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OPERATOR-VALUED FUNCTION SPACE INTEGRALS VIA CONDITIONAL INTEGRALS ON AN ANALOGUE WIENER SPACE II

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Abstract. In the present paper, using a simple formula for the conditional expectations given a generalized conditioning function over an analogue of vector-valued Wiener space, we prove that the analytic operator-valued Feynman integrals of certain classes of functions over the space can be expressed by the conditional analytic Feynman integrals of the functions. We then provide the conditional analytic Feynman integrals of several functions which are the kernels of the analytic operator-valued Feynman integrals.

1. Introduction

Let r be a positive integer and let $C_0^r[0,t]$ [7] denote the r-dimensional Wiener space. On the space $C_0^r[0,t]$ Cameron and Storvick [1] introduced a very general analytic operator-valued function space Feynman integral $J_q^{an}(F)$, which mapped an $L_2(\mathbb{R}^r)$-function ψ into an $L_2(\mathbb{R}^r)$-function $J_q^{an}(F)\psi$. In [2, 10] the existence of the analytic operator-valued Feynman integral $J_q^{an}(F)$ as an operator from $L_1(\mathbb{R})$ to $L_\infty(\mathbb{R})$ was studied, and Chung, Park and Skoug [7] showed that it can be expressed by the conditional analytic Feynman integral of F. Further work extending the above $\mathcal{L}(L_1, L_\infty)$-theory with the conditional analytic Feynman integrals was studied by the author [5] over the space $(C^r[0,t], w_\varphi^r)$ [9, 11] of the continuous $\mathbb{R}^r$-valued paths on $[0,t]$ which generalizes the space $C_0^r[0,t]$. In fact the author [4] introduced the conditional Wiener integral over $C^r[0,t]$ and derived a simple formula for the conditional Wiener integral with the conditioning function $X_n : C^r[0,t] \to \mathbb{R}^{(n+1)r}$ given by

$$X_n(x) = (x(t_0), x(t_1), \ldots, x(t_n)), \tag{1}$$

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where $0 = t_0 < t_1 < \cdots < t_n = t$, which calculates directly the conditional Wiener integral in terms of the ordinary non-conditional Wiener integral. Applying this simple formula to a certain function F defined on $C^r[0,t]$ with the conditioning function $X_1 : C^r[0,t] \to \mathbb{R}^{2r}$ defined by $X_1(x) = (x(0), x(t))$, he [5] could express the analytic operator-valued Feynman integral $J_q^{an}(F) : L_1(\mathbb{R}^r) \to L_\infty(\mathbb{R}^r)$ in terms of the conditional analytic Feynman integral $E^{anf_q}[F|X_1]$ of F given X_1.

In the present paper we further develop the concepts in [5] with more generalized conditioning function $X_n (n \geq 1)$ given by (1). For the conditioning function X_n we proceed to express the analytic $\mathcal{L}(L_1, L_\infty)$-operator valued Feynman integrals in terms of the conditional analytic Feynman w_φ^r-integrals. In fact we establish that for certain functions F on $C^r[0,t]$ and for a nonzero real q, the analytic operator-valued Feynman integral $J_q^{an}(F)$ exists as an element of $\mathcal{L}(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r))$, the space of the bounded linear operators from $L_1(\mathbb{R}^r)$ to $L_\infty(\mathbb{R}^r)$, and it is given by the formula

$$\begin{aligned}&(J_q^{an}(F)\psi)(\xi)\\&= (-iq)^{\frac{r}{2}} \int_{\mathbb{R}^{(n+1)r}} E^{anf_q}[F|X_n](\xi_0, \xi_1, \ldots, \xi_n)\psi(\xi_n)\Psi(-iq, \xi_0\\&\quad - \xi)W_r(-iq, \vec{\tau}_n, (\xi_0, \xi_1, \ldots, \xi_n))dm_L^r(\xi_0)dm_L^r(\xi_1)\cdots dm_L^r(\xi_n)\end{aligned}$$

for $\psi \in L_1(\mathbb{R}^r)$ and m_L^r-a.e. $\xi \in \mathbb{R}^r$, where m_L^r is the Lebesgue measure over $\mathbb{R}^r$, W_r is given by

$$\begin{aligned}&W_r(-iq, \vec{\tau}_n, (\xi_0, \xi_1, \ldots, \xi_n))\\&= \biggl[\prod_{j=1}^n \frac{q}{2\pi i(t_j - t_{j-1})}\biggr]^{\frac{r}{2}} \exp\biggl\{\frac{iq}{2}\sum_{j=1}^n \frac{\|\xi_j - \xi_{j-1}\|_{\mathbb{R}^r}^2}{t_j - t_{j-1}}\biggr\}\end{aligned}$$

and Ψ is the analytic extension of the probability density of φ^r. Thus $J_q^{an}(F)$ can be interpreted as an integral operator with the kernel

$$(-iq)^{\frac{r}{2}} E^{anf_q}[F|X_n](\xi_0, \xi_1, \ldots, \xi_n)\Psi(-iq, \xi_0 - \xi)W_r(-iq, \vec{\tau}_n, (\xi_0, \xi_1, \ldots, \xi_n)).$$

We then provide the conditional analytic Feynman w_φ^r-integral for the cylinder functions which are important in quantum mechanics and Feynman integration theories themselves. We note that if $\varphi^r = \delta_{\vec{0}}$, the Dirac measure concentrated at $\vec{0} \in \mathbb{R}^r$, then $C^r[0,t]$ is identified with the r-dimensional Wiener space $C_0^r[0,t]$ so that our works in this paper generalize those of [7] when $n = 1$. Furthermore if $n = 1$, then most results of this paper can be reduced to those in [5], that is, the works in this paper also extend the results in the same reference.

2. An analogue of the r-dimensional Wiener space

Throughout this paper let $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_+^\sim$ denote the sets of the complex numbers, the complex numbers with positive real parts and the nonzero complex numbers with nonnegative real parts, respectively. Furthermore let m_L

denote the Lebesgue measure on the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$. The dot product on the r-dimensional Euclidean space $\mathbb{R}^r$ is denoted by $\langle\cdot,\cdot\rangle_{\mathbb{R}^r}$.

For a positive real t let $C = C[0,t]$ be the space of all real-valued continuous functions on the closed interval $[0,t]$ with the supremum norm. Let $(C[0,t],\mathcal{B}(C[0,t]),w_\varphi)$ denote the analogue of Wiener space associated with the probability measure φ [9, 11], where φ is a probability measure on $\mathcal{B}(\mathbb{R})$. Let $C^r = C^r[0,t]$ be the product space of $C[0,t]$ with the product measure w_φ^r. Since $C[0,t]$ is a separable Banach space, $\mathcal{B}(C^r[0,t]) = \prod_{j=1}^r \mathcal{B}(C[0,t])$. This probability measure space $(C^r[0,t],\mathcal{B}(C^r[0,t]),w_\varphi^r)$ is called an analogue of the r-dimensional Wiener space. For v in $L_2[0,t]$ and x in $C[0,t]$ let (v,x) denote the Paley-Wiener-Zygmund integral of v according to x [9] and let $\langle\cdot,\cdot\rangle_2$ denote the inner product over $L_2[0,t]$.

Lemma 2.1 ([9, Lemma 2.1])**.** *If $f:\mathbb{R}^{n+1}\to\mathbb{C}$ is a Borel measurable function, then*

$$\begin{aligned}&\int_C f(x(t_0),x(t_1),\dots,x(t_n))dw_\varphi(x)\\ \overset{*}{=}&\int_{\mathbb{R}}\int_{\mathbb{R}^n} f(u_0,u_1,\dots,u_n)W_1(1,\vec{t}_n,(u_0,u_1,\dots,u_n))dm_L^n(u_1,\dots,u_n)d\varphi(u_0),\end{aligned}$$

where

$$\begin{aligned}&W_r(\lambda,\vec{t}_n,(u_0,u_1,\dots,u_n))\\ &=\Big[\prod_{j=1}^n\frac{\lambda}{2\pi(t_j-t_{j-1})}\Big]^{\frac{r}{2}}\exp\Big\{-\frac{\lambda}{2}\sum_{j=1}^n\frac{\|u_j-u_{j-1}\|_{\mathbb{R}^r}^2}{t_j-t_{j-1}}\Big\}\end{aligned}\tag{2}$$

for $r\in\mathbb{N}$, $\lambda\in\mathbb{C}_+^\sim$, $\vec{t}_n=(t_0,t_1,\dots,t_n)$ with $0=t_0<t_1<\cdots<t_n\le t$, $(u_0,u_1,\dots,u_n)\in\mathbb{R}^{(n+1)r}$, and $\overset{*}{=}$ means that if either side exists, then both sides exist and they are equal.

Now we introduce a useful lemma which plays a key role in the proof of Theorem 3.2. The proof of it is similar to the proof of Lemma 3.4 in [5].

