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ON A GENERALIZATION OF RIGHT DUO RINGS

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Abstract. We study the structure of rings whose principal right ideals contain a sort of two-sided ideals, introducing *right π-duo* as a generalization of (weakly) right duo rings. Abelian π-regular rings are π-duo, which is compared with the fact that Abelian regular rings are duo. For a right π-duo ring R, it is shown that every prime ideal of R is maximal if and only if R is a (strongly) π-regular ring with $J(R) = N_*(R)$. This result may be helpful to develop several well-known results related to *pm* rings (i.e., rings whose prime ideals are maximal). We also extend the right π-duo property to several kinds of ring which have roles in ring theory.

Throughout this note every ring is associative with identity unless otherwise specified. Given a ring R (possibly without identity), $J(R)$, $N_*(R)$, $N^*(R)$, and $N(R)$ denote the Jacobson radical, the prime radical, the upper nilradical (i.e., sum of all nil ideals), and the set of all nilpotent elements in R, respectively. It is well-known that $N^*(R) \subseteq J(R)$ and $N_*(R) \subseteq N^*(R) \subseteq N(R)$. We use $R[x]$ ($R[[x]]$) to denote the polynomial (power series) ring with an indeterminate x over R. Denote the n by n full (resp., upper triangular) matrix ring over R by $\mathrm{Mat}_n(R)$ (resp., $U_n(R)$). Use e_{ij} for the matrix unit with (i, j)-entry 1 and elsewhere 0. Denote $\{(a_{ij}) \in U_n(R) \mid$ the diagonal entries of (a_{ij}) are all equal$\}$ by $D_n(R)$. $r_R(-)$ (resp., $l_R(-)$) is used to denote a right (resp., left) annihilator in R. $\prod$ denotes the direct product of rings. $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n).

1. Right π-duo rings

In this section we introduce the concept of a right π-duo ring as a generalization of weakly right duo ring, and study the structure of right π-duo rings. Let R be a ring and M be a right R-module. Buhphang and Rege [4] called M *semicommutative* if $mRa = 0$ whenever $ma = 0$ for $m \in M$ and $a \in R$. We first consider the condition $(*)$:

$$\text{If } ma = 0 \text{ for } m \in M \text{ and } a \in R, \text{ then } mRa^n = 0 \text{ for some } n \geq 1,$$

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as a generalization of semicommutative modules.

Proposition 1.1. *For a ring R the following conditions are equivalent*:

(1) *Every right R-module satisfies the condition* $(*)$.

(2) *Every cyclic right R-module satisfies the condition* $(*)$.

(3) *For any $a \in R$ there is a positive integer n such that aR contains Ra^n.*

(4) *For any $a \in R$ there is a positive integer n such that aR contains Ra^nR.*

Proof. (1)⇒(2) and (3)⇒(4) are obvious.

(2)⇒(3): Assume that (2) holds. Consider the cyclic right R-module R/aR. Since $(1+aR)a = 0$, there exists a positive integer n such that $(1+aR)ra^n = 0$ for all $r \in R$. This implies $Ra^n \subseteq aR$.

(4)⇒(1): Assume that (4) holds. Let M be a right R-module and suppose $ma = 0$ for $m \in M$ and $a \in R$. Then $aR \subseteq r_R(m)$. By assumption, $Ra^nR \subseteq aR$ for some positive integer n. This yields $mRa^n = 0$. □

Following Feller [7], a ring (possibly without identity) is called *right duo* if every right ideal is two-sided. Left duo rings are defined similarly. A ring is called *duo* if it is both left and right duo. Let R be a ring and M be a right R-module. Buhphang and Rege proved that a ring R is right duo if and only if every right R-module is semicommutative if and only if every cyclic right R-module is semicommutative, in [4, Proposition 2.11].

Based on Proposition 1.1, a ring R (possibly without identity) will be called *right π-duo* if R satisfies the condition $(*)$. Left π-duo rings are defined similarly. A ring is called *π-duo* if it is both left and right π-duo.

According to Yao [27], a ring R (possibly without identity) is called *weakly right duo* if for each a in R there exists a positive integer n such that a^nR is a two-sided ideal of R. Weakly left duo rings are defined similarly. A ring is called *weakly duo* if it is both weakly left and weakly right duo. A ring (possibly without identity) is usually called *Abelian* if every idempotent is central. Weakly right duo rings are Abelian by [27, Lemma 4].

According to Yu [28], a ring R (possibly without identity) is called *right quasi-duo* if every maximal right ideal of R is two-sided. Left quasi-duo rings are defined similarly. A ring is called *quasi-duo* if it is both left and right quasi-duo. One may immediately observe that a ring R is right quasi-duo if and only if so is $R/J(R)$, and that every factor ring of a right quasi-duo ring is again right quasi-duo. It is also straightforward that $\prod_{i\in I} R_i$ of rings R_i is right quasi-duo if and only if each R_i is right quasi-duo. Yu showed in [28, Proposition 2.1] that a ring R is right quasi-duo if and only if $U_n(R)$ is right quasi-duo, where n is allowed to be any finite or infinite cardinal number.

It is obvious that commutative rings are duo, right duo rings are weakly right duo, and weakly right duo rings are right π-duo. Yu proved that weakly right duo rings are right quasi-duo in [28, Proposition 2.2]. But every implication is irreversible by the following.

Example 1.2. (1) We find a right π-duo ring but not weakly right duo by help of constructions in [3, Section 2] and [12, Example 3].

Let F be a field and let $V_1 = F((x))[[y]]$ and $V_2 = F((y))[[x]]$, where $F((x))$ and $F((y))$ are quotient fields of the power series rings $F[[x]]$ and $F[[y]]$ respectively. Then $V_1 \nsubseteq V_2$ and $V_2 \nsubseteq V_1$. Define a ring isomorphism $\sigma : V_1 \to V_2$ by $x \mapsto y$ and $y \mapsto x$, and put $K = F((x))((y))$. Then σ can be extended to an automorphism of K, so we can form the skew power series ring $W = K[[t;\sigma]]$ with the elements $\sum_{i=0}^{\infty} t^i k_i$ for $k_i \in K$. The multiplication is only subject to $kt = t\sigma(k)$ for $k \in K$.

We next consider the subring

$$R = \{v + \sum_{i=1}^{\infty} t^i k_i \in W \mid v \in V_1, k_i \in K\}$$

of W consisting of those elements whose constant term is in V_1. Then by [3], R is a right chain ring (i.e., a ring whose lattice of right ideals is linearly ordered).

