cdots, m$. For the consistency of notation, let $v_0 = 0$, and for simplicity, assume that $v_m \ge \max\{u_j \mid j = k+1, k+2, \cdots, l\}$. Let $f(x)$ be described as follows.

$$f(x) = \begin{cases} f^1(x), \ \textit{if } \ 0 \le x_j \le v_1 \\ f^2(x), \ \textit{if } \ v_1 \le x_j \le v_2 \\ \qquad \vdots \\ f^m(x), \ \textit{if } \ v_{m-1} \le x_j \le v_m \end{cases}$$

Proposition 1: Choi and Park (2015) If jobs k and $l(k < l)$ are consecutive JIT jobs in an optimal schedule x^* of the convex-TCTP-f, then $\left(x_{k+1}^*, x_{k+2}^*, \cdots, x_l^*\right)$ is an optimal solution of the following problem.

$$CON(k, l): minimize \sum_{j=k+1}^{l} f(x_j)$$
$$subject\ to\ d_k + \sum_{j=k+1}^{l}(p_j - x_j) = d_k,$$
$$0 \le x_j \le u_j,\ \ j = k+1, k+2, \cdots, l.$$

Henceforth, we assume that jobs k and l are consecutive JIT jobs under an optimal schedule x^* of the convex-TCTP-f. Let O be the set of optimal solutions of $CON(k, l)$. It is observed that $\left(x_{k+1}^*, x_{k+2}^*, \cdots, x_l^*\right)$ has the smallest tardy weight in O. We introduce the strongly polynomial-time approach to obtain $\left(x_{k+1}^*, x_{k+2}^*, \cdots, x_l^*\right)$, which is denoted $x^*(k, l)$. Let α be the index of the last compressed job in $x^*(k, l)$, that is, $\alpha = \max\left\{ j \middle| x_j^* > 0,\ j = k+1, k+2, \cdots, l\right\}$.

Lemma 1: If $\alpha < l$, then $f^1(x)$ is a linear function.

Proof: Suppose that $f^1(x)$ is not linear. Let $\overline{x}(k, l)$ be a new schedule constructed by letting $\overline{x}_\alpha = x_\alpha^* - \varepsilon$ and $\overline{x}_l = \varepsilon$, where $\varepsilon > 0$ is a sufficiently small value. Note that $x_l^* = 0$ and $\overline{x}(k, l)$ and $x^*(k, l)$ have the same number of the tardy jobs. Then, since $f^1(x)$ is a strictly convex function from the non-linearity of $f^1(x)$,

$$\begin{aligned}&\sum_{j=k+1}^{l} f(x_j^*) - \sum_{j=k+1}^{l} f(\overline{x}_j)\\ &= \left(f(x_\alpha^*) + f(0)\right) - \left(f(x_\alpha^* - \varepsilon) + f(\varepsilon)\right)\\ &= \varepsilon\left(\frac{f(x_\alpha^*) - f(x_\alpha^* - \varepsilon)}{\varepsilon} - \frac{f(\varepsilon) - f(0)}{\varepsilon}\right) > 0\end{aligned}$$

This is a contradiction to Proposition 1. ■

Lemma 2: If $\alpha < l$, then $x_j^* \le v_1$ for $j = 1, 2, \cdots, \alpha$.

Proof: It can be proved by the similar argument of the proof of Lemma 1. ■

Theorem 1: If $\alpha < l$, then

$$x_j^* = \begin{cases} \min\{u_j, v_1\}, & for\ \ j = k+1, \cdots, \alpha - 1 \\ d_l - d_k - \sum_{j=k+1}^{\alpha} x_j^*, & for\ \ j = \alpha \end{cases}$$

Proof: It holds immediately from $f_j(x) = f(x)$ and Lemmas 1 and 2. ■

Henceforth, assume that job l is compressed, that is, $\alpha = l$. Let $P(x^*)$ be the set of the partially compressed jobs in $x^*(k, l)$.

Lemma 3: If $j_1 < j_2$ and $\{j_1, j_2\} \subseteq P(x^*)$, then $j_{j1}^* \ge j_{j2}^*$.