Lemma 2.2. *For $\vec{t}_n=(t_0,t_1,\dots,t_n)$ with $0=t_0<t_1<\cdots<t_n\le t$, $\lambda>0$ and $\xi\in\mathbb{R}^r$, let $X_n^{\lambda,\xi}:C^r[0,t]\to\mathbb{R}^{(n+1)r}$ be the function given by*

$$X_n^{\lambda,\xi}(x)=(\lambda^{-\frac12}x(t_0)+\xi,\lambda^{-\frac12}x(t_1)+\xi,\dots,\lambda^{-\frac12}x(t_n)+\xi).$$

Furthermore let $P_{X_n^{\lambda,\xi}}$ be the probability distribution of $X_n^{\lambda,\xi}$ on the Borel class $\mathcal{B}(\mathbb{R}^{(n+1)r})$ of $\mathbb{R}^{(n+1)r}$ and suppose that φ^r is absolutely continuous with respect to the Lebesgue measure m_L^r. Then $P_{X_n^{\lambda,\xi}}\ll m_L^{(n+1)r}$ and

$$\frac{dP_{X_n^{\lambda,\xi}}}{dm_L^{(n+1)r}}(\xi_0,\xi_1,\dots,\xi_n)=\lambda^{\frac{r}{2}}W_r(\lambda,\vec{t}_n,(\xi_0,\xi_1,\dots,\xi_n))\frac{d\varphi^r}{dm_L^r}(\lambda^{\frac12}(\xi_0-\xi))$$

for $m_L^{(n+1)r}$-a.e. $(\xi_0,\xi_1,\dots,\xi_n)\in\mathbb{R}^{(n+1)r}$, where W_r is given by (2).

3. A simple formula for conditional w_φ^r-integrals and the operator-valued function space integrals

In this section we introduce a simple formula for an analogue of the conditional Wiener integrals over $C^r[0,t]$ with a vector-valued conditioning function.

Let $F : C^r[0,t] \to \mathbb{C}$ be integrable and let X be a random vector on $C^r[0,t]$ assuming that the value space of X is a normed space with the Borel σ-algebra. Then we have the conditional expectation $E[F|X]$ of F given X from a well known probability theory. Furthermore there exists a P_X-integrable complex-valued function ψ on the value space of X such that $E[F|X](x) = (\psi \circ X)(x)$ for w_φ^r-a.e. $x \in C^r[0,t]$, where P_X is the probability distribution of X. The function ψ is called the conditional w_φ^r-integral of F given X and it is also denoted by $E[F|X]$.

Throughout this paper, let $\vec{\tau}_n = (t_0, t_1, \ldots, t_n)$ be given with $0 = t_0 < t_1 < \cdots < t_n = t$. For any x in $C^r[0,t]$ define the polygonal function $[x]$ by

$$[x](s) = \sum_{j=1}^{n} \chi_{(t_{j-1},t_j]}(s)\left(\frac{t_j - s}{t_j - t_{j-1}}x(t_{j-1}) + \frac{s - t_{j-1}}{t_j - t_{j-1}}x(t_j)\right) + \chi_{\{t_0\}}(s)x(t_0) \tag{3}$$

for $s \in [0,t]$, where $\chi_{(t_{j-1},t_j]}$ and $\chi_{\{t_0\}}$ denote the indicator functions. Similarly, for $\vec{\xi}_n = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{(n+1)r}$, define the polygonal function $[\vec{\xi}_n]$ by (3), where $x(t_j)$ is replaced by ξ_j for $j = 0, 1, \ldots, n$.

In the following theorem we introduce a simple formula for the conditional w_φ^r-integrals on $C^r[0,t]$ [4].

Theorem 3.1. *Let $F : C^r[0,t] \to \mathbb{C}$ be integrable and $X_n : C^r[0,t] \to \mathbb{R}^{(n+1)r}$ be given by*

$$X_n(x) = (x(t_0), x(t_1), \ldots, x(t_n)). \tag{4}$$

Then for P_{X_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{(n+1)r}$,

$$E[F|X_n](\vec{\xi}_n) = E[F(x - [x] + [\vec{\xi}_n])],$$

where P_{X_n} is the probability distribution of X_n on $(\mathbb{R}^{(n+1)r}, \mathcal{B}(\mathbb{R}^{(n+1)r}))$ and the expectation is taken over the variable x.

Let $F : C^r[0,t] \to \mathbb{C}$ be a function. For notational convenience let

$$X_n^{\lambda,\xi}(x) = X_n(\lambda^{-\frac{1}{2}}x + \xi) \tag{5}$$

and

$$F^{\lambda,\xi}(x) = F(\lambda^{-\frac{1}{2}}x + \xi) \tag{6}$$

for $\lambda > 0$ and for $\xi \in \mathbb{R}^r$. Suppose that $F^{\lambda,\xi}$ is integrable over $C^r[0,t]$. Then, by Theorem 3.1,

$$E[F^{\lambda,\xi}|X_n^{\lambda,\xi}](\vec{\xi}_n) = E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}_n])] \tag{7}$$

for $P_{X_n^{\lambda,\xi}}$-a.e. $\vec{\xi_n} = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{(n+1)r}$, where $P_{X_n^{\lambda,\xi}}$ is the probability distribution of $X_n^{\lambda,\xi}$ on $(\mathbb{R}^{(n+1)r}, \mathcal{B}(\mathbb{R}^{(n+1)r}))$. Let

$$(K_\lambda(F))(\vec{\xi_n}) = E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi_n}])]. \tag{8}$$

If $(K_\lambda(F))(\vec{\xi_n})$ has the analytic extension $J_\lambda^*(F)(\vec{\xi_n})$ on $\mathbb{C}_+$ as a function of λ, then it is called the conditional analytic Wiener w_φ^r-integral of F given X_n with parameter λ and denoted by

$$E^{anw_\lambda}[F|X_n](\vec{\xi_n}) = J_\lambda^*(F)(\vec{\xi_n})$$

for $\vec{\xi_n} \in \mathbb{R}^{(n+1)r}$. Moreover, if for a nonzero real q, $E^{anw_\lambda}[F|X_n](\vec{\xi_n})$ has the limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the conditional analytic Feynman w_φ^r-integral of F given X_n with parameter q and denoted by

$$E^{anf_q}[F|X_n](\vec{\xi_n}) = \lim_{\lambda \to -iq} E^{anw_\lambda}[F|X_n](\vec{\xi_n}).$$

Let $F : C^r[0,t] \to \mathbb{C}$ be a measurable function. For any $\lambda > 0$, ψ in $L_1(\mathbb{R}^r)$ and ξ in $\mathbb{R}^r$, let $\psi_t^{\lambda,\xi}(x) = \psi(\lambda^{-\frac{1}{2}} x(t) + \xi)$ and

$$(I_\lambda(F)\psi)(\xi) = \int_{C^r} F^{\lambda,\xi}(x) \psi_t^{\lambda,\xi}(x) dw_\varphi^r(x),$$

where $F^{\lambda,\xi}(x)$ is given by (6). If $I_\lambda(F)\psi$ is in $L_\infty(\mathbb{R}^r)$ as a function of ξ and if the correspondence $\psi \to I_\lambda(F)\psi$ gives an element of $\mathcal{L} \equiv \mathcal{L}(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r))$, we say that the operator-valued function space integral $I_\lambda(F)$ exists. Next suppose that there exists an $\mathcal{L}$-valued function which is weakly analytic in $\mathbb{C}_+$ and agrees with $I_\lambda(F)$ on $(0, \infty)$. Then this $\mathcal{L}$-valued function is denoted by $I_\lambda^{an}(F)$ and is called the analytic operator-valued Wiener w_φ^r-integral of F associated with parameter λ. Finally, for a nonzero real q, suppose that there exists an operator $J_q^{an}(F)$ in $\mathcal{L}$ such that for every ψ in $L_1(\mathbb{R}^r)$, $I_\lambda^{an}(F)\psi$ converges weakly to $J_q^{an}(F)\psi$ as λ approaches to $-iq$ through $\mathbb{C}_+$. Then $J_q^{an}(F)$ is called the analytic operator-valued Feynman w_φ^r-integral of F with parameter q. We can take ψ to be Borel measurable [8], and the weak limit and the weak analyticity are based on the weak* topology on $L_\infty(\mathbb{R}^r)$ induced by its pre-dual $L_1(\mathbb{R}^r)$ [2, 10].