We first show that R is not weakly right duo by using a slightly different method than [12, Example 3]. Assume on the contrary that R is weakly right duo. Then for $y + t \in R$, $R(y+t)^k \subseteq (y+t)^k R$ for some $k \geq 1$, and so $\frac{1}{x}(y+t)^k = (y+t)^k f(t)$ for some $f(t) = v + \sum_{i=1}^{\infty} t^i k_i \in R$. Since $(y+t)^k$ is not a unit, we get $f(t) = v$ by comparing the degrees of both sides. Thus

$$\begin{aligned}\frac{1}{x}(y+t)^k &= \frac{y^k}{x} + t\frac{x^{k-1} + \cdots + y^{k-1}}{y} + \cdots + \frac{1}{x}t^k \\ &= y^k v + t(x^{k-1} + \cdots + y^{k-1})v + \cdots + t^k v.\end{aligned}$$

From the constant terms of the equation, $\frac{y^k}{x} = y^k v$, leading to $v = \frac{1}{x}$. But, comparing the coefficients of t, we have $v = \frac{1}{y}$, a contradiction. Consequently, R is not weakly right duo.

We next show that R is right π-duo. Let $f(t) = v + \sum_{i=1}^{\infty} t^i k_i \in R$. If $f(t)$ is a unit, we are done. Letting $f(t)$ be a nonzero nonunit. If $Rf(t)^n R \subseteq f(t)R$ for some $n \geq 2$, then we are done. Assume that $Rf(t)^n R \nsubseteq f(t)R$ for all $n \geq 2$. As noted before, R is a right chain ring. Thus we have $f(t)R \subseteq Rf(t)^2 R$. Then $f(t) = \sum_{j=1}^{m} r_j f(t)^2 s_j$ for some $r_j = v_j + \sum_{i=1}^{\infty} t^i u_{ij}, s_j = v'_j + \sum_{i=1}^{\infty} t^i v_{ij} \in R$. Note that

$$\begin{aligned} & v + \sum_{i=1}^{\infty} t^i k_i \\ (1) \quad & = \sum_{j=1}^{m} r_j (v^2 + t(\sigma(v)k_1 + k_1 v) + t^2(\sigma^2(v)k_2 + \sigma(k_1)k_1 + k_2 v) + \cdots)s_j.\end{aligned}$$

Then $v = \sum_{j=1}^{m} v_j v^2 v'_j$. Since $v, v_j, v'_j \in V_1$, we have $v = v^2(\sum_{j=1}^{m} v_j v'_j)$. Thus $v(1 - v(\sum_{j=1}^{m} v_j v'_j)) = 0$. Since V_1 is a domain, either $v = 0$ or $v(\sum_{j=1}^{m} v_j v'_j) = 1$. If $v = 0$, then by simple computation from the equality (1), we have $k_i = 0$ for all i, which leads to $f(t) = 0$, a contradiction. If v is a unit in V_1, then $f(t)$

is a unit in R, which is also a contradiction. Consequently, $Rf(t)^nR \subseteq f(t)R$ for some $n \geq 2$, entailing that R is a right π-duo ring.

(2) Let D be a division ring and $R = D_3(D)$. R is easily shown to be weakly right duo since any $A \in R$ is either invertible or nilpotent, but not right duo as can be seen by

$$\begin{pmatrix} 0 & 0 & D \\ 0 & 0 & D \\ 0 & 0 & 0 \end{pmatrix} = Re_{23} \nsubseteq e_{23}R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & D \\ 0 & 0 & 0 \end{pmatrix}.$$

(3) Let D be a division ring and $R = U_2(D)$. Then R is right quasi-duo since $R/J(R) \cong D \oplus D$, but R is not weakly right duo since weakly right duo rings are Abelian.

The π-duo property is not left-right symmetric by the following.

Example 1.3. There is a right π-duo ring which is not left π-duo. We refer to [22, Example 1]. Let $F(x)$ be the quotient filed of the polynomial ring $F[x]$ with an indeterminate x over a field F. Define a ring endomorphism $\sigma : F(x) \to F(x)$ by $\sigma(f(x)/g(x)) = f(x^2)/g(x^2)$. We consider the skew power series ring $R = F(x)[[t;\sigma]]$ with the elements $\sum_{i=0}^{\infty} t^i k_i$ for $k_i \in F(x)$. The multiplication is only subject to $kt = t\sigma(k)$ for $k \in F(x)$. We first note that each coefficient of the elements in Rt^n is of the form $f(t^{2n})/g(t^{2n})$. Then R is not left π-duo since $t^m x \notin Rt$ for any $m \geq 1$. But, by the same method as in [22, Example 1], we can compute that R is right duo (so right π-duo).

We use freely the above facts among right duo rings, weakly right duo rings, right π-duo rings, and right quasi-duo rings.

In [8], a ring R is called *von Neumann regular* (simply, *regular*) if for every $x \in R$ there exists $y \in R$ such that $xyx = x$. A ring R is usually called *π-regular* if for each $a \in R$ there exist a positive integer n and $b \in R$ such that $a^n = a^n b a^n$. It is easily shown that $J(R)$ of a π-regular ring R is nil. Regular rings are clearly π-regular. Abelian regular rings are duo by [8, Theorem 3.2], comparing with the following fact that Abelian π-regular rings are π-duo.

Lemma 1.4. (1) *A ring R is Abelian π-regular if and only if so is $D_n(R)$.*

(2) *Abelian π-regular rings are weakly duo (hence π-duo).*

(3) *Every factor ring of a right π-duo ring is also right π-duo.*

(4) *Every finite direct product of right π-duo rings is also right π-duo.*

(5) *Every nil ring is right π-duo as a ring without identity.*

Proof. (1) Let R be an Abelian π-regular ring. Then $D_n(R)$ is also Abelian by [16, Lemma 2]. Define $N_n(R) = \{(a_{ij} \in D_n(R) \mid a_{ii} = 0 \text{ for all } i\}$. Note that $N_n(R)$ is a nil ideal of $D_n(R)$, and $\frac{D_n(R)}{N_n(R)} \cong R$ is π-regular. So $D_n(R)$ is π-regular by [2, Theorem 4].

Conversely, let $r \in R$ and $a = (a_{ij}) \in D_n(R)$ with $a_{ii} = r$ and $a_{ij} = 0$ for $i \neq j$. Since $D_n(R)$ is π-regular, there exist $b = (b_{ij}) \in D_n(R)$ with $b_{ii} = s$

and $k \geq 1$ such that $a^k = a^k b a^k$. This yields $r^k = r^k s r^k$. The class of Abelian rings is obviously closed under subrings.

(2) Let R be an Abelian π-regular ring and $a \in R$. Then there exist $b \in R$ and $n \geq 1$ such that $a^n = a^n b a^n$. Since R is Abelian, $a^n b$ and $b a^n$ are both central. So we have $r a^n = r a^n b a^n = a^n b r a^n \in a^n R$ and $a^n r = a^n b a^n r = a^n r b a^n \in R a^n$ for $r \in R$.

(3) and (5) are obvious.

(4) Let R_i, $i \in \{1, \ldots, n\}$, be right π-duo rings, and $R = \prod_{i=1}^n R_i$. Take $a = (a_1, \ldots, a_n) \in R$. Then there exist $k_1, \ldots, k_n \geq 1$ such that $R_i a_i^{k_i} \subseteq a_i R_i$. Note that $R_i a_i^k \subseteq a_i R_i$ for all i, where k is maximal in $\{k_1, \ldots, k_n\}$. This yields that $R a^k \subseteq aR$. □

The converse of Lemma 1.4(3) need not hold by the following.

