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SUBNORMALITY OF $S_2(a, b, c, d)$ AND ITS BERGER MEASURE

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Abstract. We introduce a 2-variable weighted shift, denoted by $S_2(a, b, c, d)$, which arises naturally from analytic function space theory. We investigate when it is subnormal, and compute the Berger measure of it when it is subnormal. And we apply the results to investigate the relationship among 2-variable subnormal, hyponormal and 2-hyponormal weighted shifts.

1. Introduction

Let H be a complex separable Hilbert space and let $B(H)$ denote the Banach space of bounded linear operators on H, an operator $T \in B(H)$ is called normal if $T^*T = TT^*$, it is called subnormal if there is a Hilbert space $K \supseteq H$ and a normal operator N on K such that $NH \subseteq H$ and $T = N|_H$, and it is called hyponormal if $T^*T \geqslant TT^*$. Clearly, one sees that T is normal $\Longrightarrow$ T is subnormal $\Longrightarrow$ T is hyponormal. The weighted shift operator is often used to investigate the relationship among these types of operators.

Recall that if $\alpha : \alpha_0, \alpha_1, \ldots$ is a bounded sequence of positive numbers, the unilateral weighted shift W_α associated with α (called weight sequence) is the operator on $l^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n = \alpha_n e_{n+1} (n \geqslant 0)$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for $l^2(\mathbb{Z}_+)$. Given weighted sequence $\alpha : \alpha_0, \alpha_1, \ldots$, the weighted shift W_α is also denoted by $\text{shift}(\alpha_0, \alpha_1, \ldots)$.

The following result gives an elegant and simple condition to determine when a weighted shift is subnormal.

Lemma 1.1 (Berger theorem, cf. [1, 14]). *Let W_α be a weighted shift with weight sequence $\alpha : \alpha_0, \alpha_1, \ldots$, and define the moment of W_α by $\gamma_0 \triangleq 1$, $\gamma_n \triangleq \alpha_0^2 \alpha_1^2 \cdots \alpha_{n-1}^2 (n \geqslant 1)$. Then W_α is subnormal if and only if there exists a*

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probability measure ν *on* $[0, \| W_\alpha \|^2]$ *such that*

$$\gamma_n = \int_{[0,\| W_\alpha \|^2]} t^n d\nu(t) \quad (n \geqslant 1).$$

The probability measure ν *is called the Berger measure of* W_α*.*

In 2005, R. Curto et al. introduced a class of weighted shifts with weight sequence $\alpha_n = \sqrt{\frac{an+b}{cn+d}}$ $(n \geqslant 0)$, which is denoted by $S(a, b, c, d)$. Using the technique of Schur products of matrixes, they showed that

Lemma 1.2 (cf. [6])**.** *Let* $a, b, c, d > 0$ *satisfy* $ad - bc > 0$*. Then* $S(a, b, c, d)$ *is subnormal.*

The method in [6] can not be used to compute the Berger measure of $S(a, b, c, d)$ and the authors posed the problem of finding out it in the same paper. Note that the Berger measure is intimately related to a subnormal weighted shift through the Berger theorem and is essential in the computation of subnormal backward extension theorem. Cui and Duan (cf. [3]) completely answered this problem through the analytic function space theory.

Lemma 1.3 (cf. [3])**.** *Let* $a, b, c, d > 0$ *satisfy* $ad - bc > 0$*. Then* $S(a, b, c, d)$ *is subnormal with the Berger measure*

$$d\xi(t) = (\frac{c}{a})^{\frac{b}{a}} \frac{\Gamma(\frac{d}{c})}{\Gamma(\frac{b}{a})\Gamma(\frac{d}{c} - \frac{b}{a})} t^{\frac{b}{a}-1}(1 - \frac{ct}{a})^{\frac{d}{c}-\frac{b}{a}-1} dt,$$

which is defined on $[0, \frac{a}{c}]$*.*

Classes of weighted shift operator in $S(a, b, c, d)$ are used to handle many kinds of problems such as investigating the relationship of operator pair between the boundary of its Taylor spectrum and its essential spectrum (cf. [7]), between its hyponormality and weak 1-hyponormality (cf. [10]), investigating propagation phenomena (cf. [9]), Aluthge transforms of weighted shifts (cf. [11]), etc. It is interesting to consider the generalization of this type of weighted shift in the 2-variable case and we will do it in this paper.

Before proceeding, let us recall some necessary definitions and facts about 2-variable weighted shifts.

Let $T_1, T_2 \in B(H)$ be a commuting operator pair, $\mathbf{T} = (T_1, T_2)$ is called normal if T_1 and T_2 are both normal, $\mathbf{T}$ is called subnormal if there is a Hilbert space $K \supseteq H$ and a normal operator pair $\mathbf{N}$ on K such that $\mathbf{N}H \subseteq H$ and $\mathbf{T} = \mathbf{N}|_H$, and $\mathbf{T}$ is called hyponormal if $[\mathbf{T}^*, \mathbf{T}] \triangleq \begin{pmatrix} [T_1^*,T_1] & [T_2^*,T_1] \\ [T_1^*,T_2] & [T_2^*,T_2] \end{pmatrix} \geqslant 0$, where $[S, T] \triangleq ST - TS$ (cf. [4, 8]).

Let $\mathbb{Z}_+^2 = \mathbb{Z}_+ \times \mathbb{Z}_+$, a 2-variable weighted shift $\mathbf{T} = (T_1, T_2)$ is a commuting operator pair defined on $l^2(\mathbb{Z}_+^2)$ as follows:

$$T_1 e_{\mathbf{k}} \triangleq \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1}, T_2 e_{\mathbf{k}} \triangleq \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\{e_{\mathbf{k}} : \mathbf{k} \in \mathbb{Z}_+^2\}$ is the canonical basis for the Hilbert space $l^2(\mathbb{Z}_+^2)$, $\varepsilon_1 = (1,0)$, $\varepsilon_2 = (0,1)$, and $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} > 0$, for each $\mathbf{k} \in \mathbb{Z}_+^2$ ($\alpha_{\mathbf{k}}, \beta_{\mathbf{k}}$ are called the weight sequence of $\mathbf{T}$).

Definition 1.4 (cf. [13])**.** Given a 2-variable weighted shift $W_{\alpha,\beta}$ with weight sequence $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}}$, and $\mathbf{k} \in \mathbb{Z}_+^2$, the moment for $W_{\alpha,\beta}$ of order $\mathbf{k}$ is defined as follows:

$$\gamma_{\mathbf{k}}(W_{\alpha,\beta}) = \begin{cases} 1 & \mathbf{k} = 0, \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 & k_1 \geqslant 1, k_2 = 0, \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 & k_1 = 0, k_2 \geqslant 1, \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2 & k_1 \geqslant 1, k_2 \geqslant 1. \end{cases}$$

The following results describe the subnormality and hyponormality of a 2-variable weighted shift.