Theorem 3.2. *Let the assumptions and notations be as given in Lemma 2.2 and X_n be given by (4). For $F : C^r[0,t] \to \mathbb{C}$ suppose that $E^{anw_\lambda}[F|X_n](\vec{\xi_n})$ exists for $\lambda \in \mathbb{C}_+$ and $m_L^{(n+1)r}$-a.e. $\vec{\xi_n} \in \mathbb{R}^{(n+1)r}$, and that for each bounded subset Ω of $\mathbb{C}_+$, there exists $M_\Omega \geq 0$ such that*

$$|E^{anw_\lambda}[F|X_n](\vec{\xi_n})| \leq M_\Omega \tag{9}$$

for all $\lambda \in \Omega$ and $m_L^{(n+1)r}$-a.e. $\vec{\xi_n} \in \mathbb{R}^{(n+1)r}$. Furthermore suppose that there exists a function Ψ on $\mathbb{C}_+ \times \mathbb{R}^r$ satisfying the following conditions:

(i) *for each $\lambda > 0$, $\Psi(\lambda, \eta) = \frac{d\varphi^r}{dm_L^r}(\lambda^{\frac{1}{2}}\eta)$ for m_L^r-a.e. $\eta \in \mathbb{R}^r$,*

(ii) *for m_L^r-a.e. $\eta \in \mathbb{R}^r$, $\Psi(\lambda, \eta)$ is analytic on $\mathbb{C}_+$ as a function of λ, and*

(iii) *for each bounded subset* Ω *of* $\mathbb{C}_+$, $\Psi(\lambda,\eta)$ *is bounded for all* $\lambda\in\Omega$ *and for* m_L^r*-a.e.* $\eta\in\mathbb{R}^r$.

Then for $\lambda\in\mathbb{C}_+$, *the analytic operator-valued Wiener* w_φ^r*-integral* $I_\lambda^{an}(F)$ *exists as an element of* $\mathcal{L}$ *and is given by*

$$(10)\qquad \begin{aligned}&(I_\lambda^{an}(F)\psi)(\xi)\\ &= \lambda^{\frac{r}{2}}\int_{\mathbb{R}^{(n+1)r}} E^{anw_\lambda}[F|X_n](\xi_0,\xi_1,\ldots,\xi_n)\psi(\xi_n)\Psi(\lambda,\xi_0-\xi)\\ &\qquad\times W_r(\lambda,\vec{\tau}_n,(\xi_0,\xi_1,\ldots,\xi_n))dm_L^r(\xi_0)dm_L^r(\xi_1)\cdots dm_L^r(\xi_n)\end{aligned}$$

for $\psi\in L_1(\mathbb{R}^r)$ *and* m_L^r*-a.e.* $\xi\in\mathbb{R}^r$, *where* W_r *is given by* (2). *In addition, suppose that* $n=1$ *and for a nonzero real* q, $E^{anf_q}[F|X_1](\xi_0,\xi_1)$ *exists for* m_L^{2r}*-a.e.* $(\xi_0,\xi_1)\in\mathbb{R}^{2r}$. *Moreover suppose that* Ψ *can be extended to* $(\mathbb{C}_+\cup\{-iq\})\times\mathbb{R}^r$ *with the following two additional conditions*:

(ii)′ *for* m_L^r*-a.e.* $\eta\in\mathbb{R}^r$, $\Psi(\lambda,\eta)$ *is continuous at* $\lambda=-iq$ *as a function of* λ, *and*

(iii)′ *there exists a* (*Borel measurable*) *function* $\Phi_q\in L_1(\mathbb{R}^r)$ *satisfying*

$$|\Psi(\lambda,\eta)|\le|\Phi_q(\eta)| \text{ for all } \lambda\in\Omega_\epsilon \text{ and } m_L^r\text{-a.e. } \eta\in\mathbb{R}^r,$$

where $\Omega_\epsilon=\{\lambda\in\mathbb{C}_+:|\lambda+iq|<\epsilon\}$ *for some* $\epsilon>0$.

Then the analytic operator-valued Feynman w_φ^r*-integral* $J_q^{an}(F)$ *exists as an element of* $\mathcal{L}$ *and it is given by*

$$(11)\quad \begin{aligned}(J_q^{an}(F)\psi)(\xi) &= (-iq)^{\frac{r}{2}}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r} E^{anf_q}[F|X_1](\xi_0,\xi_1)\psi(\xi_1)\\ &\qquad\times\Psi(-iq,\xi_0-\xi)W_r(-iq,\vec{\tau}_1,(\xi_0,\xi_1))dm_L^r(\xi_0)dm_L^r(\xi_1).\end{aligned}$$

Proof. For $\lambda>0$, $\psi\in L_1(\mathbb{R}^r)$ and $\xi\in\mathbb{R}^r$

$$\begin{aligned}(I_\lambda(F)\psi)(\xi) &= \int_{C^r} E[F^{\lambda,\xi}\psi_t^{\lambda,\xi}|X_n^{\lambda,\xi}](X_n^{\lambda,\xi}(x))dw_\varphi^r(x)\\ &= \int_{\mathbb{R}^{(n+1)r}} E[F^{\lambda,\xi}\psi_t^{\lambda,\xi}|X_n^{\lambda,\xi}](\vec{\xi}_n)dP_{X_n^{\lambda,\xi}}(\vec{\xi}_n),\end{aligned}$$

where $X_n^{\lambda,\xi}$ is given by (5) and $P_{X_n^{\lambda,\xi}}$ is the probability distribution of $X_n^{\lambda,\xi}$ on the Borel class of $\mathbb{R}^{(n+1)r}$. For $\vec{\xi}_n=(\xi_0,\xi_1,\ldots,\xi_n)\in\mathbb{R}^{(n+1)r}$

$$\begin{aligned}E[F^{\lambda,\xi}\psi_t^{\lambda,\xi}|X_n^{\lambda,\xi}](\vec{\xi}_n) &= E[F(\lambda^{-\frac12}(x-[x])+[\vec{\xi}_n])\psi(\lambda^{-\frac12}(x-[x])(t)+[\vec{\xi}_n](t))]\\ &= (K_\lambda(F))(\vec{\xi}_n)\psi(\xi_n)\end{aligned}$$

by (7), where $K_\lambda(F)$ is given by (8). By Lemma 2.2

$$\begin{aligned}(I_\lambda(F)\psi)(\xi) = \lambda^{\frac{r}{2}}\int_{\mathbb{R}^{(n+1)r}} &(K_\lambda(F))(\xi_0,\xi_1,\ldots,\xi_n)\psi(\xi_n)W_r(\lambda,\vec{\tau}_n,(\xi_0,\xi_1,\\ &\ldots,\xi_n))\frac{d\varphi^r}{dm_L^r}(\lambda^{\frac12}(\xi_0-\xi))dm_L^r(\xi_0)dm_L^r(\xi_1)\cdots dm_L^r(\xi_n).\end{aligned}$$

Now suppose that Ψ satisfies (i), (ii) and (iii). By (i)

$$(I_\lambda(F)\psi)(\xi) = \lambda^{\frac{r}{2}} \int_{\mathbb{R}^{(n+1)r}} (K_\lambda(F))(\xi_0, \xi_1, \ldots, \xi_n)\psi(\xi_n)\Psi(\lambda, \xi_0 - \xi) \times W_r(\lambda, \vec{\tau}_n, (\xi_0, \xi_1, \ldots, \xi_n))dm_L^r(\xi_0)dm_L^r(\xi_1)\cdots dm_L^r(\xi_n).$$

Let Ω_λ be a bounded subset of $\mathbb{C}_+$ containing λ. Then for m_L^r-a.e. $\xi \in \mathbb{R}^r$

$$|(I_\lambda(F)\psi)(\xi)| \leq \lambda^{\frac{r}{2}} M_{\Omega_\lambda} \|\Psi\|_{\Omega_\lambda,\infty} \|\psi\|_{L_1(\mathbb{R}^r)},$$

where $\|\Psi\|_{\Omega_\lambda,\infty}$ denotes the essential supremum of Ψ on $\Omega_\lambda \times \mathbb{R}^r$, so that $I_\lambda(F)\psi \in L_\infty(\mathbb{R}^r)$ and $I_\lambda(F) \in \mathcal{L}(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r))$. For $\psi \in L_1(\mathbb{R}^r)$ let $(Q_\lambda(F)\psi)(\xi)$ be the right hand side of (10) for $(\lambda, \xi) \in \mathbb{C}_+ \times \mathbb{R}^r$ and let Ω be any bounded subset of $\mathbb{C}_+$. By the same method

$$(12) \qquad |(Q_\lambda(F)\psi)(\xi)| \leq |\lambda|^{\frac{r}{2}} M_\Omega \|\Psi\|_{\Omega,\infty} \|\psi\|_{L_1(\mathbb{R}^r)} \left(\frac{|\lambda|}{\mathrm{Re}\lambda}\right)^{\frac{nr}{2}}$$

for all $\lambda \in \Omega$ and m_L^r-a.e. $\xi \in \mathbb{R}^r$ so that $Q_\lambda(F)\psi \in L_\infty(\mathbb{R}^r)$ and $Q_\lambda(F) \in \mathcal{L}(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r))$ for $\lambda \in \mathbb{C}_+$. Using the same method as used in the proof of Theorem 3.5 in [5] we can prove that $I_\lambda^{an}(F)$ exists and $I_\lambda^{an}(F) = Q_\lambda(F)$ for $\lambda \in \mathbb{C}_+$. The remainder part of the theorem follows from Theorem 3.5 in [5]. $\square$

Letting $\Psi(\lambda, \eta) = (\frac{1}{2\pi\alpha^2})^{\frac{r}{2}} \exp\{-\frac{\lambda}{2\alpha^2}\|\eta\|_{\mathbb{R}^r}^2\}$ for $\alpha > 0$, $\lambda \in \mathbb{C}_+$ and $\eta \in \mathbb{R}^r$, we have the following corollary from Theorem 3.2.