Example 1.5. Consider $R = U_2(D)$ over a division ring D. Then each of the non-trivial factor rings $R/J(R) \cong D \oplus D$, $R/I \cong D$, and $R/K \cong D$ is right π-duo, where $J(R) = \left(\begin{smallmatrix} 0 & D \\ 0 & 0 \end{smallmatrix}\right)$, $I = \left(\begin{smallmatrix} D & D \\ 0 & 0 \end{smallmatrix}\right)$, $K = \left(\begin{smallmatrix} 0 & D \\ 0 & D \end{smallmatrix}\right)$; but R is neither left nor right π-duo as can be seen by the computation that $e_{11}^n R = e_{11} R = \left(\begin{smallmatrix} D & D \\ 0 & 0 \end{smallmatrix}\right) \nsubseteq R e_{11} = \left(\begin{smallmatrix} D & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & D \\ 0 & D \end{smallmatrix}\right) = R e_{22} = R e_{22}^n \nsubseteq e_{22} R = \left(\begin{smallmatrix} 0 & 0 \\ 0 & D \end{smallmatrix}\right)$ for all $n \geq 1$.

Example 1.5 also shows that "R is right π-duo if and only if so is $R/J(R)$" need not be true, comparing with the fact that R is right quasi-duo if and only if $R/J(R)$ is right quasi-duo.

The following shows that infinite direct products of right π-duo rings need not be right π-duo, comparing with Lemma 1.4(4).

Example 1.6. We apply [15, Example 2.4]. Let A be a right π-duo ring and $R_n = D_n(A)$ for $n \geq 6$. Then every R_n is right π-duo by Theorem 1.7 below. Set R be the direct product of R_i's for $i = 6, 8, \ldots, 2k, \ldots$ $(k = 3, 4, \ldots)$. Take $x = (x_i), y = (y_i) \in R$ such that

$$x_i = e_{12} + \cdots + e_{(\frac{i}{2}-1)\frac{i}{2}} \text{ and } y_i = e_{(\frac{i}{2}+1)(\frac{i}{2}+2)} + \cdots + e_{(i-1)i}.$$

Then $xy = 0$, but $xRy^m \neq 0$ for all $m \geq 1$ by the same computation as in [15, Example 2.4]. This yields that R is not right π-duo by Proposition 1.9(3) to follow.

For any ring A, $\mathrm{Mat}_n(A)$ $(U_n(A))$, with $n \geq 2$, is neither left nor right π-duo by a similar method to Example 1.5. For a ring R and $n \geq 2$, let $V_n(R)$ be the ring of all matrices (a_{ij}) in $D_n(R)$ such that $a_{st} = a_{(s+1)(t+1)}$ for $s = 1, \ldots, n-2$ and $t = 2, \ldots, n-1$. Note that $V_n(R) \cong \frac{R[x]}{x^n R[x]}$.

It can be easily checked that $D_n(R)$ $(n \geq 2)$ over a division ring R is weakly right duo (and hence right π-duo) as noted in Example 1.2(2). But we have a more general result for right π-duo rings as we see in the following.

Theorem 1.7. *Let R be a ring and $n \geq 2$. Then the following conditions are equivalent*:

(1) *R is right π-duo.*

(2) $D_n(R)$ *is right* π*-duo.*
(3) $V_n(R)$ *is right* π*-duo.*

Proof. (2)⇒(1): Assume that $E = D_n(R)$ is right π-duo. Let $a \in R$ and $A = (a_{ij}) \in E$ with $a_{ii} = a$ and elsewhere zero. Then there exist $k \geq 1$ such that $EA^k \subseteq AE$. This yields $Ra^k \subseteq aR$, entailing that R is right π-duo. The proof of (3)⇒(1) is similar.

(1)⇒(2): Write $E_n = D_n(R)$. Let R be right π-duo. Let $0 \neq A = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \in E_2$. If $a = 0$, then $E_2A^2 = 0 \subset AE_2$. Assume $a \neq 0$. Since R is right π-duo, there exists $k \geq 1$ such that $Ra^k \subseteq aR$. So for $r \in R$,

$$r(a^k)^{k+1} = ra^{k(k+1)} = ra^ka^{k^2} = ar_1a^ka^{k(k-1)} = a^2r_2a^{k(k-1)} = \cdots = a^{k+1}r_{k+1}$$

for some $r_1, \ldots, r_{k+1} \in R$. Consider the following equalities:

$$\begin{aligned}
\begin{pmatrix} r & s \\ 0 & r \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^{k(k+1)} &= \begin{pmatrix} r & s \\ 0 & r \end{pmatrix}\begin{pmatrix} a^k & b_1 \\ 0 & a^k \end{pmatrix}^{k+1} \\
&= \begin{pmatrix} r & s \\ 0 & r \end{pmatrix}\begin{pmatrix} a^{k(k+1)} & b_2 \\ 0 & a^{k(k+1)} \end{pmatrix} \\
&= \begin{pmatrix} ra^{k(k+1)} & b_3 \\ 0 & ra^{k(k+1)} \end{pmatrix} \\
&= \begin{pmatrix} a^{k+1}r_{k+1} & a\alpha \\ 0 & a^{k+1}r_{k+1} \end{pmatrix},
\end{aligned}$$

where $s, b_1 \in R$ and $b_2, b_3 \in Ra^kR$ (here $b_3 = a\alpha$ for some $\alpha \in R$ since $Ra^kR \subseteq aR$). Moreover $ba^kr_{k+1} = a\beta$ for some $\beta \in R$ also since $Ra^kR \subseteq aR$. Thus we now have

$$\begin{aligned}
\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} a^kr_{k+1} & \alpha - \beta \\ 0 & a^kr_{k+1} \end{pmatrix} &= \begin{pmatrix} a^{k+1}r_{k+1} & a\alpha - a\beta + ba^kr_{k+1} \\ 0 & a^{k+1}r_{k+1} \end{pmatrix} \\
&= \begin{pmatrix} a^{k+1}r_{k+1} & a\alpha - a\beta + a\beta \\ 0 & a^{k+1}r_{k+1} \end{pmatrix} \\
&= \begin{pmatrix} a^{k+1}r_{k+1} & a\alpha \\ 0 & a^{k+1}r_{k+1} \end{pmatrix} \\
&= \begin{pmatrix} r & s \\ 0 & r \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^{k(k+1)}.
\end{aligned}$$

This concludes that $D_2(R)$ is also right π-duo.