Lemma 1.5 (2-variable Berger theorem, cf. [13])**.** *A 2-variable weighted shift $\mathbf{T} = (T_1, T_2)$ is subnormal if and only if there is a probability measure μ defined on the rectangle $R = [0, \| T_1 \|^2] \times [0, \| T_2 \|^2]$ such that*

$$\gamma_{\mathbf{k}}(\mathbf{T}) = \int_R s^{k_1} t^{k_2} d\mu(s,t), \forall \mathbf{k} = (k_1, k_2) \in \mathbb{Z}_+^2.$$

The probability measure μ is called the Berger measure of $\mathbf{T}$.

Lemma 1.6 (Six-point test, cf. [4])**.** *Let $\mathbf{T} = (T_1, T_2)$ be a 2-variable weighted shift with weight sequences $\alpha_{\mathbf{k}}$, $\beta_{\mathbf{k}}$. Then $\mathbf{T}$ is hyponormal if and only if*

$$H_{\mathbf{T}}(\mathbf{k}) \triangleq \begin{pmatrix} \alpha_{\mathbf{k}+\varepsilon_1}^2 - \alpha_{\mathbf{k}}^2 & \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta_{\mathbf{k}+\varepsilon_2}^2 - \beta_{\mathbf{k}}^2 \end{pmatrix} \geqslant 0$$

for all $\mathbf{k} \in \mathbb{Z}_+^2$.

The rest of this paper is organized as follows: In Section 2, based on the computation about the Berger measure of the 2-variable weighted shift determined by the coordinate multiplication operators on the weighted Bergman space over the unit ball in $\mathbb{C}^2$ (which is well-known subnormal), we consider its generalization and introduce the definition of $S_2(a,b,c,d)$, and investigate when it is subnormal and determine its Berger measure when it is subnormal. In Section 3, we will give some application in considering the relationship between different types of the 2-variable weighted shifts.

2. Subnormality of $\mathbf{S_2(a,b,c,d)}$ and its Berger measure

First we will recall the definition of the weighted Bergman space over the unit ball in $\mathbb{C}^2$. This is the start point of our computation.

Definition 2.1 (cf. [15])**.** Let $\mathbb{B}_2$ be the open unit ball in $\mathbb{C}^2$, if $\nu > 2$, the weighted Bergman space is defined as follows:

$$\mathcal{A}_\nu^2(\mathbb{B}_2) \triangleq \{f \in H(\mathbb{B}_2) : \| f \|_\nu^2 = \int_{\mathbb{B}_2} |f(z)|^2 d\lambda^{(\nu)}(z) < +\infty\},$$

where $d\lambda^{(\nu)}(z) = C_\nu(1-|z|^2)^{\nu-3}d\lambda(z)$, $C_\nu = \frac{(\nu-1)(\nu-2)}{2}$, λ is the normalized Lebesgue measure on $\mathbb{C}^2$ such that $\lambda(\mathbb{B}_2) = 1$. Let m be the Lebesgue measure on $\mathbb{C}^2$, then $\lambda(E) = \frac{2}{\pi^2}m(E)$ for any measurable subset of $\mathbb{B}_2$.

It is well-known that $\mathcal{A}^2_\nu(\mathbb{B}_2)$ has a canonical orthonormal basis $\{e_{\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^2_+\}$, where $e_{\mathbf{k}} = [a_{|\mathbf{k}|}\frac{|\mathbf{k}|!}{\mathbf{k}!}]^{\frac{1}{2}}z^{\mathbf{k}}$, $|\mathbf{k}| = k_1 + k_2$, $\mathbf{k}! = k_1!k_2!$, and $a_0 = 1$, $a_n = \frac{\nu\cdots(\nu+n-1)}{n!} > 0 (n \geqslant 1)$. Then $\| z^{\mathbf{k}} \|^2 = \frac{1}{a_{|\mathbf{k}|}\frac{|\mathbf{k}|!}{\mathbf{k}!}} = \frac{\mathbf{k}!}{\nu\cdots(\nu+|\mathbf{k}|-1)} \triangleq w_{\mathbf{k}}$ (cf. [12]).

Example 2.2. Let $\mathbf{T} = (T_1, T_2) = (M_{z_1}, M_{z_2})$, M_{z_i} denotes the coordinate multiplication operator on $\mathcal{A}^2_\nu(\mathbb{B}_2)$. Then

$$T_1 e_{(k_1,k_2)} = z_1(a_{|\mathbf{k}|}\frac{|\mathbf{k}|!}{\mathbf{k}!})^{\frac{1}{2}}z_1^{k_1}z_2^{k_2} = \sqrt{\frac{a_{|\mathbf{k}|}\frac{|\mathbf{k}|!}{\mathbf{k}!}}{a_{|\mathbf{k}|+1}\frac{(|\mathbf{k}|+1)!}{(k_1+1)!k_2!}}}e_{\mathbf{k}+\varepsilon_1} = \sqrt{\frac{k_1+1}{\nu+|\mathbf{k}|}}e_{\mathbf{k}+\varepsilon_1},$$

$$T_2 e_{(k_1,k_2)} = z_2(a_{|\mathbf{k}|}\frac{|\mathbf{k}|!}{\mathbf{k}!})^{\frac{1}{2}}z_1^{k_1}z_2^{k_2} = \sqrt{\frac{a_{|\mathbf{k}|}\frac{|\mathbf{k}|!}{\mathbf{k}!}}{a_{|\mathbf{k}|+1}\frac{(|\mathbf{k}|+1)!}{(k_2+1)!k_1!}}}e_{\mathbf{k}+\varepsilon_2} = \sqrt{\frac{k_2+1}{\nu+|\mathbf{k}|}}e_{\mathbf{k}+\varepsilon_2}.$$

Let $\alpha_k = \sqrt{\frac{k_1+1}{\nu+|\mathbf{k}|}}$, $\beta_k = \sqrt{\frac{k_2+1}{\nu+|\mathbf{k}|}}$, then $T_1 e_{\mathbf{k}} = \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1}$, $T_2 e_{\mathbf{k}} = \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2}$, so $\mathbf{T}$ is a 2-variable weighted shift (cf. [3]).

It is well-known that the weighted shift $\mathbf{T}$ in the above example determined by the multiplication operator on $\mathcal{A}^2_\nu(\mathbb{B}_2)$ is subnormal, in the following theorem we will compute the Berger measure for it.

Recall that the Gamma function $\Gamma(x)$ and Beta function $B(x,y)$ is defined as follows (cf. [2]):

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, x > 0,$$

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, x, y > 0.$$

Moreover, it holds that $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, whenever $x, y > 0$ (cf. [2]).

Set $\Delta = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x, y \leqslant 1, 0 \leqslant x + y \leqslant 1\}$. We have the following result.