Corollary 3.3. *Let X_n be given by* (4). *Moreover let φ^r be normally distributed with the mean vector $\vec{0} \in \mathbb{R}^r$ and the nontrivial variance-covariance matrix $\alpha^2 I_r$, where $\alpha > 0$ and I_r is the r-dimensional identity matrix. For $F : C^r[0,t] \to \mathbb{C}$ suppose that $E^{anw_\lambda}[F|X_n](\vec{\xi}_n)$ satisfies* (9) *in Theorem 3.2. Then for $\lambda \in \mathbb{C}_+$, the analytic operator-valued Wiener w_φ^r-integral $I_\lambda^{an}(F)$ exists as an element of $\mathcal{L}$ and is given by*

$$(13) \qquad (I_\lambda^{an}(F)\psi)(\xi) = \left(\frac{\lambda}{2\pi\alpha^2}\right)^{\frac{r}{2}} \int_{\mathbb{R}^{(n+1)r}} E^{anw_\lambda}[F|X_n](\xi_0, \xi_1, \ldots, \xi_n)\psi(\xi_n)W_r(\lambda, \vec{\tau}_n, (\xi_0, \xi_1, \ldots, \xi_n)) \exp\left\{-\frac{\lambda\|\xi_0 - \xi\|_{\mathbb{R}^r}^2}{2\alpha^2}\right\} dm_L^r(\xi_0)dm_L^r(\xi_1)\cdots dm_L^r(\xi_n)$$

for $\psi \in L_1(\mathbb{R}^r)$ and m_L^r-a.e. $\xi \in \mathbb{R}^r$, where W_r is given by (2).

Theorem 3.4. *If the conditions* (iii) *and* (iii)$'$ *in Theorem 3.2 are replaced by the condition: for each bounded subset Ω of $\mathbb{C}_+$, there exists a* (*Borel measurable*) *function $\Phi_\Omega \in L_1(\mathbb{R}^r)$ satisfying*

$$(14) \qquad |\Psi(\lambda, \eta)| \leq |\Phi_\Omega(\eta)| \text{ for all } \lambda \in \Omega \text{ and } m_L^r\text{-a.e. } \eta \in \mathbb{R}^r,$$

then the conclusions of Theorem 3.2 hold true.

Proof. Suppose that φ^r is absolutely continuous and Ψ satisfies (i) and (ii) of Theorem 3.2. By the same method as used in the proof of Theorem 3.2

$$\begin{aligned}(I_\lambda(F)\psi)(\xi) = \lambda^{\frac{r}{2}} \int_{\mathbb{R}^{(n+1)r}} &(K_\lambda(F))(\xi_0,\xi_1,\ldots,\xi_n)\psi(\xi_n)\Psi(\lambda,\xi_0-\xi)\\ &\times W_r(\lambda,\vec{\tau}_n,(\xi_0,\xi_1,\ldots,\xi_n))dm_L^r(\xi_0)dm_L^r(\xi_1)\cdots dm_L^r(\xi_n)\end{aligned}$$

for $\lambda > 0$, $\psi \in L_1(\mathbb{R}^r)$ and $\xi \in \mathbb{R}^r$, where $K_\lambda(F)$ is given by (8). Let Ω_λ be a bounded subset of $\mathbb{C}_+$ containing λ. Then for m_L^r-a.e. $\xi \in \mathbb{R}^r$

$$\begin{aligned}|(I_\lambda(F)\psi)(\xi)| \le \lambda^{\frac{r}{2}} M_{\Omega_\lambda} \int_{\mathbb{R}^{(n+1)r}} &|\psi(\xi_n)|\,|\Phi_{\Omega_\lambda}(\xi_0-\xi)| W_r(\lambda,\vec{\tau}_n,\\ &(\xi_0,\xi_1,\ldots,\xi_n))dm_L^r(\xi_0)dm_L^r(\xi_1)\cdots dm_L^r(\xi_n)\end{aligned}$$

by (14), where Ω is replaced by Ω_λ. For $j = 1,\ldots,n-1$, a simple calculation shows that

$$\begin{aligned}&\left[\frac{\lambda}{2\pi\sqrt{(t_{j+1}-t_j)(t_j-t_{j-1})}}\right]^r \int_{\mathbb{R}^r} \exp\left\{-\frac{\lambda}{2}\left(\frac{\|\xi_{j+1}-\xi_j\|_{\mathbb{R}^r}^2}{t_{j+1}-t_j}\right.\right.\\ &\quad\left.\left.+\frac{\|\xi_j-\xi_{j-1}\|_{\mathbb{R}^r}^2}{t_j-t_{j-1}}\right)\right\}dm_L^r(\xi_j)\\ &=\left[\frac{\lambda}{2\pi(t_{j+1}-t_{j-1})}\right]^{\frac{r}{2}} \exp\left\{-\frac{\lambda\|\xi_{j+1}-\xi_{j-1}\|_{\mathbb{R}^r}^2}{2(t_{j+1}-t_{j-1})}\right\}.\end{aligned}$$

Hence

$$\begin{aligned}|(I_\lambda(F)\psi)(\xi)| &\le \lambda^{\frac{r}{2}} M_{\Omega_\lambda}\left(\frac{\lambda}{2\pi t}\right)^{\frac{r}{2}} \int_{\mathbb{R}^{2r}} |\psi(\xi_n)|\,|\Phi_{\Omega_\lambda}(\xi_0-\xi)|\\ &\qquad\times \exp\left\{-\frac{\lambda\|\xi_n-\xi_0\|_{\mathbb{R}^r}^2}{2t}\right\}dm_L^{2r}(\xi_0,\xi_n)\\ &\le M_{\Omega_\lambda}\|\Phi_{\Omega_\lambda}\|_{L_1(\mathbb{R}^r)}\|\psi\|_{L_1(\mathbb{R}^r)}\left(\frac{\lambda}{\sqrt{2\pi t}}\right)^r,\end{aligned}$$

so that $I_\lambda(F)\psi \in L_\infty(\mathbb{R}^r)$ and $I_\lambda(F) \in \mathcal{L}(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r))$. For $\psi \in L_1(\mathbb{R}^r)$ let $(Q_\lambda(F)\psi)(\xi)$ be the right hand side of (10) for $(\lambda,\xi) \in \mathbb{C}_+ \times \mathbb{R}^r$. Using the same method as used in the proof of Theorem 3.2, we can prove the existence of $I_\lambda^{an}(F)$ and the equality $I_\lambda^{an}(F) = Q_\lambda(F)$ for $\lambda \in \mathbb{C}_+$ if (12) is replaced by the following inequality

$$|(Q_\lambda(F)\psi)(\xi)| \le M_\Omega|\lambda|^{\frac{(n+1)r}{2}}(\mathrm{Re}\lambda)^{-\frac{(n-1)r}{2}}\|\Phi_\Omega\|_{L_1(\mathbb{R}^r)}\|\psi\|_{L_1(\mathbb{R}^r)}\left(\frac{1}{2\pi t}\right)^{\frac{r}{2}}$$

which is easily obtained for $\lambda \in \Omega$ and m_L^r-a.e. $\xi \in \mathbb{R}^r$, where Ω is arbitrary bounded subset of $\mathbb{C}_+$. The remainder part of the theorem follows from Theorems 3.5 and 3.7 in [5]. □

Theorem 3.5. *Let $n \ge 2$, the assumptions be as given in Lemma 2.2 and X_n be given by* (4)*. For $F : C^r[0,t] \to \mathbb{C}$ suppose that $E^{anw_\lambda}[F|X_n](\vec{\xi}_n)$ exists for*