Proof: It can be proved by the similar argument of the proof of Lemma 1. ■

For simplicity of explanation, we introduce the following property. Let the i-th domain have the *partition* property in $x^*(k, l)$, if

$$\begin{cases} v_{i-1} \le x_j^* \le v_i, & if\ \ u_i \ge v_{i-1} \\ x_j^* \le u_i, & otherwise \end{cases}$$

Lemma 4: There exists the i-th domain with the partition property in $x^*(k, l)$.

Proof: If all jobs are fully compressed in $x^*(k, l)$, then the m-th domain has the partition property. Thus, hence forth, we assume that $x^*(k, l)$ has at least one partially compressed job. Let job j_1 be the partially compressed job such that

$$j_1 = argmin\left\{x_j^* \middle| j \in P(x^*)\right\}.$$

Assume that $v_{h-1} \le x_{j_1}^* < v_h$. If $h=m$, then Lemma 4 holds immediately from the definition of j_1. Thus, assume that $h < m$. Then, there exists a job j_2 such that $v_h < x_{j_2}^*$.
Let $\overline{x}(k, l)$ be a new schedule constructed by letting $\overline{x}_{j_1} = x_{j_1}^* + \varepsilon$ and $\overline{x}_{j_2} = x_{j_2}^* - \varepsilon$, where $\varepsilon > 0$ is a sufficiently small value. Note that $\overline{x}(k, l)$ and $x^*(k, l)$ have the same number of the tardy jobs. Then, since $x_{j_1}^* < v_h < x_{j2}^*$,

$$\begin{aligned}&\sum_{j=k+1}^{l} f(x_j^*) - \sum_{j=k+1}^{l} f(\overline{x}_j)\\ &= \left(f^h\left(x_{j_1}^*\right) + f^{h+1}\left(x_{j_2}^*\right)\right) - \left(f^h\left(x_{j_1}^* + \varepsilon\right) + f^{h+1}\left(x_{j_2}^* - \varepsilon\right)\right)\\ &= \varepsilon\left(\frac{f^{h+1}\left(x_{j_2}^*\right) - f^{h+1}\left(x_{j_2}^* - \varepsilon\right)}{\varepsilon} - \frac{f^h\left(x_{j_1}^* + \varepsilon\right) - f^h\left(x_{j_1}^*\right)}{\varepsilon}\right)\\ &> 0\end{aligned}$$

This is a contradiction to the optimality of $x^*(k, l)$. ■

Lemma 5: Suppose that the l-th domain has the partition property in $x^*(k, l)$.

i) $f^h(x)$ is a linear function, then

$$x_j^* = \begin{cases} \min\{u_j, v_h\}, & for\ \ j = k+1, \cdots, \beta - 1 \\ \lambda_j, & for\ \ j = \beta \\ \min\{u_j, v_{h-1}\}, & for\ \ j = \beta + 1, \cdots, l \end{cases}$$

where

- $\lambda_j = d_l - d_k - \sum_{j'=k+1}^{j-1} \min\{u_{j'}, v_h\}$
  $-\sum_{j'=j+1}^{l} \min\{u_{j'}, v_{h-1}\}$

- β is the first index j such that $v_{h-1} \le \lambda_j \le \min\{u_j, v_h\}$.

ii) If $f^h(x)$ is a strictly convex function,

$$x_j^* = \begin{cases} u_j, & if \ u_j \le t^* \\ t^*, & otherwise, \end{cases}$$

where t^* be such that

$$\sum_{j \in \{j | u_j \le t^*\}} u_j + \sum_{j \in \{j | u_j > t^*\}} t^* = d_l - d_k.$$

Proof: i) It holds immediately from the linearity of $f^h(x)$ and Lemma 3.