Theorem 2.3. *When $\nu > 2$, $\mathbf{T}$ in Example 2.2 is subnormal with the Berger measure*

$$d\mu(s,t) = (\nu-1)(\nu-2)(1-s-t)^{\nu-3}\chi_\Delta(s,t)dsdt.$$

Proof. First we will show that $\gamma_{\mathbf{k}}(\mathbf{T}) = \| z^{\mathbf{k}} \|^2$, where $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2_+$.

In fact, when $k_1 \geqslant 1$, $k_2 \geqslant 1$, for $i = 0, 1, 2, \ldots$, we have

$$M_{z_1}e_{(i,0)} = M_{z_1}\frac{z_1^i}{\sqrt{w_{(i,0)}}} = \frac{z_1^{i+1}}{\sqrt{w_{(i+1,0)}}}\frac{\sqrt{w_{(i+1,0)}}}{\sqrt{w_{(i,0)}}} = \sqrt{\frac{w_{(i+1,0)}}{w_{(i,0)}}}e_{(i+1,0)}.$$

Then, $\alpha_{(i,0)} = \sqrt{\frac{w_{(i+1,0)}}{w_{(i,0)}}}$.

Similarly,

$$M_{z_2}e_{(k_1,i)} = M_{z_2}\frac{z_1^{k_1}z_2^i}{\sqrt{w_{(k_1,i)}}} = \frac{z_1^{k_1}z_2^{i+1}}{\sqrt{w_{(k_1,i+1)}}}\frac{\sqrt{w_{(k_1,i+1)}}}{\sqrt{w_{(k_1,i)}}} = \sqrt{\frac{w_{(k_1,i+1)}}{w_{(k_1,i)}}}e_{(k_1,i+1)}.$$

Then $\beta_{(k_1,i)} = \sqrt{\frac{w_{(k_1,i+1)}}{w_{(k_1,i)}}}$.

Thus,

$$\begin{aligned}\gamma_{\mathbf{k}}(\mathbf{T}) &= \alpha^2_{(0,0)}\cdots\alpha^2_{(k_1-1,0)}\beta^2_{(k_1,0)}\cdots\beta^2_{(k_1,k_2-1)}\\ &= \frac{w_{(1,0)}}{w_{(0,0)}}\cdots\frac{w_{(k_1,0)}}{w_{(k_1-1,0)}}\frac{w_{(k_1,1)}}{w_{(k_1,0)}}\cdots\frac{w_{(k_1,k_2)}}{w_{(k_1,k_2-1)}}\\ &= \frac{w_{(k_1,k_2)}}{w_{(0,0)}} = \frac{\| z^{\mathbf{k}} \|^2}{\| z^{\mathbf{0}} \|^2}.\end{aligned}$$

Note that $\| z^{\mathbf{0}} \|^2 = 1$, we have $\gamma_{\mathbf{k}}(\mathbf{T}) = \| z^{\mathbf{k}} \|^2$.

When $k_1 = 0$, $k_2 \geqslant 1$, being similar to the proof above, we have

$$M_{z_2}e_{(0,i)} = M_{z_2}\frac{z_2^i}{\sqrt{w_{(0,i)}}} = \frac{z_2^{i+1}}{\sqrt{w_{(0,i+1)}}}\frac{\sqrt{w_{(0,i+1)}}}{\sqrt{w_{(0,i)}}} = \sqrt{\frac{w_{(0,i+1)}}{w_{(0,i)}}}e_{(0,i+1)}.$$

Then, $\beta_{(0,i)} = \sqrt{\frac{w_{(0,i+1)}}{w_{(0,i)}}}$.

Thus,

$$\gamma_{\mathbf{k}}(\mathbf{T}) = \beta^2_{(0,0)}\cdots\beta^2_{(0,k_2-1)} = \frac{w_{(0,1)}}{w_{(0,0)}}\cdots\frac{w_{(0,k_2)}}{w_{(0,k_2-1)}} = \frac{w_{(0,k_2)}}{w_{(0,0)}} = \| z^{\mathbf{k}} \|^2 .$$

When $k_1 \geqslant 1$, $k_2 = 0$, using the result of the case that $k_1 \geqslant 1$, $k_2 \geqslant 1$, we get

$$\gamma_{\mathbf{k}}(\mathbf{T}) = \alpha^2_{(0,0)}\cdots\alpha^2_{(k_1-1,0)} = \frac{w_{(1,0)}}{w_{(0,0)}}\cdots\frac{w_{(k_1,0)}}{w_{(k_1-1,0)}} = \frac{w_{(k_1,0)}}{w_{(0,0)}} = \| z^{\mathbf{k}} \|^2 .$$

Finally, it is obvious that $\gamma_{\mathbf{k}}(\mathbf{T}) = 1 = \| z^{\mathbf{k}} \|^2$ when $\mathbf{k} = 0$.

To summarize, we conclude that for all $\mathbf{k} \in \mathbb{Z}^2_+$, $\gamma_{\mathbf{k}}(\mathbf{T}) = \| z^{\mathbf{k}} \|^2$.

In the following, we will compute $\| z^{\mathbf{k}} \|^2$.

Note that

$$\| z^{\mathbf{k}} \|^2 = \frac{(\nu-1)(\nu-2)}{\pi^2}\int_{\mathbb{B}_2} |z_1^{k_1}z_2^{k_2}|^2(1-|z_1|^2-|z_2|^2)^{\nu-3}dm(z).$$

Let $z_1 = r_1e^{i\theta_1}, z_2 = r_2e^{i\theta_2}$, then we have

$$\| z^{\mathbf{k}} \|^2 = 4(\nu-1)(\nu-2)\int_{\Delta_1} r_1^{2k_1}r_2^{2k_2}(1-r_1^2-r_2^2)^{\nu-3}r_1r_2dr_1dr_2,$$

where $\Delta_1 = \{(x,y) \in \mathbb{R}^2 : x, y \geqslant 0, 0 \leqslant x^2+y^2 \leqslant 1\}$.

Let $s = r_1^2, t = r_2^2$, we get

$$\| z^{\mathbf{k}} \|^2 = (\nu-1)(\nu-2)\int_{\Delta} s^{k_1}t^{k_2}(1-s-t)^{\nu-3}dsdt.$$

Let

$$d\mu(s,t) = (\nu-1)(\nu-2)(1-s-t)^{\nu-3}\chi_\Delta(s,t)dsdt,$$

then

$$\gamma_{\mathbf{k}}(\mathbf{T}) = \| z^{\mathbf{k}} \|^2 = \int_{[0,1]^2} s^{k_1} t^{k_2} d\mu(s,t).$$

Then it follows from Berger theorem that when $\nu > 2$, $\mathbf{T}$ is subnormal with the Berger measure $d\mu(s,t) = (\nu-1)(\nu-2)(1-s-t)^{\nu-3}\chi_\Delta(s,t)dsdt$. $\square$

Based on the computations on the weighted Bergman space $\mathcal{A}^2_\nu(\mathbb{B}_2)$ and the 1-variable case, we introduce a new kind of operator pairs $S_2(a,b,c,d)$ as following.