$\lambda \in \mathbb{C}_+$ and $m_L^{(n+1)r}$-a.e. $\vec{\xi}_n \in \mathbb{R}^{(n+1)r}$, and that for each bounded subset Ω of $\mathbb{C}_+$, there exists a (Borel measurable) function $\Psi_\Omega \in L_1(\mathbb{R}^{nr})$ such that

$$|E^{anw_\lambda}[F|X_n](\xi_0, \xi_1, \ldots, \xi_{n-1}, \xi_n)| \le |\Psi_\Omega(\xi_0, \xi_1, \ldots, \xi_{n-1})| \tag{15}$$

for all $\lambda \in \Omega$ and $m_L^{(n+1)r}$-a.e. $(\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{(n+1)r}$. Furthermore suppose that there exists a function Ψ on $\mathbb{C}_+ \times \mathbb{R}^r$ satisfying conditions (i), (ii), (iii) *of Theorem 3.2. Then for $\lambda \in \mathbb{C}_+$, the analytic operator-valued Wiener w_φ^r-integral $I_\lambda^{an}(F)$ exists as an element of $\mathcal{L}$ and is given by* (10)*. In addition, suppose that for a nonzero real q, $E^{anf_q}[F|X_n](\vec{\xi}_n)$ exists for $m_L^{(n+1)r}$-a.e. $\vec{\xi}_n \in \mathbb{R}^{(n+1)r}$. Moreover suppose that Ψ can be extended to $(\mathbb{C}_+ \cup \{-iq\}) \times \mathbb{R}^r$ with the condition* (ii)$'$ *of Theorem 3.2. Then the analytic operator-valued Feynman w_φ^r-integral $J_q^{an}(F)$ exists as an element of $\mathcal{L}$ and it is given by* (10) *replacing λ and E^{anw_λ} by $-iq$ and E^{anf_q}, respectively.*

Proof. Suppose that φ^r is absolutely continuous and Ψ satisfies (i), (ii) and (iii) of Theorem 3.2. By the same method as used in the proof of Theorem 3.2

$$\begin{aligned}(I_\lambda(F)\psi)(\xi) = \lambda^{\frac{r}{2}} \int_{\mathbb{R}^{(n+1)r}} &(K_\lambda(F))(\xi_0, \xi_1, \ldots, \xi_{n-1}, \xi_n)\psi(\xi_n)\Psi(\lambda, \xi_0 - \xi) \\ &\times W_r(\lambda, \vec{\tau}, (\xi_0, \xi_1, \ldots, \xi_n)) dm_L^r(\xi_0) dm_L^r(\xi_1) \cdots dm_L^r(\xi_n)\end{aligned}$$

for $\lambda > 0$, $\psi \in L_1(\mathbb{R}^r)$ and $\xi \in \mathbb{R}^r$, where $K_\lambda(F)$ is given by (8). Let Ω_λ be a bounded subset of $\mathbb{C}_+$ containing λ. Then for m_L^r-a.e. $\xi \in \mathbb{R}^r$

$$\begin{aligned}|(I_\lambda(F)\psi)(\xi)| &\le \lambda^{\frac{r}{2}} \|\Psi\|_{\Omega_\lambda,\infty} \left[\prod_{j=1}^n \frac{\lambda}{2\pi(t_j - t_{j-1})}\right]^{\frac{r}{2}} \int_{\mathbb{R}^{(n+1)r}} |\psi(\xi_n)| \\ &\qquad \times |\Psi_{\Omega_\lambda}(\xi_0, \xi_1, \ldots, \xi_{n-1})| dm_L^r(\xi_0) dm_L^r(\xi_1) \cdots dm_L^r(\xi_n) \\ &= \lambda^{\frac{r}{2}} \|\Psi\|_{\Omega_\lambda,\infty} \|\Psi_{\Omega_\lambda}\|_{L_1(\mathbb{R}^{nr})} \|\psi\|_{L_1(\mathbb{R}^r)} \left[\prod_{j=1}^n \frac{\lambda}{2\pi(t_j - t_{j-1})}\right]^{\frac{r}{2}},\end{aligned}$$

where $\|\Psi\|_{\Omega_\lambda,\infty}$ denotes the essential supremum of Ψ on $\Omega_\lambda \times \mathbb{R}^r$ so that $I_\lambda(F)\psi \in L_\infty(\mathbb{R}^r)$ and $I_\lambda(F) \in \mathcal{L}(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r))$. For $\psi \in L_1(\mathbb{R}^r)$ let $(Q_\lambda(F)\psi)(\xi)$ be the right hand side of (10) for $(\lambda, \xi) \in \mathbb{C}_+ \times \mathbb{R}^r$. Using the same method as used in the proof of Theorem 3.2, we can prove the existence of $I_\lambda^{an}(F)$ and the equality $I_\lambda^{an}(F) = Q_\lambda(F)$ for $\lambda \in \mathbb{C}_+$ if (12) is replaced by the following inequality

$$|(Q_\lambda(F)\psi)(\xi)| \le |\lambda|^{\frac{r}{2}} \|\Psi\|_{\Omega,\infty} \|\Psi_\Omega\|_{L_1(\mathbb{R}^{nr})} \|\psi\|_{L_1(\mathbb{R}^r)} \left[\prod_{j=1}^n \frac{|\lambda|}{2\pi(t_j - t_{j-1})}\right]^{\frac{r}{2}}$$

which is easily obtained for $\lambda \in \Omega$ and m_L^r-a.e. $\xi \in \mathbb{R}^r$, where Ω is arbitrary bounded subset of $\mathbb{C}_+$. Furthermore suppose that Ψ satisfies (ii)$'$ of Theorem 3.2. For $\psi \in L_1(\mathbb{R}^r)$ let $(J_q^{an}(F)\psi)(\xi)$ be the right hand side of (10) for $\xi \in \mathbb{R}^r$ where λ and E^{anw_λ} are replaced by $-iq$ and E^{anf_q}, respectively, and let $\Omega_\epsilon = \{\lambda \in \mathbb{C}_+ : |\lambda + iq| < \epsilon\}$ for some $\epsilon > 0$. Then, by (ii)$'$ of Theorem 3.2,

$$\begin{aligned}
&|(J_q^{an}(F)\psi)(\xi)| \\
&\leq |q|^{\frac{r}{2}} \|\Psi\|_{\Omega_\epsilon,\infty} \|\Psi_{\Omega_\epsilon}\|_{L_1(\mathbb{R}^{nr})} \|\psi\|_{L_1(\mathbb{R}^r)} \left[\prod_{j=1}^n \frac{|q|}{2\pi(t_j - t_{j-1})}\right]^{\frac{r}{2}} \\
&\leq (|q| + \epsilon)^{\frac{r}{2}} \|\Psi\|_{\Omega_\epsilon,\infty} \|\Psi_{\Omega_\epsilon}\|_{L_1(\mathbb{R}^{nr})} \|\psi\|_{L_1(\mathbb{R}^r)} \left[\prod_{j=1}^n \frac{|q| + \epsilon}{2\pi(t_j - t_{j-1})}\right]^{\frac{r}{2}}
\end{aligned}$$

which implies $J_q^{an}(F)\psi \in L_\infty(\mathbb{R}^r)$ and $J_q^{an}(F) \in \mathcal{L}(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r))$. The remainder part of the theorem follows from Theorem 3.5 in [5]. $\square$

By Theorem 3.5 we can obtain the following corollary.

Corollary 3.6. *Let $n \geq 2$, X_n be given by* (4) *and φ^r be the measure as given in Corollary* 3.3. *Suppose that $F : C^r[0,t] \to \mathbb{C}$ satisfies the conditions in Theorem 3.5. Then for nonzero real q, the analytic operator-valued Feynman w_φ^r-integral $J_q^{an}(F)$ exists as an element of $\mathcal{L}$ and it is given by the right hand side of* (13) *replacing λ and E^{anw_λ} by $-iq$ and E^{anf_q}, respectively.*

Using the same method as used in the proof of Theorem 3.5, we can prove the following theorem.

Theorem 3.7. *If the conditions* (iii) *and* (15) *in Theorems 3.2 and 3.5, respectively, are replaced by* (14) *in Theorem 3.4 and the following condition: for each bounded subset Ω of $\mathbb{C}_+$, there exists a* (*Borel measurable*) *function $\Psi_\Omega \in L_1(\mathbb{R}^{(n-1)r})$ such that*

$$|E^{anw_\lambda}[F|X_n](\xi_0, \xi_1, \ldots, \xi_{n-1}, \xi_n)| \leq |\Psi_\Omega(\xi_1, \ldots, \xi_{n-1})|$$

for all $\lambda \in \Omega$ and $m_L^{(n+1)r}$-a.e. $(\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{(n+1)r}$, respectively, then the statement of Theorem 3.5 holds true.