ii) Suppose that $x_j^* \ne x_{j_2}^*$, where $\{j_1, j_2\} \subseteq \{j | u_j > t^*\}$. For simplicity, assume that $x_j^* < x_{j_2}^*$. Let $\bar{x}(k, l)$ be anew schedule constructed by letting $x_{j_1} = x_{j_1}^* + \varepsilon$ and $x_{j_2} = x_{j_2}^* + \varepsilon$, where $\varepsilon > 0$ is a sufficiently small value. Note that $x^*(k, l)$ and $\bar{x}(k, l)$ have the same number of the tardy jobs. Then, since $f^h(x)$ is strictly convex and $x_j^* < x_{j_2}^*$,

$$\begin{aligned} &\sum_{j=k+1}^{l} f(x_j^*) - \sum_{j=k+1}^{l} f(\bar{x}_j) \\ &= \left(f^h\left(x_{j_1}^*\right) + f^h\left(x_{j_2}^*\right)\right) - \left(f^h\left(x_{j_1}^* + \varepsilon\right) + f^h\left(x_{j_2}^* - \varepsilon\right)\right) \\ &= \varepsilon \left(\frac{f^h\left(x_{j_2}^*\right) - f^h\left(x_{j_2}^* - \varepsilon\right)}{\varepsilon} - \frac{f^h\left(x_{j_1}^* + \varepsilon\right) - f^h\left(x_{j_1}^*\right)}{\varepsilon} \right) \\ &> 0 \end{aligned}$$

This is a contradiction. ■

Lemma 6: $x^*(k, l)$ can be obtained in strongly polynomial time.

Proof: Lemma 6 can be proven from showing that β and t^* in Lemma 5 can be obtained in strongly polynomial time. It is observed that β can be obtained trivially in strongly polynomial time, and, furthermore, t^* can be obtained through the following procedure.

Procedure t^*

Step 1: Sort the elements in $P_h = \{u_j | v_{h-1} < u_j < v_h\}$ in the increasing of u_j, and the resulting sequence is $(u_{\pi(1)}, u_{\pi(2)}, \cdots, u_{\pi(\bar{h})})$. where $\bar{h} = |P_h|$. Note that $u_{\pi(i)} < u_{\pi(i+1)}$, $i = 1, 2, \cdots, \bar{h} - 1$.

Step 2: Partition the interval $[v_{h-1}, v_h)$ into $(\bar{h}+1)$ sub-intervals in $\{I_1, \cdots, I_{\bar{h}+1}\}$, where $I_i = [u_{\pi(i-1)}, u_{\pi(i)})$, $i = 1, 2, \cdots, \bar{h}+1$. For consistency of notation, let $u_{\pi(0)} = v_{h-1}$ and $u_{\pi(\bar{h}+1)} = v_h$.

Step 3: For each $i = 1, 2, \cdots, \bar{h}+1$, construct $N_i = \{j | u_j < u_{\pi(i-1)}\}$ and calculate

$$\gamma_i = \frac{d_l - d_k - \sum_{j \in N_i} u_j}{l - k - |N_i|}$$

Step 4: If there exists an index i' such that $u_\pi(i'-1) \le \gamma_{i'} \le u_\pi(i')$ for some i', then $t^* = \gamma_{i'}$, while there exists no domain with the partition property in $x^*(k, l)$, otherwise.

Note that Procedure t^* can be done in $O(n)$. Thus, β and t^* in Lemma 5 can be obtained in strongly polynomial time. ■

Numerical Example: Consider the instance such that

$$f(x) = \begin{cases} \frac{x^2}{2}, & if \ 0 \le x \le 2 \\ 4x - 4, & if \ 2 \le x \le 4 \\ \frac{x^2}{2}, & if \ 4 \le x \le 6 \end{cases}$$

and

Job j	p_j	u_j	d_j
1	0	0	0
2	2	1	100
3	4	3	100
4	6	5	100
5	2	1	100
6	8	6	100
7	5	4	d_7

For simplicity, assume that jobs 1 and 7 are consecutive JIT jobs under an optimal schedule.

i) $d_7 = 12$

Since jobs 1 and 7 are consecutive JIT jobs, the total compression amount should be 15, which implies that the second domain has the partition property. By Lemma 5-i), $\beta = 6$. Thus $x_1^* = 0$, $x_2^* = 1$, $x_3^* = 3$, $x_4^* = 4$, $x_5^* = 1$, $x_6^* = 4$ and $x_7^* = 2$.

ii) $d_7 = 9$
In this case, the third domain has the partition property. By Lemma 5-ii), $t^* = 4.5$. Thus $x_1^* = 0$, $x_2^* = 1$, $x_3^* = 3$, $x_4^* = 4.5$, $x_5^* = 1$, $x_6^* = 4.5$ and $x_7^* = 4$.