Definition 2.4. Let $a,b,c,d > 0$ satisfy $ad - 2bc \neq 0$. We denote the 2-variable weighted shift with weight sequences $\alpha_{\mathbf{k}} = \sqrt{\frac{ak_1+b}{c(k_1+k_2)+d}}$, $\beta_{\mathbf{k}} = \sqrt{\frac{ak_2+b}{c(k_1+k_2)+d}}$ by $S_2(a,b,c,d)$.

In the following, we will investigate the subnormality of $S_2(a,b,c,d)$ and its Berger measure. Generalizing Theorem 2.3, we have:

Theorem 2.5. *The 2-variable weighted shift $W_{\alpha,\beta}$ with weight sequences $\alpha_{\mathbf{k}} = \sqrt{\frac{k_1+q}{k_1+k_2+p}}$, $\beta_{\mathbf{k}} = \sqrt{\frac{k_2+q}{k_1+k_2+p}}$ $(p > 2q > 0)$ is subnormal with the Berger measure*

$$d\mu_{(p,q)}(s,t) = \frac{s^{q-1}t^{q-1}(1-s-t)^{p-2q-1}\chi_\Delta(s,t)dsdt}{B(q,p-q)B(q,p-2q)}.$$

Proof. Let

$$d\mu'_{(p,q)}(s,t) = \frac{s^{q-1}t^{q-1}(1-s-t)^{p-2q-1}\chi_\Delta(s,t)dsdt}{B(q,p-q)B(q,p-2q)}.$$

Since

$$\int_{[0,1]^2} d\mu'_{(p,q)}(s,t) = \frac{\int_0^1 ds \int_0^{1-s} s^{q-1}t^{q-1}(1-s-t)^{p-2q-1}dt}{B(q,p-q)B(q,p-2q)},$$

let $t = (1-s)u$, then,

$$\begin{aligned}\int_{[0,1]^2} d\mu'_{(p,q)}(s,t) &= \frac{\int_0^1 ds \int_0^1 s^{q-1}[(1-s)u]^{q-1}[(1-s)(1-u)]^{p-2q-1}(1-s)du}{B(q,p-q)B(q,p-2q)} \\ &= \frac{\int_0^1 s^{q-1}(1-s)^{p-q-1}ds \int_0^1 u^{q-1}(1-u)^{p-2q-1}du}{B(q,p-q)B(q,p-2q)} \\ &= 1.\end{aligned}$$

Therefore, $d\mu'_{(p,q)}$ is a probability measure on $[0,1]^2$.

On the other hand, let $\gamma^{(p,q)}_{\mathbf{k}} = \int_{[0,1]^2} s^{k_1}t^{k_2} d\mu'_{(p,q)}(s,t)$, then,

$$\gamma^{(p,q)}_{\mathbf{k}} = \frac{\int_0^1 ds \int_0^{1-s} s^{k_1}t^{k_2}s^{q-1}t^{q-1}(1-s-t)^{p-2q-1}dt}{B(q,p-q)B(q,p-2q)},$$

let $t=(1-s)u$, then,

$$\begin{aligned}\gamma_{\mathbf{k}}^{(p,q)} &= \frac{\int_0^1 ds\int_0^1 s^{k_1+q-1}[(1-s)u]^{k_2+q-1}[(1-s)(1-u)]^{p-2q-1}(1-s)du}{B(q,p-q)B(q,p-2q)}\\ &= \frac{\int_0^1 s^{k_1+q-1}(1-s)^{k_2+p-q-1}ds\int_0^1 u^{k_2+q-1}(1-u)^{p-2q-1}du}{B(q,p-q)B(q,p-2q)}\\ &= \frac{B(k_1+q,k_2+p-q)B(k_2+q,p-2q)}{B(q,p-q)B(q,p-2q)}\\ &= \frac{\Gamma(p)\Gamma(k_1+q)\Gamma(k_2+q)}{\Gamma^2(q)\Gamma(k_1+k_2+p)}.\end{aligned}$$

In the following, we will prove that $\gamma_{\mathbf{k}}^{(p,q)}=\gamma_{\mathbf{k}}(W_{\alpha,\beta})$.

In fact, when $k_1\geqslant 1$, $k_2\geqslant 1$,

$$\begin{aligned}\gamma_{\mathbf{k}}^{(p,q)} &= \frac{\Gamma(p)\Gamma(k_1+q)\Gamma(k_2+q)}{\Gamma^2(q)\Gamma(k_1+k_2+p)}\\ &= \frac{[q\cdots(q+k_1-1)][q\cdots(q+k_2-1)]}{[p\cdots(p+k_1-1)][(p+k_1)\cdots(p+k_1+k_2-1)]}\\ &= \alpha^2_{(0,0)}\cdots\alpha^2_{(k_1-1,0)}\beta^2_{(k_1,0)}\cdots\beta^2_{(k_1,k_2-1)}\\ &= \gamma_{\mathbf{k}}(W_{\alpha,\beta}).\end{aligned}$$

When $k_1=0$, $k_2\geqslant 1$,

$$\gamma_{\mathbf{k}}^{(p,q)}=\frac{\Gamma(p)\Gamma(k_2+q)}{\Gamma(q)\Gamma(k_2+p)}=\frac{q\cdots(q+k_2-1)}{p\cdots(p+k_2-1)}=\beta^2_{(0,0)}\cdots\beta^2_{(0,k_2-1)}=\gamma_{\mathbf{k}}(W_{\alpha,\beta}).$$

When $k_1\geqslant 1$, $k_2=0$,

$$\gamma_{\mathbf{k}}^{(p,q)}=\frac{\Gamma(p)\Gamma(k_1+q)}{\Gamma(q)\Gamma(k_1+p)}=\frac{q\cdots(q+k_1-1)}{p\cdots(p+k_1-1)}=\alpha^2_{(0,0)}\cdots\alpha^2_{(k_1-1,0)}=\gamma_{\mathbf{k}}(W_{\alpha,\beta}).$$

Finally, it is obvious that $\gamma_{\mathbf{k}}^{(p,q)}=1$ when $\mathbf{k}=0$.

To summarize, we get that $\gamma_{\mathbf{k}}^{(p,q)}=\gamma_{\mathbf{k}}(W_{\alpha,\beta})$.