Next we show that E_3 is right π-duo. Let $0 \neq A = \left(\begin{smallmatrix} a & b_1 & c \\ 0 & a & b_2 \\ 0 & 0 & a \end{smallmatrix}\right) \in E_3$. If $a = 0$, then $E_3A^3 = 0 \subset AE_3$. Assume $a \neq 0$. Since R is right π-duo, there exists $k \geq 1$ such that $Ra^k \subseteq aR$. So for $r \in R$,

$$\begin{aligned}
r((a^k)^{k+1})^{k+1} &= ra^{k(k+1)(k+1)} = ra^ka^{k((k+1)^2-1)} \\
&= ar_1a^ka^{k((k+1)^2-2)} = a^2r_2a^{k((k+1)^2-2)}
\end{aligned}$$

$$= \cdots = a^{k+1}r_{k+1}a^{k((k+1)^2-(k+1))} = a^{k+1}r_{k+1}a^{k^2(k+1)}$$

for some $r_1, \ldots, r_{k+1} \in R$. Consider the following equalities:

$$\begin{pmatrix} r & s_1 & t \\ 0 & r & s_2 \\ 0 & 0 & r \end{pmatrix} \begin{pmatrix} a & b_1 & c \\ 0 & a & b_2 \\ 0 & 0 & a \end{pmatrix}^{k(k+1)^3}$$

$$= \begin{pmatrix} r & s_1 & t \\ 0 & r & s_2 \\ 0 & 0 & r \end{pmatrix} \left[\begin{pmatrix} a & b_1 & c \\ 0 & a & b_2 \\ 0 & 0 & a \end{pmatrix}^{k(k+1)^2} \right]^{k+1}$$

$$= \begin{pmatrix} r & s_1 & t \\ 0 & r & s_2 \\ 0 & 0 & r \end{pmatrix} \begin{pmatrix} a^{k(k+1)^2} & b_1' & c' \\ 0 & a^{k(k+1)^2} & b_2' \\ 0 & 0 & a^{k(k+1)^2} \end{pmatrix}^{k+1}$$

$$= \begin{pmatrix} r & s_1 & t \\ 0 & r & s_2 \\ 0 & 0 & r \end{pmatrix} \begin{pmatrix} a^{k(k+1)^3} & b_1'' & c'' \\ 0 & a^{k(k+1)^3} & b_2'' \\ 0 & 0 & a^{k(k+1)^3} \end{pmatrix}$$

$$= \begin{pmatrix} a^{k+1}r_{k+1}a^{k^2(k+1)(k+2)} & al_1 & al_2 \\ 0 & a^{k+1}r_{k+1}a^{k^2(k+1)(k+2)} & al_3 \\ 0 & 0 & a^{k+1}r_{k+1}a^{k^2(k+1)(k+2)} \end{pmatrix},$$

where $s_1, s_2, t, b_1', b_2', c' \in R$ and $b_1'', b_2'', c'' \in Ra^{k(k+1)^2}R$. Here

$$rb_1'' + s_1a^{k(k+1)^2} = al_1, \ rb_2'' + s_2a^{k(k+1)^2} = al_3, \ \text{ and } \ rc'' + s_1b_2'' + ta^{k(k+1)^2} = al_2$$

for some $l_1, l_2, l_3 \in Ra^kR$ by referring to the computation "$r((a^k)^{k+1})^{k+1} = a^{k+1}r_{k+1}a^{k^2(k+1)}$" above. Similarly we have

$$b_1a^kr_{k+1}a^{k^2(k+1)(k+2)} = a\alpha \ \text{ and } \ b_2a^kr_{k+1}a^{k^2(k+1)(k+2)} = a\beta$$

for some $\alpha, \beta \in Ra^kR$. Then we also have $b_1(l_3-\beta)+ca^kr_{k+1}a^{k^2(k+1)(k+2)} = a\gamma$ for some $\gamma \in R$. Thus we now have

$$\begin{pmatrix} a & b_1 & c \\ 0 & a & b_2 \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} a^kr_{k+1}a^{k^2(k+1)(k+2)} & l_1 - \alpha & l_2 - \gamma \\ 0 & a^kr_{k+1}a^{k^2(k+1)(k+2)} & l_3 - \beta \\ 0 & 0 & a^kr_{k+1}a^{k^2(k+1)(k+2)} \end{pmatrix}$$

$$= \begin{pmatrix} a^{k+1}r_{k+1}a^{k^2(k+1)(k+2)} & al_1 & al_2 \\ 0 & a^{k+1}r_{k+1}a^{k^2(k+1)(k+2)} & al_3 \\ 0 & 0 & a^{k+1}r_{k+1}a^{k^2(k+1)(k+2)} \end{pmatrix}$$

$$= \begin{pmatrix} r & s_1 & t \\ 0 & r & s_2 \\ 0 & 0 & r \end{pmatrix} \begin{pmatrix} a & b_1 & c \\ 0 & a & b_2 \\ 0 & 0 & a \end{pmatrix}^{k(k+1)^3}.$$

This concludes that $D_3(R)$ is also right π-duo.

Lastly we extend the computing method for $D_3(R)$ to show that E_n is also right π-duo. We proceed by induction on n. Let $0 \neq A = (a_{ij}) \in E_n$ with $a_{ii} = a$. If $a = 0$, then $E_n A^n = 0 \subset AE_n$. Assume $a \neq 0$. Consider two $n-1$ by $n-1$ matrices

$$B = \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1(n-1)} \\ 0 & a_{22} & \cdots & \cdots & a_{2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_{(n-1)(n-1)} \end{pmatrix}$$

and

$$C = \begin{pmatrix} a_{22} & \cdots & \cdots & a_{2(n-1)} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{(n-1)(n-1)} & a_{(n-1)n} \\ 0 & 0 & \cdots & 0 & a_{nn} \end{pmatrix}$$

which are taken in part from the matrix A. Then $B, C \in E_{n-1}$. Let $S = (s_{ij}) \in E_n$. Take two $n-1$ by $n-1$ matrices S_1 and S_2 from S as above. In the following we use the computation related to $D_3(R)$ repeatedly. Then, by induction hypothesis, we can take m sufficiently large such that

$$S_1 B^m = (b_{ij}) = B \begin{pmatrix} a^k\alpha & ag_{12} & \cdots & \cdots & ag_{1(n-1)} \\ 0 & a^k\alpha & \cdots & \cdots & ag_{2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a^k\alpha \end{pmatrix}$$

and

$$S_2 C^m = (c_{ij}) = C \begin{pmatrix} a^k\alpha & ah_{23} & \cdots & \cdots & ah_{2n} \\ 0 & a^k\alpha & \cdots & \cdots & ah_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a^k\alpha \end{pmatrix} \in E_{n-1}$$

with $b_{ii}, c_{ii} \in a^{k+1}R$ and $\alpha, g_{ij}, h_{ij} \in Ra^kR$ for $i \neq j$. Similarly we can let $SA^m = (d_{ij}) \in E_n$ with $d_{1n} \in aRa^kR$, say $d_{1n} = ad$ with $d \in Ra^kR$. Note that $b_{ij} = c_{ij} = d_{ij}$ and $g_{ij} = h_{ij}$ for i, j with $2 \leq i, j \leq n-1$. Since $a_{12}h_{2n} + a_{13}h_{3n} + \cdots + a_{1(n-1)}h_{(n-1)n} + a_{1n}a^k\alpha \in Ra^kR \subseteq aR$, we can write $a_{12}h_{2n} + a_{13}h_{3n} + \cdots + a_{1(n-1)}h_{(n-1)n} + a_{1n}a^k\alpha = a\epsilon_1$ for some $\epsilon_1 \in R$.