Theorem 3.8. *Let $n \geq 2$, the assumptions be as given in Lemma 2.2 and X_n be given by* (4). *For $F : C^r[0,t] \to \mathbb{C}$ suppose that $E^{anw_\lambda}[F|X_n](\vec{\xi_n})$ satisfies* (9) *in Theorem 3.2. Let*

$$B(x) = f(x(t_0), \ldots, x(t_{n-1}))F(x)$$

for w_φ^r-a.e. $x \in C^r[0,t]$, where $f \in L_1(\mathbb{R}^{nr})$. Furthermore suppose that there exists a function Ψ on $\mathbb{C}_+ \times \mathbb{R}^r$ satisfying conditions (i), (ii) *and* (iii) *of Theorem 3.2. Then for $\lambda \in \mathbb{C}_+$, the analytic operator-valued Wiener w_φ^r-integral $I_\lambda^{an}(B)$ exists as an element of $\mathcal{L}$ and is given by*

$$\begin{aligned}
(16) \quad &(I_\lambda^{an}(B)\psi)(\xi) \\
&= \lambda^{\frac{r}{2}} \int_{\mathbb{R}^{(n+1)r}} f(\xi_0, \xi_1, \ldots, \xi_{n-1}) E^{anw_\lambda}[F|X_n](\xi_0, \xi_1, \ldots, \xi_{n-1}, \xi_n)\psi(\xi_n) \\
&\quad \times \Psi(\lambda, \xi_0 - \xi) W_r(\lambda, \vec{\tau}, (\xi_0, \xi_1, \ldots, \xi_n)) dm_L^r(\xi_0) dm_L^r(\xi_1) \cdots dm_L^r(\xi_n)
\end{aligned}$$

for $\psi \in L_1(\mathbb{R}^r)$ and m_L^r-a.e. $\xi \in \mathbb{R}^r$. In addition, suppose that for a nonzero real q, $E^{anf_q}[F|X_n](\vec{\xi_n})$ exists for $m_L^{(n+1)r}$-a.e. $\vec{\xi_n} \in \mathbb{R}^{(n+1)r}$. Moreover suppose that Ψ can be extended to $(\mathbb{C}_+ \cup \{-iq\}) \times \mathbb{R}^r$ with the condition (ii)$'$ *of Theorem 3.2. Then the analytic operator-valued Feynman w_φ^r-integral $J_q^{an}(B)$ exists as an element of $\mathcal{L}$ and it is given by the right hand side of* (16), *where λ and E^{anw_λ} are replaced by $-iq$ and E^{anf_q}, respectively.*

Proof. For $\lambda > 0$ and $\vec{\xi_n} = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{(n+1)r}$

$$\begin{aligned}(K_\lambda(B))(\vec{\xi_n}) &= \int_{C^r} f(\lambda^{-\frac{1}{2}}(x-[x])(t_0) + [\vec{\xi_n}](t_0), \dots, \lambda^{-\frac{1}{2}}(x-[x])(t_{n-1}) \\ &\qquad + [\vec{\xi_n}](t_{n-1}))F(\lambda^{-\frac{1}{2}}(x-[x]) + [\vec{\xi_n}])dw_\varphi^r(x) \\ &= f(\xi_0, \dots, \xi_{n-1}) \int_{C^r} F(\lambda^{-\frac{1}{2}}(x-[x]) + [\vec{\xi_n}])dw_\varphi^r(x) \\ &= f(\xi_0, \dots, \xi_{n-1})(K_\lambda(F))(\vec{\xi_n}),\end{aligned}$$

where K_λ is given by (8). Since $f(\xi_0, \dots, \xi_{n-1})$ is independent of $\lambda \in \mathbb{C}_+$, the existence of $E^{anw_\lambda}[B|X_n](\vec{\xi_n})$ follows from the existence of $E^{anw_\lambda}[F|X_n](\vec{\xi_n})$ and

$$E^{anw_\lambda}[B|X_n](\vec{\xi_n}) = f(\xi_0, \xi_1, \dots, \xi_{n-1})E^{anw_\lambda}[F|X_n](\vec{\xi_n}).$$

Furthermore, for any bounded subset Ω of $\mathbb{C}_+$,

$$\begin{aligned}|E^{anw_\lambda}[B|X_n](\vec{\xi_n})| &= |f(\xi_0, \xi_1, \dots, \xi_{n-1})|\,|E^{anw_\lambda}[F|X_n](\vec{\xi_n})| \\ &\le M_\Omega |f(\xi_0, \xi_1, \dots, \xi_{n-1})|\end{aligned}$$

for all $\lambda \in \Omega$ and $m_L^{(n+1)r}$-a.e. $\vec{\xi_n} = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{(n+1)r}$. The theorem now follows from Theorem 3.5. □

By Corollary 3.3 and Theorem 3.8, we have the following corollary.

Corollary 3.9. *Let $n \ge 2$ and X_n be given by* (4). *Moreover let φ^r be the measure as given in Corollary 3.3. Suppose that B is as given in Theorem 3.8. Then for nonzero real q, the analytic operator-valued Feynman w_φ^r-integral $J_q^{an}(B)$ exists as an element of $\mathcal{L}$ and it is given by*

(17)

$$\begin{aligned}&(J_q^{an}(B)\psi)(\xi) \\ &= \left(\frac{q}{2\pi i \alpha^2}\right)^{\frac{r}{2}} \int_{\mathbb{R}^{(n+1)r}} f(\xi_0, \xi_1, \dots, \xi_{n-1})E^{anf_q}[F|X_n](\xi_0, \xi_1, \dots, \xi_{n-1}, \xi_n) \\ &\quad \times \psi(\xi_n)W_r(-iq, \vec{\tau}_n(\xi_0, \xi_1, \dots, \xi_n)) \exp\left\{\frac{qi}{2}\frac{\|\xi_0 - \xi\|_{\mathbb{R}^r}^2}{\alpha^2}\right\} dm_L^r(\xi_0)dm_L^r(\xi_1) \\ &\quad \cdots dm_L^r(\xi_n)\end{aligned}$$

for $\psi \in L_1(\mathbb{R}^r)$ and m_L^r-a.e. $\xi \in \mathbb{R}^r$, where W_r is given by (2).

By the same method as used in the proof of Theorem 3.8, we obtain the following theorem from Theorem 3.7.

Theorem 3.10. *Let $n \geq 2$, the assumptions be as given in Lemma 2.2 and X_n be given by* (4). *For $F : C^r[0,t] \to \mathbb{C}$ suppose that $E^{anw_\lambda}[F|X_n](\vec{\xi}_n)$ satisfies* (9) *in Theorem 3.2. Let*

$$D(x) = f(x(t_1), \ldots, x(t_{n-1}))F(x)$$

for w_φ^r-a.e. $x \in C^r[0,t]$, where $f \in L_1(\mathbb{R}^{(n-1)r})$. Furthermore suppose that there exists a function Ψ on $\mathbb{C}_+ \times \mathbb{R}^r$ satisfying conditions (i), (ii) *of Theorem 3.2 and* (14) *of Theorem 3.4. Then for $\lambda \in \mathbb{C}_+$, the analytic operator-valued Wiener w_φ^r-integral $I_\lambda^{an}(D)$ exists as an element of $\mathcal{L}$ and is given by* (16), *where $f(\xi_0, \xi_1, \ldots, \xi_{n-1})$ is replaced by $f(\xi_1, \ldots, \xi_{n-1})$. In addition, suppose that for a nonzero real q, $E^{anf_q}[F|X_n](\vec{\xi}_n)$ exists for $m_L^{(n+1)r}$-a.e. $\vec{\xi}_n \in \mathbb{R}^{(n+1)r}$. Moreover suppose that Ψ can be extended to $(\mathbb{C}_+ \cup \{-iq\}) \times \mathbb{R}^r$ with the condition* (ii)$'$ *of Theorem 3.2. Then the analytic operator-valued Feynman w_φ^r-integral $J_q^{an}(D)$ exists as an element of $\mathcal{L}$ and it is given by the right hand side of* (16) *where λ, E^{anw_λ} and $f(\xi_0, \xi_1, \ldots, \xi_{n-1})$ are replaced by $-iq$, E^{anf_q} and $f(\xi_1, \ldots, \xi_{n-1})$, respectively.*

Remark 3.11. The function Ψ satisfying the conditions in Theorems 3.2, 3.4, 3.5, 3.7, 3.8 and 3.10 exists [5]. We note that such a function which is not a normal density can be obtained.

4. The conditional w_φ-integrals of bounded functions and the operator-valued function space integrals

Throughout this section, we assume that $r = 1$. Let $\mathcal{M} = \mathcal{M}(L_2[0,t])$ be the class of all $\mathbb{C}$-valued Borel measures of bounded variation over $L_2[0,t]$ and let $\mathcal{S}_{w_\varphi}$ be the space of all functions F of the form; for $\sigma \in \mathcal{M}$

$$F(x) = \int_{L_2[0,t]} \exp\{i(v,x)\} d\sigma(v) \tag{18}$$

for w_φ-a.e. $x \in C[0,t]$. Using the same method in [3], it can be shown that $\mathcal{S}_{w_\varphi}$ is a Banach algebra. Let $\hat{\mathrm{M}}(\mathbb{R}^\gamma)$ be the set of all functions ϕ on $\mathbb{R}^\gamma$ defined by

$$\phi(u) = \int_{\mathbb{R}^\gamma} \exp\{i\langle z, u\rangle_{\mathbb{R}^\gamma}\} d\rho(z), \tag{19}$$

for $u \in \mathbb{R}^\gamma$, where ρ is a complex Borel measure of bounded variation over $\mathbb{R}^\gamma$.