Combing the above computation with the 2-variable Berger theorem, we conclude that the 2-variable weighted shift $W_{\alpha,\beta}$ is subnormal with the Berger measure $d\mu'_{(p,q)}(s,t)=\frac{s^{q-1}t^{q-1}(1-s-t)^{p-2q-1}\chi_\Delta(s,t)dsdt}{B(q,p-q)B(q,p-2q)}$. □

Proposition 2.6. *If 2-variable weighted shift $W_{\alpha,\beta}=(T_1,T_2)$ with weight sequences $\alpha_{\mathbf{k}}$, $\beta_{\mathbf{k}}$ is subnormal with the Berger measure μ, and $W'_{\alpha,\beta}=(T'_1,T'_2)$ is a 2-variable weighted shift with weight sequences $\alpha'_{\mathbf{k}}$, $\beta'_{\mathbf{k}}$. Furthermore, suppose that $\parallel T_1\parallel=\parallel T_2\parallel=1$ and $\exists M>0$ such that $\alpha'_{\mathbf{k}}=\sqrt{M}\alpha_{\mathbf{k}}$, $\beta'_{\mathbf{k}}=\sqrt{M}\beta_{\mathbf{k}}$. Then $W'_{\alpha,\beta}$ is subnormal with the Berger measure $d\mu'(s,t)=d\mu(\frac{s}{M},\frac{t}{M})$, which is defined on $[0,M]^2$.*

Proof. Obviously, $\parallel T'_1\parallel=\parallel T'_2\parallel=\sqrt{M}$, and $\gamma_{\mathbf{k}}(W'_{\alpha,\beta})=M^{k_1+k_2}\gamma_{\mathbf{k}}(W_{\alpha,\beta})$.

Since $W_{\alpha,\beta}$ is subnormal with the Berger measure μ, from the 2-variable Berger theorem, we get

$$\gamma_{\mathbf{k}}(W_{\alpha,\beta}) = \int_{[0,1]^2} s^{k_1}t^{k_2}d\mu(s,t).$$

Therefore,

$$\gamma_{\mathbf{k}}(W'_{\alpha,\beta}) = M^{k_1+k_2}\int_{[0,1]^2} s^{k_1}t^{k_2}d\mu(s,t) = \int_{[0,M]^2} s^{k_1}t^{k_2}d\mu(\frac{s}{M},\frac{t}{M}).$$

Let $d\mu'(s,t) = d\mu(\frac{s}{M},\frac{t}{M})$, then we see that $\gamma_{\mathbf{k}}(W'_{\alpha,\beta}) = \int_{[0,M]^2} s^{k_1}t^{k_2}d\mu'(s,t)$, and it follows from the 2-variable Berger theorem that the 2-variable weighted shift $W'_{\alpha,\beta}$ is subnormal with the Berger measure μ'. $\square$

Now we can obtain the main result of this section.

Theorem 2.7. *$S_2(a,b,c,d)$ is subnormal if and only if $ad-2bc>0$. When $S_2(a,b,c,d)$ is subnormal, it has the Berger measure*

$$d\mu(s,t) = \frac{s^{\frac{b}{a}-1}t^{\frac{b}{a}-1}(1-\frac{cs}{a}-\frac{ct}{a})^{\frac{d}{c}-\frac{2b}{a}-1}}{(\frac{a}{c})^{\frac{2b}{a}}B(\frac{b}{a},\frac{d}{c}-\frac{b}{a})B(\frac{b}{a},\frac{d}{c}-\frac{2b}{a})}\chi_{\Omega}(s,t)dsdt,$$

which is defined on $[0,\frac{a}{c}]^2$, where

$$\Omega = \{(x,y)\in\mathbb{R}^2 : 0\leqslant x,y\leqslant\frac{a}{c}, 0\leqslant x+y\leqslant\frac{a}{c}\}.$$

Proof. Note that $\alpha_{\mathbf{k}} = \sqrt{\frac{a}{c}}\sqrt{\frac{k_1+\frac{b}{a}}{k_1+k_2+\frac{d}{c}}}$, $\beta_{\mathbf{k}} = \sqrt{\frac{a}{c}}\sqrt{\frac{k_2+\frac{b}{a}}{k_1+k_2+\frac{d}{c}}}$.

Let $\sqrt{\frac{a}{c}} = \sqrt{M}$, $\frac{b}{a}=q$, $\frac{d}{c}=p$. On the one hand, when $ad-2bc>0$, we have $p-2q>0$. From Theorem 2.5, it follows that the 2-variable weighted shift with weight sequences $\alpha'_{\mathbf{k}} = \sqrt{\frac{k_1+\frac{b}{a}}{k_1+k_2+\frac{d}{c}}}$, $\beta'_{\mathbf{k}} = \sqrt{\frac{k_2+\frac{b}{a}}{k_1+k_2+\frac{d}{c}}}$ is subnormal with the Berger measure

$$d\mu_1(s,t) = \frac{s^{\frac{b}{a}-1}t^{\frac{b}{a}-1}(1-s-t)^{\frac{d}{c}-\frac{2b}{a}-1}\chi_{\Delta}(s,t)dsdt}{B(\frac{b}{a},\frac{d}{c}-\frac{b}{a})B(\frac{b}{a},\frac{d}{c}-\frac{2b}{a})}.$$

Combining the above result with Proposition 2.6, we conclude that $S_2(a,b,c,d)$ is subnormal with the Berger measure

$$d\mu(s,t) = \frac{s^{\frac{b}{a}-1}t^{\frac{b}{a}-1}(1-\frac{cs}{a}-\frac{ct}{a})^{\frac{d}{c}-\frac{2b}{a}-1}}{(\frac{a}{c})^{\frac{2b}{a}}B(\frac{b}{a},\frac{d}{c}-\frac{b}{a})B(\frac{b}{a},\frac{d}{c}-\frac{2b}{a})}\chi_{\Omega}(s,t)dsdt.$$

On the other hand, when $ad-2bc<0$, since

$$\alpha^2_{\mathbf{k}+\varepsilon_1} - \alpha^2_{\mathbf{k}} = \frac{ack_2+ad-bc}{[c(k_1+k_2)+d][c(k_1+k_2+1)+d]},$$

$$(\alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}})^2 = \frac{(ak_1+b)(ak_2+b)c^2}{[c(k_1+k_2)+d]^2[c(k_1+k_2+1)+d]^2},$$

$$\beta^2_{\mathbf{k}+\varepsilon_2} - \beta^2_{\mathbf{k}} = \frac{ack_1 + ad - bc}{[c(k_1+k_2)+d][c(k_1+k_2+1)+d]},$$

then,

$$\det(H_{S_2(a,b,c,d)}(\mathbf{k})) = \frac{[ad + ac(k_1+k_2)](ad-2bc)}{[c(k_1+k_2)+d]^2[c(k_1+k_2+1)+d]^2} < 0.$$

The six-point test shows that $S_2(a,b,c,d)$ is not hyponormal, and in particular it is not subnormal. □

3. Some applications

In this section, we will use the operator pair $S_2(a,b,c,d)$ to give some examples to show the relationship of subnormality, hyponormality, and k-hyponormality. First we will recall some definitions and facts that will be used in the sequel.