Now letting

$$D = \begin{pmatrix} a^k\alpha & g_{12} - \alpha_{12} & \cdots & \cdots & g_{1(n-1)} - \alpha_{12} & d - \epsilon_1 \\ 0 & a^k\alpha & \cdots & \cdots & h_{2(n-1)} - \alpha_{2(n-1)} & h_{2n} - \epsilon_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a^k\alpha & h_{(n-1)n} - \epsilon_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & a^k\alpha \end{pmatrix} \in E_n,$$

the product AD is equal to

$$\begin{pmatrix} a & a_{12} & \cdots & \cdots & a_{1(n-1)} & a_{1n} \\ 0 & a & \cdots & \cdots & a_{2(n-1)} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a & a_{(n-1)n} \\ 0 & 0 & 0 & \cdots & 0 & a \end{pmatrix}$$

$$\begin{pmatrix} a^k\alpha & g_{12}-\alpha_{12} & \cdots & \cdots & g_{1(n-1)}-\alpha_{12} & d-\epsilon_1 \\ 0 & a^k\alpha & \cdots & \cdots & h_{2(n-1)}-\alpha_{2(n-1)} & h_{2n}-\epsilon_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a^k\alpha & h_{(n-1)n}-\epsilon_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & a^k\alpha \end{pmatrix}$$

$$= \begin{pmatrix} a^{k+1}\alpha & ag_{12} & \cdots & \cdots & ag_{1(n-1)} & ad \\ 0 & a^{k+1}\alpha & \cdots & \cdots & ag_{2(n-1)} & ah_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & & 0\cdots & 0 & a^{k+1}\alpha & ah_{(n-1)n} \\ 0 & 0 & 0 & \cdots & 0 & a^{k+1}\alpha \end{pmatrix} = SA^m,$$

where $a_{23}h_{3n}+\cdots+a_{2(n-1)}h_{(n-1)n}+a_{2n}a^k\alpha = a\epsilon_2$ for some $\epsilon_2 \in R$ ($\epsilon_3, \ldots, \epsilon_{n-1}$ can be obtained similarly), and $a_{12}a^k\alpha = a\alpha_{12}$ for some $\alpha_{12} \in R$ ($\alpha_{13}, \ldots, \alpha_{1(n-1)}, \ldots, \alpha_{(n-2)(n-1)}$ can be obtained similarly). Therefore $D_n(R)$ is right π-duo.

The proof of (1) $\Rightarrow$ (3) is similar. $\square$

Recall that $V_n(R) \cong \frac{R[x]}{x^nR[x]}$. So Theorem 1.7 gives the following.

Corollary 1.8. *Let $n \geq 2$. A ring R is right π-duo if and only if so is $R[x]/x^nR[x]$.*

In [8], a ring R is called *strongly regular* if for any $a \in R$, $a \in a^2R$. A ring is strongly regular if and only if it is Abelian regular by [8, Theorem 3.5]. An ideal I of R is called *completely primary* if for elements $a, b \in R$ so that $ab \in I$ with $a \notin I$, then $b^m \in I$ for some $m \geq 1$.

An element a in a ring R is usually called *regular* if $l_R(a) = 0 = r_R(a)$. Following the literature, a ring R is called right *Ore* if given $a, b \in R$ with b regular there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. It is a well-known fact that R is a right Ore ring if and only if there exists the classical right quotient ring of R. Right Noetherian domains are also well-known as an example of right Ore rings. A ring R is usually called *right p.p.* if every principal right ideal of R is projective. Left p.p. rings are defined similarly, and a ring is called p.p. if it is both left and right p.p.. An Abelian right p.p. ring is easily shown to be reduced as we see in [16, Lemma 1(1)], and so it also left p.p.. We use this fact freely.

Proposition 1.9. (1) *Right π-duo rings are right quasi-duo.*

(2) *If R is a right π-duo ring, then every prime ideal of R is completely primary.*

(3) *Let R be a right π-duo ring. If $ab = 0$ for $a, b \in R$, then $aRb^m = 0$ for some $m \geq 1$.*

(4) *Right π-duo rings are Abelian.*

(5) *Right π-duo rings are right Ore.*

(6) *If a ring R is right π-duo and right p.p., then the classical right quotient ring $Q(R)$ of R is strongly regular.*

Proof. (1) Let R be a right π-duo ring. We apply the proof of [28, Proposition 2.2]. Let M be a maximal right ideal of R. Assume on the contrary that M is not a left ideal of R. Then $aM \nsubseteq M$ for some $a \in R \backslash M$, entailing $aM + M = R$. Say $ax + y = 1$ for $x, y \in M$. Since R is right π-duo, there exists $m \geq 1$ such that $R(xa)^m \subseteq (xa)R \subseteq M$. Thus we have

$$\begin{aligned} y(1 + ax + \cdots + (ax)^m) &= (1 - ax)(1 + ax + \cdots + (ax)^m) \\ &= 1 - (ax)^{m+1} = 1 - a(xa)^m x, \end{aligned}$$

entailing $1 \in M$ since $y, a(xa)^m x \in M$. This induces a contradiction. Therefore R is right quasi-duo.

(2) Suppose P is a prime ideal of a ring R such that $ab \in P$ with $a \notin P$. If R is right π-duo, then $aRb^m \subseteq abR \subseteq P$ for some $m \geq 1$. This yields $b^m \in P$.

(3) Let R be a right π-duo ring and $ab = 0$. Then $aRb^m \subseteq abR = 0$ for some $m \geq 1$, entailing $aRb^m = 0$.

(4) Let R be a right π-duo ring and $e^2 = e \in R$. Then $eR(1 - e) = 0 = (1 - e)Re$ by (3) since $e(1 - e) = 0 = (1 - e)e$. This leads to R being Abelian.

(5) Let R be a right π-duo ring and $a, b \in R$ with b regular. Then there exists $m \geq 1$ such that $Rb^m \subseteq bR$ and thus $ab^m = bc$ for some $c \in R$, noting that b^m is regular.

(6) Let R be a ring that is right π-duo and right p.p.. So R is reduced by (4), entailing R being p.p.. By (5), R has a classical right quotient ring $Q(R)$. Moreover $Q(R)$ is also reduced by [23, Theorem 16]. We will use this fact freely.

We next show that $Q(R)$ is p.p.. Let $ab^{-1} \in Q(R)$. Since R is right p.p., $r_R(a) = eR$ for some $e^2 = e \in R$. Then $ab^{-1}e = aeb^{-1} = 0$, so $eQ \subseteq r_{Q(R)}(ab^{-1})$. Next let $cd^{-1} \in r_{Q(R)}(ab^{-1})$. Then we have

$$\begin{aligned} ab^{-1}cd^{-1} = 0 \ &\Rightarrow\ ab^{-1}c = 0 \ \Rightarrow\ cab^{-1} = 0 \\ &\Rightarrow\ ca = 0 = ac \ \Rightarrow\ c \in eR \ \Rightarrow\ c = ec, \end{aligned}$$

entailing $cd^{-1} = ecd^{-1} \in eQ$. Since e is depending on not c but a, we can get $r_{Q(R)}(ab^{-1}) \subseteq eQ$. Thus $r_{Q(R)}(ab^{-1}) = eQ$.