Theorem 2: The convex-TCTP-f is solved in strong polynomial time if the compression cost function of each job is identical.

Proof: $x^*(k, l)$ satisfying Lemma 5 can be obtained in $O(n^2)$ by Lemma 6. Let $v\left(x^*(k, l)\right)$ be the total cost of $x^*(k, l)$. Then, we can obtain the optimal schedule by reduction to the shortest path problem. The reduced graph is same with the one in the proof of Theorem 4 except that the length of edge from $N(k)$ to $N(l)$ is $v\left(x^*(k, l)\right)$. Since the number of the edges in the reduced graph is $O(n^2)$ and the graph is acyclic, the reduced problem can be solved in $O(n^4)$ by the algorithm of (Ahuja *et al.*, 1990). The proof is complete.■

3. CONCAVE-TCTP

In this section, we show that the concave-TCTP-f remains NP-hard while the one with concave-TCTP-u can be solved in strongly polynomial time.

Theorem 3: The decision version of the concave-TCTP-f is NP-complete.

Proof: The proof is by reduction from the partition problem which is known to be NP-complete (Garey and Johnson, 1979).

Partition problem: Given m positive integers $a_1, a_2, \cdots, a_m$ such that $\sum_{j=1}^{m} a_j = A$, is there a subset $I \subset \{1, 2, \cdots, m\}$ such that $\sum_{j \in I} a_j = A/2$?

The decision version of the concave-TCTP-f is stated as follows: Given a threshold K, find a schedule x such that

$$\sum_{j \in T(x)} w_j + \sum_{j=1}^{n} f_j(x_j) \le K.$$

It is clear that the decision version of the concave-TCTP-f is in NP. Now, we reduce the partition problem to the decision version of the concave-TCTP-f. Given an instance of the partition problem, we construct an instance of the concave-TCTP-f as follows. Let $n = 2m+1$. Given $(2m+1)$ jobs such that for $j = 1, 2, \cdots, m$, let

- $p_{2j-1} = M^j$, $w_{2j-1} = a_j$, and $d_{2j-1} = \sum_{i=1}^{j-1} M^i + A$;
- $p_{2j} = M^j + a_j$, $w_{2j} = (m+1)A^2$, and $d_{2j} = \sum_{i=1}^{j} M^i + A$;
- $p_{2m+1} = 0$, $w_{2m+1} = (m+1)A^2$, and $d_{2m+1} = \sum_{i=1}^{m} M^i + A/2$,

where $M = A^{2m}$.

Let $u_j = p_j$, $j = 1, 2, \cdots, 2m+1$ and

$$f(x_j) = \begin{cases} \dfrac{A^2}{\varepsilon}, & \text{for } 0 \le x_j \le \varepsilon \\ A^2, & \text{for } \varepsilon \le x_j \le u_j \end{cases}$$

where $\in > 0$ is a sufficiently small value. Note that $f(x_j)$ is continuous and concave, and $f(u_j) = A^2$ for $j = 1, 2, \cdots, 2m+1$. Let $K = mA^2 + A/2$.

Suppose there exists a set $\bar{I}$ such that $\sum_{j \in I} a_j = \frac{A}{2}$. We can construct a schedule $\bar{x}$ by letting $\bar{x}_{2j-1} = u_{2j-1}$ and $\bar{x}_{2j} = 0$ if $j \in \bar{I}$, while $\bar{x}_{2j-1} = 0$ and $\bar{x}_{2j} = u_{2j}$, otherwise. Then, for $j \in \bar{I}$