Lemma 3.1 (Subnormal backward extension of a 1-variable weighted shift, cf. [8])**.** *Let $T = W_\alpha$ be a weighted shift whose restriction to $\mathcal{M} \triangleq \bigvee\{e_1, e_2, \ldots\}$ is subnormal, with Berger measure $\mu_{\mathcal{M}}$. Then W_α is subnormal* (*with associated measure μ*) *if and only if*

(1) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;

(2) $\alpha_0^2 \leqslant (\| \frac{1}{t} \|_{L^1(\mu_{\mathcal{M}})})^{-1}$.

In this case, $d\mu(t) = \frac{\alpha_0^2}{t} d\mu_{\mathcal{M}}(t) + (1 - \alpha_0^2 \| \frac{1}{t} \|_{L^1(\mu_{\mathcal{M}})}) d\delta_0(t)$, where δ_0 denotes Dirac measure at 0.

Definition 3.2 (cf. [8])**.** Let μ be a probability measure on $X \times Y = \mathbb{R}_+ \times \mathbb{R}_+$, and assume that $\frac{1}{t} \in L^1(\mu)$. The extremal measure μ_{ext} on $X \times Y$ is given by $d\mu_{ext}(s,t) \triangleq (1 - \delta_0(t)) \frac{1}{t\| \frac{1}{t} \|_{L^1(\mu)}} d\mu(s,t)$, where δ_0 denotes Dirac measure at 0.

Obviously, the extremal measure is also a probability measure (cf. [8]).

Definition 3.3 (cf. [8])**.** Given a measure μ on $X \times Y$, the marginal measure μ^X is given by $\mu^X \triangleq \mu \circ \pi_X^{-1}$, where $\pi_X : X \times Y \to X$ is the canonical projection onto X.

In fact, for each $E \subseteq X$, $\mu^X(E) = \mu(E \times Y)$, and if μ is a probability measure, so does μ^X (cf. [8]).

Lemma 3.4 (Subnormal backward extension of a 2-variable weighted shift, cf. [8])**.** *Consider the 2-variable weighted shift $\mathbf{T} = W_{\alpha,\beta}$, and let $\mathcal{M} = span\{e_{\mathbf{k}} : k_2 \geqslant 1\}$ be an invariant subspace of $\mathbf{T}$. Assume that $\mathbf{T}_{\mathcal{M}} \triangleq \mathbf{T}|_{\mathcal{M}}$ is subnormal with the Berger measure $\mu_{\mathcal{M}}$ and that $W_0 \triangleq shift(\alpha_{00}, \alpha_{10}, \ldots)$ is subnormal with Berger measure ν. Then $\mathbf{T}$ is subnormal if and only if*

(1) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;

(2) $\beta_{00}^2 \leqslant (\| \frac{1}{t} \|_{L^1(\mu_{\mathcal{M}})})^{-1}$;

(3) $\beta_{00}^2 \parallel \frac{1}{t} \parallel_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leqslant \nu$.
Moreover, if $\beta_{00}^2 \parallel \frac{1}{t} \parallel_{L^1(\mu_{\mathcal{M}})} = 1$*, then* $(\mu_{\mathcal{M}})_{ext}^X = \nu$*. In the case when* $\mathbf{T}$ *is subnormal, the Berger measure* μ *of* $\mathbf{T}$ *is given by*

$$d\mu(s,t) = Ad(\mu_{\mathcal{M}})_{ext}(s,t) + (d\nu(s) - Ad(\mu_{\mathcal{M}})_{ext}^X(s))d\delta_0(t),$$

where $A = \beta_{00}^2 \parallel \frac{1}{t} \parallel_{L^1(\mu_{\mathcal{M}})}$.

As applications of the results in Section 2, we will use the subnormality and the Berger measure of $S(a,b,c,d)$ and $S_2(a,b,c,d)$ to construct an example to show that there exists a 2-variable weighted shift which is hyponormal but not subnormal. Our method is some different from [5, 8], where it is the first time in the literatures to construct such type of examples.

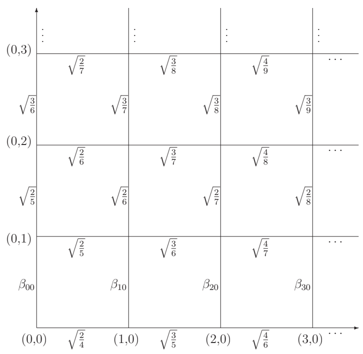

Fig. 1

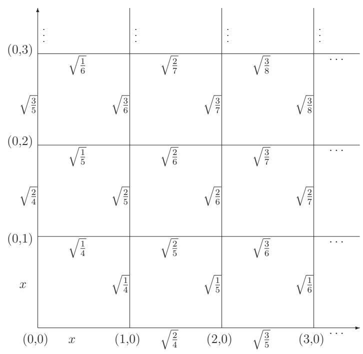

Fig. 2

Figure 1 and Figure 2. Weight diagram of the 2-variable weighted shift $\mathbf{T}$ in Example 3.5 and weight diagram of the 2-variable weighted shift $\mathbf{T}(x)$ in Example 3.9.

Example 3.5. In Lemma 3.4, we assume that $\mathbf{T}_{\mathcal{M}} = S_2(1,2,1,5)$, $W_0 = S(1,2,1,4)$ (see Fig. 1), then
(1) $\mathbf{T}$ is subnormal if and only if $\beta_{00} \in (0, \frac{1}{2}]$;
(2) $\mathbf{T}$ is hyponormal if and only if $\beta_{00} \in (0, \frac{\sqrt{3}}{3}]$.

Proof. (1) From Theorem 2.5, we get

$$d\mu_{\mathcal{M}}(s,t) = \frac{st}{B(2,3)B(2,1)}\chi_\Delta(s,t)dsdt = 24st\chi_\Delta(s,t)dsdt.$$

From Lemma 1.3, we get

$$d\nu(s) = \frac{s(1-s)ds}{B(2,2)} = 6s(1-s)ds.$$

Thus,

$$\| \frac{1}{t} \|_{L^1(\mu_{\mathcal{M}})} = 24 \int_{\Delta} sdsdt = 24 \int_0^1 ds \int_0^{1-s} sdt = 4,$$

and

$$d(\mu_{\mathcal{M}})_{ext}(s,t) = \frac{1-\delta_0(t)}{4t} d\mu_{\mathcal{M}}(s,t) = 6s(1-\delta_0(t))\chi_{\Delta}(s,t)dsdt.$$

Therefore,

$$d(\mu_{\mathcal{M}})_{ext}^X(s) = \int_0^{1-s} 6s(1-\delta_0(t))dsdt = 6s(1-s)ds.$$

According to the theorem of subnormal backward extension of a 2-variable weighted shift, $\mathbf{T}$ is subnormal if and only if

$$\beta_{00}^2 \leqslant \frac{1}{4}, \quad 4\beta_{00}^2 6s(1-s)ds \leqslant 6s(1-s)ds,$$

thus, $\beta_{00} \in (0, \frac{1}{2}]$.