We now have that $Q(R)$ is a reduced p.p. ring. Therefore $Q(R)$ is strongly regular by [13, Lemma 3.3]. $\square$

Following the literature, a ring is called *locally finite* if every finite subset generates a finite multiplicative semigroup. It is shown that a ring is locally finite if every finite subset generates a finite subring in [17, Theorem 2.2(1)]. It is easily checked that the class of locally finite rings contains finite rings and algebraic closures of finite fields.

Proposition 1.10. *Let R be a locally finite ring. Then the following conditions are equivalent*:

(1) *R is Abelian.*
(2) *If $ab = 0$ for $a, b \in R$, then $aRb^m = 0$ for some $m \geq 1$.*
(3) *R is right* (*left*) *π-duo.*
(4) *R is right* (*left*) *weakly duo.*
(5) *R is weakly duo.*

Proof. (1)⇒(5) is proved by [22, Proposition 15]. The proof of Proposition 1.9(4) is applicable to show (2)⇒(1). (3)⇒(2) is shown by Proposition 1.9(3), and (5)⇒(4)⇒(3) are obvious. □

Each of the converses of Proposition 1.9(1, 4, 5) need not be true by the following.

Example 1.11. (1) $R = U_2(D)$ over a division ring D is right quasi-duo, but it is neither left nor right π-duo since left or right π-duo rings are Abelian by Proposition 1.9(4).

(2) We refer to [26, Theorem 1.3.5, Corollary 2.1.14, and Theorem 2.1.15]. Let $F\langle x, y\rangle$ be the free algebra with noncommuting indetermiantes x, y over a field F of characteristic zero. The first Weyl algebra $A_1(F) \cong \frac{F\langle x,y\rangle}{(yx-xy-1)}$, R say, is a domain whose invertible elements are nonzero elements in F (hence central), where $(yx - xy - 1)$ is the ideal of $F\langle x, y\rangle$ generated by $yx - xy - 1$. We identify x and y with their images in R for simplicity. R is neither left nor right π-duo. In fact, Rx^n (resp., x^nR) is not contained in xR (resp., Rx) for all $n \geq 1$ since $yx^n \notin xR$ (resp., $x^ny \notin Rx$). But R is a right Noetherian domain, and so R is right Ore.

(3) The ring $A_1(F)$ in (2) is a domain (hence Abelian) but not right π-duo.

Related to Proposition 1.9(4), one may ask whether a right π-duo ring is Abelian even for the case of without identity. However the answer is negative by the following.

Example 1.12. Consider the ring $R = \left(\begin{smallmatrix} D & D \\ 0 & 0 \end{smallmatrix}\right) \subset U_2(D)$ over a right π-duo ring D. Then R is clearly non-Abelian. Let $A = \left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) \in R$. Since D is right π-duo, there exists $m \geq 1$ such that $Da^m \subseteq aD$. Take $B \in RA^{2m}$. Then $B = \left(\begin{smallmatrix} u & v \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)^{2m}$ for some $\left(\begin{smallmatrix} u & v \\ 0 & 0 \end{smallmatrix}\right) \in R$. Note $B = \left(\begin{smallmatrix} ua^{2m} & ua^{2m-1}b \\ 0 & 0 \end{smallmatrix}\right)$.

Now since $ua^{2m}, ua^{2m-1}b \in aD$, we have that $ua^{2m} = ac$ and $ua^{2m-1}b = ad$ for some $c, d \in D$. Hence $B = \left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} c & d \\ 0 & 0 \end{smallmatrix}\right) \in AR$, showing that R is right π-duo.

Following the literature, a ring R is called *strongly π-regular* if for each $a \in R$, there exists positive integer $n = n(a)$ such that $a^n \in a^{2n}R$. Dischinger

[5] showed that strong π-regularity is left-right symmetric. It is well-known that the endomorphism ring of V over a division ring D is strongly π-regular if and only if V is finite dimensional over D, where V is a right D-module. Strongly π-regular rings are π-regular by Azumaya [1], and these concepts are equivalent for right π-duo rings by help of Proposition 1.9(1) and [11, Theorem 7].

Theorem 1.13. *Let R be a right π-duo ring. Then the following conditions are equivalent*:

(1) *Every prime ideal of R is maximal.*
(2) *Every prime ideal of R is right primitive.*
(3) *R is strongly π-regular and $J(R) = N_*(R)$.*
(4) *R is π-regular and $J(R) = N_*(R)$.*

Proof. Since R is right π-duo, R is right quasi-duo by Proposition 1.9(1). So $N(R) \subseteq J(R)$ by [28, Lemma 2.3].

(1)⇒(2) is obvious. (3) and (4) are equivalent by [9, Theorem 1].

(2)⇒(3): Assume that (2) holds. We first show that $R/N_*(R)$ is strongly π-regular. Since R is right π-duo, $R/N_*(R)$ is also right π-duo by Lemma 1.4(3). Note that $R/N_*(R)$ is a subdirect product of R/P's, where P runs over all prime (hence right primitive by (2)) ideals of R. R is right quasi-duo by Proposition 1.9(1). So every R/P is a division ring by [14, Proposition 1], entailing that $R/N_*(R)$ is reduced. Note that each prime factor ring of R coincides with one of $R/N_*(R)$. So $R/N_*(R)$ is strongly π-regular by [18, Lemma 4] since every prime factor ring of $R/N_*(R)$ is a division ring.

Next since R is right π-duo, R is Abelian by Proposition 1.9(4). So R is strongly π-regular by [9, Theorem 2]. This yields $J(R)$ being nil. Since $R/N_*(R)$ is reduced, we must have $J(R) = N_*(R)$.

(3)⇒(1): Assume that (3) holds. Then $J(R) = N_*(R) = N(R)$ since $N(R) \subseteq J(R)$, and so $R/N_*(R)$ is reduced strongly π-regular. Hence every prime factor ring of $R/N_*(R)$ is a division ring by [18, Lemma 4]. Thus every prime ideal of R is maximal, noting that each prime factor ring of R coincides with one of $R/N_*(R)$. □

Note that in the proof of Theorem 1.13, $J(R) = N_*(R)$ implies $J(R) = N_*(R) = N(R)$ since R is right π-duo (hence right quasi-duo).

The condition "$J(R) = N_*(R)$" in Theorem 1.13(3, 4) is not superfluous as we see in the following argument.

Example 1.14. We apply the construction of ring and argument in [20, Example 1.2] and [21, Theorem 2.2(2)]. Let K be a division ring and $R_n = D_{2^n}(K)$ for $n \geq 1$. Define a map $\sigma : R_n \to R_{n+1}$ by $B \mapsto \left(\begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix}\right)$, then R_n can be considered as a subring of R_{n+1} via σ (i.e., $B = \sigma(B)$ for $B \in R_n$). Set R be the direct limit of the direct system (R_n, σ_{ij}), where $\sigma_{ij} = \sigma^{j-i}$. Then R is a right π-duo ring by help of Theorem 1.7. But

$$J(R) = \{A \in R \mid \text{ the diagonal entries of } A \text{ are zero}\} \neq 0.$$

However R is a prime ring by applying the proof of [20, Proposition 1.3]. Thus the condition "Every prime ideal of R is maximal" in Theorem 1.13(3) does not hold. Note $N_*(R) = 0$ and R is easily shown to be π-regular.