For each $j = 1, \ldots, n$, let $\alpha_j(s) = \frac{1}{\sqrt{t_j - t_{j-1}}}\chi_{(t_{j-1}, t_j]}(s)$ where $0 \leq s \leq t$. Let V be the subspace of $L_2[0,t]$ generated by $\{\alpha_1, \ldots, \alpha_n\}$ and let $V^\perp$ denote the orthogonal complement of V. Let $\mathcal{P}$ and $\mathcal{P}^\perp$ be the orthogonal projections from $L_2[0,t]$ to V and $V^\perp$, respectively. Then for $v \in L_2[0,t]$

$$v - \mathcal{P}v = \mathcal{P}^\perp v.$$

Moreover it is not difficult to show that

$$(v,[x]) = (\mathcal{P}v, x)$$

for $x \in C[0,t]$. For $v \in L_2[0,t]$ and $\vec{\xi_n} = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$, we can show that

$$(v,[\vec{\xi_n}]) = \sum_{j=1}^{n} (\mathcal{P}v)(t_j)(\xi_j - \xi_{j-1}). \tag{20}$$

Throughout this paper, let $\{v_1, v_2, \dots, v_\gamma\}$ be an orthonormal subset of $L_2[0,t]$ such that $\{\mathcal{P}^\perp v_1, \dots, \mathcal{P}^\perp v_\gamma\}$ are independent. Let $\{e_1, \dots, e_\gamma\}$ be the orthonormal set obtained from $\{\mathcal{P}^\perp v_1, \dots, \mathcal{P}^\perp v_\gamma\}$ by the Gram-Schmidt orthonormalization process. Now, for $l = 1, \dots, \gamma$, let

$$\mathcal{P}^\perp v_l = \sum_{j=1}^{\gamma} \alpha_{lj} e_j \tag{21}$$

be the linear combinations of the e_js and let

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1\gamma} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2\gamma} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{\gamma 1} & \alpha_{\gamma 2} & \cdots & \alpha_{\gamma\gamma} \end{bmatrix}$$

be the coefficient matrix of the combinations. Let $T_A : \mathbb{R}^\gamma \to \mathbb{R}^\gamma$ and $T_{A^T} : \mathbb{R}^\gamma \to \mathbb{R}^\gamma$ be given by

$$T_A(z) = zA \text{ and } T_{A^T}(z) = zA^T \tag{22}$$

where z is arbitrary row-vector in $\mathbb{R}^\gamma$ and A^T is the transpose of A. We note that A is invertible so that T_A and T_{A^T} are isomorphisms.

For convenience we introduce useful notations from the Gram-Schmidt orthonormalization process. For $v \in L_2[0,t]$, we obtain an orthonormal set $\{e_1, \dots, e_\gamma, e_{\gamma+1}\}$ as follows; let

$$c_j(v) = \begin{cases} \langle v, e_j \rangle_2 & \text{for } j = 1, \dots, \gamma \\ \sqrt{\|v\|_2^2 - \sum_{l=1}^{\gamma} \langle v, e_l \rangle_2^2} & \text{for } j = \gamma + 1 \end{cases} \tag{23}$$

and

$$e_{\gamma+1} = \frac{1}{c_{\gamma+1}(v)} \left[v - \sum_{j=1}^{\gamma} c_j(v) e_j \right]$$

if $c_{\gamma+1}(v) \neq 0$. Then

$$v = \sum_{j=1}^{\gamma+1} c_j(v) e_j \text{ and } \|v\|_2^2 = \sum_{j=1}^{\gamma+1} [c_j(v)]^2. \tag{24}$$

We note that the equalities in (24) hold trivially for the case $c_{\gamma+1}(v)=0$. For $v\in L_2[0,t]$ let

$$(\vec{v},x)=((v_1,x),\dots,(v_\gamma,x)) \text{ for } x\in C[0,t].$$

Theorem 4.1. *Let X_n be given by (4) with $r=1$ and $G(x)=F(x)\phi(\vec{v},x)$ for w_φ-a.e. $x\in C[0,t]$, where $F\in\mathcal{S}_{w_\varphi}$ and $\phi\in\hat{\mathrm{M}}(\mathbb{R}^\gamma)$ are given by (18) and (19), respectively. For $v\in L_2[0,t]$ let $c_j(\mathcal{P}^\perp v)$ be given by (23), where v is replaced by $\mathcal{P}^\perp v$ for $j=1,\dots,\gamma$. Then for $\lambda\in\mathbb{C}_+$, $E^{anw_\lambda}[G|X_n](\vec{\xi_n})$ exists for m_L^{n+1}-a.e. $\vec{\xi_n}\in\mathbb{R}^{n+1}$, and it is given by*

$$\begin{aligned}(25)\qquad & E^{anw_\lambda}[G|X_n](\vec{\xi_n})\\ &=\int_{L_2[0,t]}\int_{\mathbb{R}^\gamma}\exp\bigg\{i[(v,[\vec{\xi_n}])+\langle z,(\vec{v},[\vec{\xi_n}])\rangle_{\mathbb{R}^\gamma}]-\frac{1}{2\lambda}[\|T_A(z)\|_{\mathbb{R}^\gamma}^2\\ &\quad+2\langle\vec{c}(\mathcal{P}^\perp v),T_A(z)\rangle_{\mathbb{R}^\gamma}+\|\mathcal{P}^\perp v\|_2^2]\bigg\}d\rho(z)d\sigma(v),\end{aligned}$$

where $\vec{c}(\mathcal{P}^\perp v)=(c_1(\mathcal{P}^\perp v),\dots,c_\gamma(\mathcal{P}^\perp v))$ and T_A is given by (22). Furthermore, for nonzero real q, $E^{anf_q}[G|X_n](\vec{\xi_n})$ exists and it is given by the right-hand side of (25) where λ is replaced by $-iq$. In particular, if $n=1$, then $E^{anf_q}[G|X_1](\xi_0,\xi_1)$ is given by

$$\begin{aligned}(26)\qquad & E^{anf_q}[G|X_1](\xi_0,\xi_1)\\ &=\int_{L_2[0,t]}\int_{\mathbb{R}^\gamma}\exp\bigg\{i\frac{\xi_1-\xi_0}{t}\bigg[\int_0^t v(s)ds+\sum_{j=1}^{\gamma}z_j\int_0^t v_j(s)ds\bigg]+\frac{1}{2qi}\bigg[\sum_{j=1}^{\gamma}\\ &\bigg(\sum_{l=1}^{\gamma}z_l\alpha_{lj}\bigg)^2+2\sum_{j=1}^{\gamma}\bigg(\int_0^t v(s)e_j(s)ds-\frac{1}{t}\int_0^t v(s)ds\int_0^t e_j(s)ds\bigg)\\ &\times\bigg(\sum_{l=1}^{\gamma}z_l\alpha_{lj}\bigg)+\int_0^t[v(s)]^2ds-\frac{1}{t}\bigg[\int_0^t v(s)ds\bigg]^2\bigg]\bigg\}d\rho(z)d\sigma(v)\end{aligned}$$

for m_L^2-a.e. $(\xi_0,\xi_1)\in\mathbb{R}^2$, where $z=(z_1,\dots,z_\gamma)$ and the α_{lj}s are as given in (21).

Proof. The equation (25) and existence of $E^{anf_q}[G|X_n]$ follow from Theorem 2.6 in [6]. To prove the remainder part of the theorem, suppose that $n=1$ and let $\vec{\xi_1}=(\xi_0,\xi_1)$. Then for $v\in L_2[0,t]$

$$(27)\qquad (v,[\vec{\xi_1}])=\frac{\xi_1-\xi_0}{t}\int_0^t v(s)ds.$$

Furthermore

$$\mathcal{P}^\perp v=v-\mathcal{P}v=v-\langle v,\alpha_1\rangle_2\alpha_1=v-\bigg(\frac{1}{t}\int_0^t v(s)ds\bigg)\chi_{(0,t]}$$

so that for $j = 1, \ldots, \gamma$,

$$c_j(\mathcal{P}^\perp v) = \langle \mathcal{P}^\perp v, e_j\rangle_2 = \int_0^t v(s)e_j(s)ds - \frac{1}{t}\int_0^t v(s)ds\int_0^t e_j(s)ds \tag{28}$$

and

$$\|\mathcal{P}^\perp v\|_2^2 = \|v\|_2^2 - \langle v, \alpha_1\rangle_2^2\|\alpha_1\|_2^2 = \int_0^t [v(s)]^2 ds - \frac{1}{t}\bigg[\int_0^t v(s)ds\bigg]^2. \tag{29}$$