(2) According to the six-point test, we are supposed to give the necessary and sufficient condition when

$$H_{\mathbf{T}}(\mathbf{k}) = \begin{pmatrix} \alpha_{\mathbf{k}+\varepsilon_1}^2 - \alpha_{\mathbf{k}}^2 & \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta_{\mathbf{k}+\varepsilon_2}^2 - \beta_{\mathbf{k}}^2 \end{pmatrix} \geqslant 0.$$

Since $\mathbf{T}_{\mathcal{M}}$ is subnormal, in particular it is hyponormal, therefore, when $k_2 \geqslant 1$, we have $H_{\mathbf{T}}(\mathbf{k}) \geqslant 0$. Thus, we need only to consider the necessary and sufficient condition when $H_{\mathbf{T}}(\mathbf{k}) \geqslant 0$ for $\mathbf{k} = (k,0), k \in \mathbb{Z}_+$.

When $\mathbf{k} = (k,0)$, we get

$$\alpha_{(k,0)} = \sqrt{\frac{k+2}{k+4}}, \quad \alpha_{(k+1,0)} = \sqrt{\frac{k+3}{k+5}}, \quad \alpha_{(k,1)} = \sqrt{\frac{k+2}{k+5}},$$

$$\beta_{(k,0)} = \beta_{00}\sqrt{\frac{4}{k+4}}, \quad \beta_{(k+1,0)} = \beta_{00}\sqrt{\frac{4}{k+5}}, \quad \beta_{(k,1)} = \sqrt{\frac{2}{k+5}},$$

then

$$H_{\mathbf{T}}((k,0)) = \begin{pmatrix} \frac{k+3}{k+5} - \frac{k+2}{k+4} & \frac{2\beta_{00}\sqrt{k+2}}{k+5} - \frac{2\beta_{00}\sqrt{k+2}}{k+4} \\ \frac{2\beta_{00}\sqrt{k+2}}{k+5} - \frac{2\beta_{00}\sqrt{k+2}}{k+4} & \frac{2}{k+5} - \frac{4\beta_{00}^2}{k+4} \end{pmatrix}.$$

We obtain that

$$\det(H_{\mathbf{T}}((k,0))) = \frac{4(k+4)-(12k+48)\beta_{00}^2}{(k+5)^2(k+4)^2}.$$

Therefore, $H_{\mathbf{T}}((k,0)) \geqslant 0$ if and only if

$$\frac{k+3}{k+5} - \frac{k+2}{k+4} \geqslant 0, \quad \frac{4(k+4)-(12k+48)\beta_{00}^2}{(k+5)^2(k+4)^2} \geqslant 0,$$

thus, $\beta_{00} \in (0, \frac{\sqrt{3}}{3}]$. $\square$

It follows that $\mathbf{T}$ is hyponormal but not subnormal if and only if $\beta_{00} \in (\frac{1}{2}, \frac{\sqrt{3}}{3}]$.

Before we construct the next example, we will recall some definitions and results.

Definition 3.6 (cf. [8])**.** Let $T_1, T_2, \ldots, T_n \in B(H)$, a commuting operator tuple $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ is called hyponormal if

$$[\mathbf{T}^*, \mathbf{T}] \triangleq \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix} \geqslant 0,$$

where $[S, T] \triangleq ST - TS$.

In [5], the authors introduced the following definition.

Definition 3.7 (cf. [5])**.** A commuting pair $\mathbf{T} = (T_1, T_2)$ is called k-hyponormal if

$$\mathbf{T}(k) \triangleq (T_1, T_2, T_1^2, T_2T_1, T_2^2, \ldots, T_1^k, T_2T_1^{k-1}, \ldots, T_2^k)(k \geqslant 2)$$

is hyponormal.

Obviously, $\mathbf{T}$ is normal $\Longrightarrow$ $\mathbf{T}$ is subnormal $\Longrightarrow$ $\mathbf{T}$ is $(k+1)$-hyponormal $\Longrightarrow$ $\mathbf{T}$ is k-hyponormal $\Longrightarrow \cdots \Longrightarrow$ $\mathbf{T}$ is 2-hyponormal $\Longrightarrow$ $\mathbf{T}$ is hyponormal $(k \geqslant 2)$. Moreover, $\mathbf{T}$ is subnormal if and only if $\mathbf{T}$ is k-hyponormal, $\forall k \geq 1$ (cf. [5]).

The following result describes when a 2-variable weighted shift is k-hyponormal.

Lemma 3.8 (cf. [5])**.** *Let $\mathbf{T} = (T_1, T_2)$ be a 2-variable weighted shift with weight sequences $\alpha_{\mathbf{k}}$, $\beta_{\mathbf{k}}$. Then $\mathbf{T}$ is k-hyponormal if and only if*

$$M_{\mathbf{k}}(k) \triangleq (\gamma_{\mathbf{k}+(m,n)+(p,q)})_{0 \leqslant m+n, p+q \leqslant k} \geqslant 0$$

for each $\mathbf{k} \in \mathbb{Z}_+^2$, where $k \geqslant 2$.

In the following, we will use the operator pairs $S_2(a, b, c, d)$ to construct an example to show that there are gaps among subnormal, hyponormal and 2-hyponormal 2-variable weighted shifts.

Example 3.9. We change $\alpha_{\mathbf{0}}$ and $\beta_{\mathbf{0}}$ to x in $S_2(1, 1, 1, 3)$, denote the new operator pair by $\mathbf{T}(x)$ (see Fig. 2), then

(1) $\mathbf{T}(x)$ is subnormal if and only if $x \in (0, \frac{\sqrt{3}}{3}]$;

(2) $\mathbf{T}(x)$ is hyponormal if and only if $x \in (0, \frac{\sqrt{6}}{4}]$;

(3) $\mathbf{T}(x)$ is 2-hyponormal if and only if $x \in (0, \frac{2\sqrt{105}}{35}]$.

Proof. Let $\mathcal{M}_1 = span\{e_{\mathbf{k}} : k_2 \geqslant 1\}$ and $\mathcal{M}_2 = span\{e_{\mathbf{k}} : k_1 \geqslant 1\}$ be invariant subspaces of $\mathbf{T}(x)$, and $\mathcal{M}_3 = span\{e_{\mathbf{k}} : k_1 \geqslant 1, k_2 = 0\}$.