$$\begin{aligned} C_{2j-1}(\bar{x}) &= \sum_{i=1}^{2j-2} \left(p_i - \bar{x}_i\right) \\ &= \sum_{i=1}^{j-1} M^i + \sum_{i \in \{1, 2, \cdots, j-1\} \cap \bar{I}} a_i \\ &\le \sum_{i=1}^{j-1} M^i + \frac{A}{2} \le d_{2j-1} \end{aligned}$$

while for $j \notin \bar{I}$

$$C_{2j-1}(\bar{x}) \ge p_{2j-1} = M^j > \sum_{i=1}^{j-1} M^i + A = d_{2j-1}.$$

For $j = 1, 2, \cdots, m$, furthermore,

$$\begin{aligned} C_{2j}(\bar{x}) &= \sum_{i=1}^{2j} (p_i - \bar{x}_i) \\ &= \sum_{i=1}^{j} M^i + \sum_{j \in \{1, 2, \cdots, j\} \cap \hat{I}} a_i \\ &\le \sum_{i=1}^{j} M^i + \frac{A}{2} \le d_{2j}, \end{aligned}$$

and, since $p_{2m+1} = 0$ and $\sum_{j \in \{1, 2, \cdots, m\} \setminus \hat{I}} a_j = A/2$,

$$\begin{aligned} C_{2m+1}(\bar{x}) &= C_{2m}(\bar{x}) \\ &= \sum_{i=1}^{m} M^i + \sum_{j \in \{1, 2, \cdots, m\} \cap \hat{I}} a_i \\ &= \sum_{i=1}^{m} M^i + \frac{A}{2} = d_{2m+1}. \end{aligned}$$

Note that $T(\bar{x}) = \left\{2j-1 \middle| j \in \{1, 2, \cdots, m\} \setminus \bar{I}\right\}$.

Since $\sum_{j \in \{1, 2, \cdots, m\} \setminus \hat{I}} a_i = A/2$,

$$\begin{aligned} &\sum_{j \in T(\bar{x})} w_j + \sum_{j=1}^{2m+1} f(\bar{x}_j) \\ &\sum_{j \in \{1, 2, \cdots, m\} \setminus \hat{I}} a_j + mA^2 = K. \end{aligned}$$

Suppose there exists a schedule $\hat{x}$ such that

$$\sum_{j\in T(\hat{x})} w_j + \sum_{j=1}^{2m+1} f(\hat{x}_j) \le K. \quad (1)$$

Let $\hat{P}$ be the sets of partially compressed jobs in $\hat{x}$. If there exists some partially compressed job j' such that $\hat{x}_{j'} \ge \varepsilon$, then we can construct a schedule $\tilde{x}$ by letting $\tilde{x}_{j'} = u_{j'}$ without the increase of the compression and tardiness costs. Thus, with out loss of generality, assume that $\hat{x}_j < \varepsilon$ for $j \in \hat{P}$.

Claim: $\left|\hat{P}\right| \le 1$.

Proof: If $\left|\hat{P}\right| > 1$, then we can select two jobs in $\{j, j'\}$, where $j < j'$, and construct a new schedule $\hat{x}'$ by letting $\hat{x}'_j = \hat{x}_j + \delta$ and $\hat{x}'_{j'} = \hat{x}_{j'} + \delta$, where $\delta = \min\{\varepsilon - \hat{x}'_j, \hat{x}_{j'}\}$. If $\hat{x}'_j \ge \varepsilon$, then $\hat{x}'_j = u_j$, which does not increase the compression and tardiness costs. Note that the total compression cost is unchanged and $T(\hat{x}') \subseteq (\hat{x})$. By repeatedly applying this argument, we can obtain the schedule satisfying Claim. □