The ring R in Example 1.14 is Abelian by [16, Lemma 2], and hence R is weakly duo by Lemma 1.4(2). A ring is usually called *pm* if every prime ideal is maximal. It is well-known that if a weakly right duo ring is pm, then it is π-regular. Thus Example 1.14 provides an example of a weakly duo π-regular ring but not pm.

2. More properties of right π-duo rings

In this section we study various properties of right π-duo rings. According to Marks [25], a ring R is called *NI* if $N^*(R) = N(R)$. Note that R is NI if and only if $N(R)$ forms a two-sided ideal if and only if $R/N^*(R)$ is reduced. It is well-known that duo rings are NI, but not conversely. Note that $U_2(\mathbb{Z})$ is clearly NI, but not right π-duo by Proposition 1.9(4). Recall that `Köthe's conjecture` means "*the sum of two nil left ideals is nil*".

Theorem 2.1. *For a right π-duo ring R, we have the following.*
(1) *If $a \in N(R)$, then both aR and Ra are nil.*
(2) *Köthe's conjecture holds if and only R is NI.*

Proof. (1) We apply the proof of Yao [27, Lemma 2]. Let $a^n = 0$ for some $n \geq 1$. For any $r \in R$, we have $R(ar)^k \subseteq (ar)R$ for some $k \geq 1$ since R is right π-duo. Then there exists $s_\alpha \in R$ such that $\alpha(ar)^k = (ar)s_\alpha$ for any $\alpha \in R$. Then

$$\begin{aligned}(ar)^{nk} &= 1(ar)^k(ar)^{(n-1)k} = a(rs_1)(ar)^{(n-1)k}\\ &= a(ar)s_2(ar)^{(n-2)k} = a^2(rs_2)(ar)^{(n-2)k}\\ &\quad \cdots\cdots\\ &= a^n(rs_n) = 0.\end{aligned}$$

Thus aR is a nil right ideal of R. It is obvious that aR is nil if and only if Ra is nil.

(2) It is obvious that Köthe's conjecture holds for NI rings. Assume that Köthe's conjecture holds. Let $a, b \in N(R)$. Then $Ra, Rb \subseteq N(R)$ by (1), and moreover $Ra + Rb \subseteq N(R)$ by assumption. This yields $a + b \in N(R)$, and so R is NI by help of (1). □

By help of Theorem 2.1(2), one can say that *if a right π-duo ring, but not NI, does exist, then Köthe's conjecture does not hold for the ring.*

Proposition 2.2. *For a right π-duo ring R, we have the following.*
(1) *If $J(R) = 0$, then R is reduced.*
(2) *A ring R is simple if and only if R is a division ring.*

Proof. (1) The fact that R is reduced comes from Theorem 2.1(1) when $J(R) = 0$.

(2) It suffices to establish the necessity. Let R be simple. Then, by (1), R is reduced. Since R is right π-duo, there exists $k \geq 1$ such that $0 \neq Ra^k \subseteq aR$ for any $0 \neq a \in R$. This yields $Ra^kR = R$ since R is simple, entailing $R = aR$. So R is a division ring. $\square$

One may ask whether the class of right π-duo rings is also closed under subrings. But the answer is negative by the following.

Example 2.3. Let R be a noncommutative division ring. Then $R[x]$ is not right π-duo by [22, Proposition 8], in spite of R being π-duo.

Next we refer to [26, Corollary 2.1.14 and Theorem 2.1.15]. Since $R[x]$ is a Noetherian domain, the classical quotient ring of $R[x]$ is a division ring (hence duo). However the subring $R[x]$ is not right π-duo.

In Example 2.3, we also notice that the right π-duo property does not go up to polynomial rings. But the converse is always true.

Proposition 2.4. *Let R be a ring. If $R[x]$ is right π-duo, then so is R.*

Proof. Let $R[x]$ be right π-duo and $a \in R$. Then $R[x]a^k \subseteq aR[x]$ for some $k \geq 1$. So for $r \in R$, $(r + x)a^k = a(b + cx)$ for some $b + cx \in R[x]$. This yields $ra^k = ab \in aR$. $\square$

Let A be an algebra (not necessarily with identity) over a commutative ring S. Following Dorroh [6], the *Dorroh extension* of A by S is the Abelian group $A \oplus S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in A$ and $s_i \in S$.

Proposition 2.5. (1) *Let A be an algebra* (*not necessarily with identity*) *over a commutative ring S. Then A is right π-duo if so is the Dorroh extension D of A by S.*

(2) *Let A be a nil algebra of characteristic a prime p. Then the Dorroh extension D of A by $\mathbb{Z}$ is weakly duo* (*hence π-duo*).

(3) *Let A be a nilpotent algebra of characteristic a prime p, and consider be the Dorroh extension D of A by $\mathbb{Z}$. Then $D[x]$ is weakly duo* (*hence π-duo*).

Proof. (1) Assume $1 \in A$. Then $s \in S$ is identified with $s1 \in A$, and so $A = \{r + s \mid (r, s) \in D\}$.

Assume that D is right π-duo and let $a \in A$. Then there exists $k \geq 1$ such that $D(a, 0)^k \subseteq (a, 0)D$. Thus, for $r \in A$,

$$(ra^k, 0) = (r, 0)(a, 0)^k = (a, 0)(r_1, s_1) = (ar_1 + as_1, 0)$$

for some $(r_1, s_1) \in D$. This yields $ra^k = a(r_1 + s_1) \in aA$, noting $s_1 = s_11 \in A$.

Assume that A does not have an identity. Then we have $ra^k = ar_1 + as_1$ by the computation above. Multiplying this equality by a on the right side, we get

$$ra^{k+1} = ra^ka = ar_1a + as_1a = a(r_1a + s_1a) \in aA,$$

noting $s_1 a \in A$.

Therefore A is right π-duo.

(2) Let $f = (a, k) \in D$, $a^n = 0$ say. Since the characteristic of A is p,

$$(a,k)^{p^n} = (a^{p^n} + {}_{p^n}C_1 k a^{p^n-1} + \cdots + {}_{p^n}C_{p^n-1} k^{p^n-1} a, k^{p^n}) = (a^{p^n}, k^{p^n}) = (0, k^{p^n})$$

by [19, Exercise 3.1.10(e)], where ${}_sC_t$ means the combination. So $(a,k)^{p^n}$ is central in D, entailing $Df^{p^n} = f^{p^n} D$. Thus D is weakly duo.