(1)Obviously, $\mathbf{T}(x)|_{\mathcal{M}_3} = S(1,2,1,4)$ with the Berger measure $d\mu_{\mathcal{M}_3}(s) = 6s(1-s)ds$, and in this case $\| \frac{1}{s} \|_{L^1(\mu_{\mathcal{M}_3})} = \int_0^1 6(1-s)ds = 3$. According to the theorem of subnormal backward extension of a 1-variable weighted shift, T_1 is subnormal if and only if $x^2 \leqslant \frac{1}{3}$, thus, $x \in (0, \frac{\sqrt{3}}{3}]$. Moreover, T_1 has the Berger measure

$$d\nu(s) = 6x^2(1-s)ds + (1-3x^2)d\delta_0(s).$$

We denote the weighted sequences of $S_2(1,1,1,3)$ by $\alpha'_{\mathbf{k}}$ and $\beta'_{\mathbf{k}}$, then it follows from the 2-variable Berger theorem that $\mathbf{T}(x)|_{\mathcal{M}_1}$ is subnormal with the Berger measure $d\mu_{\mathcal{M}_1}(s,t) = \frac{t}{\beta'^2_{00}} d\mu(s,t) = \frac{3t\chi_\Delta(s,t)dsdt}{B(1,2)B(1,1)} = 6t\chi_\Delta(s,t)dsdt$. Moreover, $\| \frac{1}{t} \|_{L^1(\mu_{\mathcal{M}_1})} = \int_{[0,1]^2} 6\chi_\Delta(s,t)dsdt = 3$.

From the definition, we obtain that

$$d(\mu_{\mathcal{M}_1})^X_{ext} = \int_0^{1-s} \frac{1-\delta_0(t)}{3t} 6tdsdt = 2(1-s)ds.$$

When $x \in (0, \frac{\sqrt{3}}{3}]$, we have

$$\beta^2_{00} = x^2 \leqslant \frac{\sqrt{3}}{3} = (\| \frac{1}{t} \|_{L^1(\mu_{\mathcal{M}_1})})^{-1},$$

$$\begin{aligned}\beta^2_{00} \| \frac{1}{t} \|_{L^1(\mu_{\mathcal{M}_1})} d(\mu_{\mathcal{M}_1})^X_{ext} &= 3x^2 \cdot 2(1-s)ds \\ &\leqslant 6x^2(1-s)ds + (1-3x^2)d\delta_0(s) = d\nu,\end{aligned}$$

it follows from the theorem of subnormal backward extension of a 2-variable weighted shift that $\mathbf{T}(x)$ is subnormal.

On the other hand, when $x \in (\frac{\sqrt{3}}{3}, +\infty)$, T_1 is not subnormal, in particular, $\mathbf{T}(x)$ is not subnormal.

Thus $\mathbf{T}(x)$ is subnormal if and only if $x \in (0, \frac{\sqrt{3}}{3}]$.

(2) According to the six-point test, we need to find out when

$$H_{\mathbf{T}}(\mathbf{k}) = \begin{pmatrix} \alpha^2_{\mathbf{k}+\varepsilon_1} - \alpha^2_{\mathbf{k}} & \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta^2_{\mathbf{k}+\varepsilon_2} - \beta^2_{\mathbf{k}} \end{pmatrix} \geqslant 0, \forall \mathbf{k} \in \mathbb{Z}^2_+.$$

Since $\mathbf{T}(x)|_{\mathcal{M}_1}$ and $\mathbf{T}(x)|_{\mathcal{M}_2}$ are subnormal, in particular they are hyponormal, therefore, when $k_1 \geqslant 1$ or $k_2 \geqslant 1$, we have $H_{\mathbf{T}}(\mathbf{k}) \geqslant 0$. Thus, we need only to consider the necessary and sufficient condition when $H_{\mathbf{T}}(\mathbf{0}) \geqslant 0$.

Since

$$H_{\mathbf{T}}(\mathbf{0}) = \begin{pmatrix} \frac{1}{2} - x^2 & \frac{1}{4} - x^2 \\ \frac{1}{4} - x^2 & \frac{1}{2} - x^2 \end{pmatrix},$$

it is easy to see that $H_{\mathbf{T}}(\mathbf{0}) \geqslant 0$ if and only if $x \in (0, \frac{\sqrt{6}}{4}]$. Thus $\mathbf{T}(x)$ is hyponormal if and only if $x \in (0, \frac{\sqrt{6}}{4}]$.

(3) From Lemma 3.8, $\mathbf{T}(x)$ is 2-hyponormal if and only if

$$M_{\mathbf{k}}(2) = (\gamma_{\mathbf{k}+(m,n)+(p,q)})_{0\leqslant m+n,p+q\leqslant 2} \geqslant 0,$$

for each $\mathbf{k} \in \mathbb{Z}_+^2$.

Since $\mathbf{T}(x)|_{\mathcal{M}_1}$ and $\mathbf{T}(x)|_{\mathcal{M}_2}$ are subnormal, and hence are 2-hyponormal. Therefore, when $k_1 \geqslant 1$ or $k_2 \geqslant 1$, we have

$$M_{\mathbf{k}}(2) = (\gamma_{\mathbf{k}+(m,n)+(p,q)})_{0\leqslant m+n,p+q\leqslant 2} \geqslant 0.$$

Thus, we need only to find out when

$$M_{\mathbf{0}}(2) = (\gamma_{(m,n)+(p,q)})_{0\leqslant m+n,p+q\leqslant 2} \geqslant 0.$$

Since

$$M_{\mathbf{0}}(2) = x^2 \begin{pmatrix} \frac{1}{x^2} & 1 & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{3}{10} & \frac{1}{10} & \frac{1}{10} \\ 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{10} & \frac{1}{10} & \frac{3}{10} \\ \frac{1}{2} & \frac{3}{10} & \frac{1}{10} & \frac{1}{5} & \frac{1}{20} & \frac{1}{30} \\ \frac{1}{4} & \frac{1}{10} & \frac{1}{10} & \frac{1}{20} & \frac{1}{30} & \frac{1}{20} \\ \frac{1}{2} & \frac{1}{10} & \frac{3}{10} & \frac{1}{30} & \frac{1}{20} & \frac{1}{5} \end{pmatrix},$$

from the matrix theory, we see that $M_{\mathbf{0}}(2) \geqslant 0$ if and only if $x \in (0, \frac{2\sqrt{105}}{35}]$ using an easy computation.

Then, $\mathbf{T}(x)$ is 2-hyponormal if and only if $x \in (0, \frac{2\sqrt{105}}{35}]$. $\square$

It follows that $\mathbf{T}(x)$ is hyponormal but not 2−hyponormal if and only if $x \in (\frac{2\sqrt{105}}{35}, \frac{\sqrt{6}}{4}]$ and $\mathbf{T}(x)$ is 2-hyponormal but not subnormal if and only if $x \in (\frac{\sqrt{3}}{3}, \frac{2\sqrt{105}}{35}]$.

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