By Claim, assume that $\left|\hat{P}\right| \le 1$. If $\left|\hat{P}\right| = 1$, then we can construct a new schedule $\tilde{x}$ by letting $\tilde{x}_j = 0$ for $j \in \hat{P}$. Due to the integrality of the elements in $\{p_j, u_j, d_j \mid j = 1, 2, \cdots, 2m+1\}$, the tardy set is unchanged, that is, $T(\tilde{x}) = T(\hat{x})$. Thus, we can obtain a schedule $\tilde{x}$ such that $\hat{P} = \varnothing$ and

$$\sum_{j\in T(\tilde{x})} w_j + \sum_{j=1}^{2m+1} f(\tilde{x}_j) \le K.$$

Thus, without loss of generality, assume that $\hat{P} = \varnothing$. Let $\hat{I}$ be set of indices j such that job $2j$ is fully compressed in $\hat{x}$. In order to satisfy inequality (1), at least one job in $\{2j-1, 2j\}$ must be fully compressed. Otherwise, some job in $\{2, 4, \cdots, 2m\}$ becomes tardy, which implies that inequality (1) is violated. Furthermore, the number of compressed jobs should be at most m. Otherwise, the total compression cost is larger than or equal to $(m+1)A^2$, which implies that inequality (1) is violated. Thus, exactly one job in $\{2j-1, 2j\}$ must become pressed. Then, inequality (1) can be rewritten as follows.

$$\sum_{j\in\hat{I}} a_j \le \frac{A}{2}. \quad (2)$$

Since job $2m+1$ should not be tardy in order to keep inequality (2), furthermore,

$$C_{2m+1}(\hat{x}) = \sum_{i=1}^{m} M^i + \sum_{j\in\{1,2,\cdots,m\}\setminus\hat{I}} a_j$$
$$\le \sum_{i=1}^{m} M^i + \frac{A}{2}. \quad (3)$$

By inequalities (2) and (3),

$$\sum_{j\in\hat{I}} a_j \sum_{j\in\{1,2,\cdots,m\}\setminus\hat{I}} a_j = \frac{A}{2}.$$

Thus, $\hat{I}$ is a solution to the partition problem. ■

The following optimality property for the concave-TCTP is induced from the proof of Lemma 3 in (Choi and Park, 2015).

Proposition 2: The number of a partially compressed job is at most one between consecutive JIT jobs in an optimal schedule of the concave-TCTP.

Based on Proposition 2, henceforth, we can reduce the concave-TCTP-u to the shortest path problem.

Theorem 4: The concave-TCTP-u can be solved in strong polynomial time.

Proof: We can reduce the concave-TCTP-u to the shortest path problem as follows. Let $N(k)$ be the node that when jobs in $\{1, 2, \cdots, k\}$ have been considered, job k is the current last JIT job. Let $N(0)$ and $N(n+1)$ be the source and the sink nodes, respectively. For $l = k+1, k+2, \cdots, n+1$, let $N(k)$ be connected to $N(l)$. The length between $N(k)$ and $N(l)$ can be calculated as the approach introduced in Claim below.

Claim: Suppose that there exists an optimal schedule such that jobs k and l are consecutive JIT jobs. Then, it can be done in $O(n^3)$ to determine the set of fully compressed jobs and at most one partially job in $\{k+1, k+2, \cdots, l\}$ under an optimal schedule.

Proof: Let $\Delta_{k,l} = d_k + \sum_{j=k+1}^{l} p_j - d_l$. Note that if $\Delta_{k,l}$ is the multiple of u, there exists no partially job in $\{k+1, k+2, \cdots, l\}$ under an optimal schedule. Let

$$\bar{n} = \left\lfloor \frac{\Delta_{k,l}}{u} \right\rfloor \text{ and } \delta_{k,l} = \Delta_{k,l} - \bar{n}u.$$

Suppose that job α is a partially compressed job in $\{k+1, k+2, \cdots, l\}$ in an optimal schedule. Henceforth, we reduce this case to the shortest path problem, referred to as $SPP(\alpha)$. Let $N_\alpha(i, n_{k,l})$ denote a node representing that the number of the fully compressed jobs is $n_{k,l}$, when the jobs in $\{k+1, k+2, \cdots, i\}$ have been considered. Note that $n_{k,l} \le \bar{n}$. Let $N_\alpha(k, 0)$ and $N_\alpha(l+1, \bar{n})$ be the source and the sink nodes, respectively. For $i = k+1, k+2, \cdots, l-1$, let $N_\alpha(i, n_{k,l})$ be connected to the following nodes:

i) $i+1 < \alpha$

- $N_\alpha(i+1, n_{k,l})$ with length

$$\begin{cases} w_{i+1}, & \text{if } d_k + \sum_{j=k+1}^{i+1} p_j - n_{k,l}u > d_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