(3) Note that $D[x]$ is isomorphic to the Dorroh extension of $A[x]$ by $\mathbb{Z}[x]$ via the corresponding

$$\sum_{i=0}^{m} (r_i, k_i) x^i \mapsto (\sum_{i=0}^{m} r_i x^i, \sum_{i=0}^{m} k_i x^i).$$

Then $D[x]$ is weakly duo by applying the proof of (2) on $A[x]$, noting that $A[x]$ is a nil algebra of characteristic a prime p. $\square$

In [10, Lemma 3], Hirano et al. proved that if $R[x]$ is right duo, then R is commutative. But this result is not valid for weakly right duo rings. Let $n \geq 3$ and

$$A = \{(a_{ij}) \in D_n(\mathbb{Z}_p) \mid \text{ the diagonal entries of } (a_{ij}) \text{ are zero}\},$$

where p is a prime. Then A is a noncommutative nilpotent ring, entailing that the Dorroh extension D of A by $\mathbb{Z}$ is also noncommutative. But $D[x]$ is weakly duo by Proposition 2.5(3).

For the case of $R[x]$ being weakly right duo, we have a similar result to [10, Lemma 3] as follows.

Proposition 2.6. *Let R be a ring and $a \in R$. If $R[x]$ is weakly right duo, then a^k is central for some $k \geq 1$.*

Proof. Suppose that $R[x]$ is weakly right duo and $a \in R$. Consider $a+x \in R[x]$. Then $R[x](a+x)^k \subseteq (a+x)^k R[x]$ for some $k \geq 1$, and so $R(a+x)^k \subseteq (a+x)^k R$. Thus for any $b \in R$, we have $b(a+x)^k = (a+x)^k c$ for some $c \in R$. Then $ba^k + 2bax + \cdots + bx^k = b(a^k + 2ax + \cdots + x^k) = (a^k + 2ax + \cdots + x^k)c = a^k c + 2acx + \cdots + cx^k$. This yields $b = c$ and $ba^k = a^k c = a^k b$. $\square$

Note that the power k of a depends only on $a + x$ (hence a) in the proof of Proposition 2.6.

The converse of the case of without identity in Proposition 2.5(1) need not hold by the following.

Example 2.7. Let A be the semigroup on the set $\{a, b\}$ satisfying the relations $a^2 = a = ba$ and $b^2 = b = ab$. Let $R = \mathbb{Z}_2[A]$ be the semigroup ring of A over $\mathbb{Z}_2$. Then $R = \{0, a, b, a+b\}$ without identity.

We first claim that R is right π-duo. In fact, $aR = bR = R$ with $a^2 = a, b^2 = b, (a+b)^2 = 0$. This implies that R is right π-duo.

We next consider the Dorroh extension of R by $\mathbb{Z}_2$, say D. Then $(a,1)D = \{(0,0),(a,1)\}$, and $(a,1)$ is an idempotent in D which is not central since $(b,1)(a,1) \neq (a,1)(b,1)$. By Proposition 1.9(4), D is not right π-duo.

Proposition 2.8. *Let M be a multiplicatively closed subset of a ring R consisting of central regular elements. If R is right π-duo, then so is $M^{-1}R$.*

Proof. Suppose that R is right π-duo and let $u^{-1}a \in M^{-1}R$. Then there exists $k \geq 1$ such that $Ra^k \subseteq aR$. Let $\epsilon \in M^{-1}R(u^{-1}a)^k$. Then $\epsilon = (v^{-1}b)(u^{-1}a)^k = v^{-1}(u^{-1})^k ba^k$ for some $v^{-1}b \in M^{-1}R$. Since $ba^k \in Ra^k \subseteq aR$, $ba^k = ac$ for some $c \in R$. Then

$$\epsilon = v^{-1}(u^{-1})^{k-1}u^{-1}ba^k = w^{-1}u^{-1}ac = (u^{-1}a)w^{-1}c \in (u^{-1}a)M^{-1}R,$$

letting $v^{-1}(u^{-1})^{k-1} = w^{-1}$. Thus $M^{-1}R$ is right π-duo. $\square$

Recall the ring of *Laurent polynomials* in x, written by $R[x;x^{-1}]$. Letting $M = \{1, x, x^2, \ldots\}$, M is clearly a multiplicatively closed subset of central regular elements in $R[x]$ such that $R[x;x^{-1}] = M^{-1}R[x]$. So Proposition 2.8 yields the following.

Corollary 2.9. *Let R be a ring. If $R[x]$ is right π-duo, then so is $R[x;x^{-1}]$.*

Recall that a ring R is called *local* if $R/J(R)$ is a division ring, and R is *semilocal* if $R/J(R)$ is semisimple Artinian. A ring R is usually called *right (left) weakly π-regular* if for each $a \in R$, there exists a positive integer $n = n(a)$, depending on a, such that $a^n \in a^nRa^nR$ ($a^n \in Ra^nRa^n$). Any π-regular ring is clearly both left and right weakly π-regular.

Proposition 2.10. (1) *Let R be a semilocal ring with $J(R)$ nil. Then R is weakly right duo if and only if R is right π-duo if and only if R is Abelian and right quasi-duo.*

(2) *Let R be a right* (*or left*) *weakly π-regular ring. Then R is weakly right duo if and only if R is right π-duo if and only if R is Abelian and right quasi-duo.*

(3) *Let $e \in R$ be a central idempotent of a ring R. Then R is right π-duo if and only if eR and $(1-e)R$ are right π-duo rings.*

(4) *Let R be a regular ring. Then R is right π-duo if and only if R is weakly right duo if and only if R is right duo.*

Proof. (1) It comes from Proposition 1.9(1, 4) and [22, Theorem 3].

(2) It follows from Proposition 1.9(1, 4) and [22, Corollary 11].

(3) Let $r \in R$ and suppose that R is right π-duo. Then $R(er)^n \subseteq erR$ for some $n \geq 1$. Since e is central, we also have

$$eR(er)^n = R(er)^n \subseteq erR = ereR,$$

entailing that eR is right π-duo. Similarly, $(1-e)R$ is also right π-duo.

The converse comes from Lemma 1.4(4).

(4) is shown by Proposition 1.9(4) and [8, Theorem 3.2]. $\square$

A ring R is called *semiperfect* if R is semilocal and idempotents can be lifted modulo $J(R)$. Local rings are Abelian and semilocal.

Proposition 2.11. *A ring R is right π-duo and semiperfect if and only if R is a finite direct sum of local right π-duo rings.*

Proof. Suppose that R is right π-duo and semiperfect. Since R is semiperfect, R has a finite orthogonal set $\{e_1, e_2, \ldots, e_n\}$ of local idempotents whose sum is 1 by [24, Proposition 3.7.2], say $R = \sum_{i=1}^{n} e_i R$ such that each $e_i R e_i$ is a local ring. By Proposition 1.9(4), R is Abelian and so $e_i R = e_i R e_i$ for each i. But each $e_i R$ is also right π-duo ring by Proposition 2.10(3).

Conversely assume that R is a finite direct sum of local right π-duo rings. Then R is semiperfect since local rings are semiperfect by [24, Corollary 3.7.1], and moreover R is right π-duo by Lemma 1.4(4). □

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