- If $n_{k,l} < \bar{n}$, then $N_\alpha(i+1, n_{k,l}+1)$ with length

$$\begin{cases} w_{i+1}, & if \;\; d_k + \sum_{j=k+1}^{i+1} p_j - (n_{k,l}+1)u > d_{i+1} \\ 0, & otherwise \end{cases}$$

ii) $i+1=\alpha$

- $N_\alpha(i+1, n_{k,l})$ with length

$$\begin{cases} w_{i+1}, & if \;\; d_k + \sum_{j=k+1}^{i+1} p_j - n_{k,l}u - \delta_{k,l} > d_{i+1} \\ 0, & otherwise \end{cases}$$

iii) $i+1>\alpha$

- $N_\alpha(i+1, n_{k,l})$ with length

$$\begin{cases} w_{i+1}, & if \;\; d_k + \sum_{j=k+1}^{i+1} p_j - n_{k,l}u - \delta_{k,l} > d_{i+1} \\ 0, & otherwise \end{cases}$$

- If $n_{k,l} < \bar{n}$, then $N_\alpha(i+1, n_{k,l}+1)$ with length

$$\begin{cases} w_{i+1}, & if \;\; d_k + \sum_{j=k+1}^{i+1} p_j - (n_{k,l}+1)u - \delta_{k,l} > d_{i+1} \\ 0, & otherwise \end{cases}$$

For $i=1$, let $N_\alpha(i, n_{k,l})$ be connected to the sink node with length

$$\begin{cases} \infty, & if \;\; n_{k,l} \neq \bar{n} \\ 0, & otherwise \end{cases}$$

The objective is to find the shortest path between the source and the sink nodes. The number of nodes in $SPP(\alpha)$ is at most $O(n^2)$ and the number of the edges is at most $O(n^2)$ since the number of edges coming from each node is at most 2. Thus, since the reduced graph is acyclic, $SPP(\alpha)$ can be solved in $O(n^2)$ by the algorithm of (Ahuja *et al.*, 1990). Let $x^*(\alpha)$ be the sub schedule corresponding to the shortest path of $SPP(\alpha)$, and $v\left(x^*(\alpha)\right)$ be the total cost of $x^*(\alpha)$. Then, the one with $\min\left\{v\left(x^*(\alpha)\right) \middle| \alpha = k+1, k+2, \cdots, l\right\}$ becomes the corresponding sub-schedule in an optimal schedule. This procedure can be done in $O(n^3)$. □

Note that $\min\left\{v\left(x^*(\alpha)\right) \middle| \alpha = k+1, k+2, \cdots, l\right\}$ becomes the length between $N(k)$ and $N(l)$. The objective is to find the shortest path between the source and the sink nodes. The number of nodes in the reduced graph is at most $O(n)$, and the number of the edges is at most $O(n^2)$ since the number of edges coming from each node is at most n. Note that since the length of each edge can be obtained in $O(n^3)$ by the above claim, the reduced graph can be constructed in $O(n^5)$. Since the reduced graph is acyclic, the reduced problem can be solved in $O(n^2)$ by the algorithm of (Ahuja *et al.*, 1990). Thus, the concave-TCTP-u can be solved in $O(n^5)$. The proof is complete. ■

4. CONCLUDING REMARKS

We consider a TCTP with multiple milestones and completely ordered jobs, in which the objective is to minimize the sum of the total penalty cost and the total compression cost. In this paper, the compression cost is expressed as the convex or the concave function. Since the computational complexities of these problems have been established, we consider the cases with special properties that the compression functions or maximal compression amounts of each job are identical. For the case with the convex cost function, we introduce a procedure to solve the problem in strongly polynomial time if the compression functions are identical. For the case with the concave cost function, we show that the problem is NP-hard even if the compression cost functions are identical, and it can be solved in strongly polynomial time if the maximal compression amounts are identical.

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