The equation (26) now follows and hence the proof is completed. □

Corollary 4.2. *Let the assumptions and notations be as given in Theorem 4.1.*

(1) *If σ is concentrated on V, then for nonzero q and m_L^{n+1}-a.e. $\vec{\xi_n} = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$,*

$$\begin{aligned} E^{anf_q}[G|X_n](\vec{\xi_n}) = &\int_{L_2[0,t]}\int_{\mathbb{R}^\gamma} \exp\bigg\{i\bigg[\sum_{j=1}^n v(t_j)(\xi_j - \xi_{j-1}) \\ &+ \langle z, (\vec{v}, [\vec{\xi_n}])\rangle_{\mathbb{R}^\gamma}\bigg] + \frac{1}{2qi}\|T_A(z)\|_{\mathbb{R}^\gamma}^2\bigg\} d\rho(z)d\sigma(v) \end{aligned}$$

and

$$\begin{aligned} E^{anf_q}[G|X_1](\xi_0, \xi_1) = &\int_{L_2[0,t]}\int_{\mathbb{R}^\gamma} \exp\bigg\{i\bigg[v(t)(\xi_1 - \xi_0) + \frac{\xi_1 - \xi_0}{t}\sum_{j=1}^\gamma z_j \\ &\times \int_0^t v_j(s)ds\bigg] + \frac{1}{2qi}\sum_{j=1}^\gamma\bigg(\sum_{l=1}^\gamma z_l\alpha_{lj}\bigg)^2\bigg\} d\rho(z)d\sigma(v). \end{aligned}$$

(2) *If σ is concentrated on $V^\perp$, then for nonzero real q,*

$$\begin{aligned} E^{anf_q}[G|X_n](\vec{\xi_n}) = &\int_{L_2[0,t]}\int_{\mathbb{R}^\gamma} \exp\bigg\{i\langle z, (\vec{v}, [\vec{\xi_n}])\rangle_{\mathbb{R}^\gamma} + \frac{1}{2qi}[\|T_A(z)\|_{\mathbb{R}^\gamma}^2 \\ &+ 2\langle \vec{c}(v), T_A(z)\rangle_{\mathbb{R}^\gamma} + \|v\|_2^2]\bigg\} d\rho(z)d\sigma(v) \end{aligned}$$

and

$$\begin{aligned} &E^{anf_q}[G|X_1](\xi_0, \xi_1) \\ &= \int_{L_2[0,t]}\int_{\mathbb{R}^\gamma} \exp\bigg\{i\frac{\xi_1 - \xi_0}{t}\sum_{j=1}^\gamma z_j\int_0^t v_j(s)ds + \frac{1}{2qi}\bigg[\sum_{j=1}^\gamma\bigg(\sum_{l=1}^\gamma z_l\alpha_{lj}\bigg)^2 \\ &\quad + 2\sum_{j=1}^\gamma\bigg(\int_0^t v(s)e_j(s)ds\bigg)\bigg(\sum_{l=1}^\gamma z_l\alpha_{lj}\bigg) + \int_0^t [v(s)]^2 ds\bigg]\bigg\} d\rho(z)d\sigma(v). \end{aligned}$$

(3) *If $v_l \in V^\perp$ for $l = 1, \ldots, \gamma$, then for nonzero real q,*

$$E^{anf_q}[G|X_n](\vec{\xi_n})$$

$$= \int_{L_2[0,t]} \int_{\mathbb{R}^\gamma} \exp\bigg\{ i(v, [\vec{\xi}_n]) + \frac{1}{2qi}[\|z\|_{\mathbb{R}^\gamma}^2 + 2\langle \vec{c}(\mathcal{P}^\perp v), z\rangle_{\mathbb{R}^\gamma} + \|\mathcal{P}^\perp v\|_2^2] \bigg\}$$
$$d\rho(z)d\sigma(v),$$

where $\vec{c}(\mathcal{P}^\perp v) = (\langle \mathcal{P}^\perp v, v_1\rangle_2, \ldots, \langle \mathcal{P}^\perp v, v_\gamma\rangle_2)$, *and*

$$E^{anf_q}[G|X_1](\xi_0, \xi_1)$$
$$= \int_{L_2[0,t]} \int_{\mathbb{R}^\gamma} \exp\bigg\{ i\frac{\xi_1 - \xi_0}{t} \int_0^t v(s)ds + \frac{1}{2qi}\bigg[\sum_{j=1}^\gamma z_j^2 + 2\sum_{j=1}^\gamma z_j \bigg(\int_0^t v(s)$$
$$\times v_j(s)ds - \frac{1}{t}\int_0^t v(s)ds \int_0^t v_j(s)ds\bigg) + \int_0^t [v(s)]^2 ds - \frac{1}{t}\bigg[\int_0^t v(s)ds\bigg]^2\bigg]\bigg\}$$
$$d\rho(z)d\sigma(v).$$

Proof. (1) If σ is concentrated on V, then for σ-a.e. $v \in L_2[0,t]$, $\mathcal{P}v = v$ and $\mathcal{P}^\perp v = 0$ so that by (20)

$$(v, [\vec{\xi}_n]) = \sum_{j=1}^n (\mathcal{P}v)(t_j)(\xi_j - \xi_{j-1}) = \sum_{j=1}^n v(t_j)(\xi_j - \xi_{j-1})$$

and for $j = 1, \ldots, \gamma$

$$c_j(\mathcal{P}^\perp v) = \langle \mathcal{P}^\perp v, e_j\rangle_2 = 0.$$

The results now follow from Theorem 4.1.

(2) If σ is concentrated on $V^\perp$, then for σ-a.e. $v \in L_2[0,t]$, $\mathcal{P}^\perp v = v$ and $\mathcal{P}v = 0$ so that $(v, [\vec{\xi}_n]) = 0$ by (20). The results now follow from Theorem 4.1.

(3) If $v_l \in V^\perp$ for $l = 1, \ldots, \gamma$, then $\mathcal{P}v_l = 0$ so that by (20)

$$(\vec{v}, [\vec{\xi}_n]) = ((v_1, [\vec{\xi}_n]), \ldots, (v_\gamma, [\xi_n])) = (0, \ldots, 0)$$

which implies $\langle z, (\vec{v}, [\vec{\xi}_n])\rangle_{\mathbb{R}^\gamma} = 0$. Furthermore, $\mathcal{P}^\perp v_l = v_l$ and $e_l = v_l$ which implies that A is the identity matrix. By Theorem 4.1, the results follow. □

Remark 4.3. (1) We note that there exist orthonormal vectors $v_1, v_2, \ldots, v_\gamma$ in $L_2[0,t]$ such that $\mathcal{P}^\perp v_1, \mathcal{P}^\perp v_2, \ldots, \mathcal{P}^\perp v_\gamma$ are independent [6].

(2) If $v_l \in V$ for some l, then $\mathcal{P}^\perp v_l = 0$ and hence $\mathcal{P}^\perp v_1, \ldots, \mathcal{P}^\perp v_\gamma$ are dependent. In this case, the proof of Theorem 4.1 can be modified.

(3) Letting $\rho = \delta_0$ or $\sigma = \delta_0$, we can obtain $E^{anf_q}[F|X_n]$ or $E^{anf_q}[\phi(\vec{v}, \cdot)|X_n]$, respectively [6, Theorems 2.1 and 2.4].

Since $E^{anw_\lambda}[G|X_n](\vec{\xi}_n)$ is bounded by $\|\rho\|\|\sigma\|$, the next theorem follows immediately from Theorems 3.2 and 4.1.

Theorem 4.4. *Let* $r = 1$*, the assumptions be as given in Lemma 2.2,* X_n *be given by* (4) *and* G *be as given in Theorem 4.1. Furthermore suppose that there exists a function* Ψ *on* $\mathbb{C}_+ \times \mathbb{R}$ *satisfying the conditions* (i), (ii) *and* (iii) *in Theorem 3.2. Then for* $\lambda \in \mathbb{C}_+$*, the analytic operator-valued Wiener* w_φ*-integral* $I_\lambda^{an}(G)$ *exists as an element of* $\mathcal{L}$ *and is given by* (10) *with* $r = 1$, *where $E^{anw_\lambda}[F|X_n]$ is replaced by $E^{anw_\lambda}[G|X_n]$ which is as given in* (25). *In addition, suppose that $n = 1$ and Ψ can be extended to $(\mathbb{C}_+ \cup \{-iq\}) \times \mathbb{R}$ with the conditions* (ii)$'$ *and* (iii)$'$ *of Theorem 3.2. Then the analytic operator-valued Feynman w_φ-integral $J_q^{an}(G)$ exists as an element of $\mathcal{L}$ and it is given by* (11) *with $r = 1$, where $E^{anf_q}[F|X_1]$ is replaced by $E^{anf_q}[G|X_1]$ which is as given in* (26).

Corollary 4.5. *Let $r = 1$, X_n be given by* (4) *and G be as given in Theorem 4.1. Moreover let φ be normally distributed with mean* 0 *and variance α^2. Then for $\lambda \in \mathbb{C}_+$, the analytic operator-valued Wiener w_φ-integral $I_\lambda^{an}(G)$ exists as an element of $\mathcal{L}$ and is given by* (13) *with $r = 1$, where $E^{anw_\lambda}[F|X_n]$ is replaced by $E^{anw_\lambda}[G|X_n]$ which is as given in* (25).

Remark 4.6. Under the assumptions as given in Corollary 4.5, we can prove the existence of the analytic operator-valued Feynman w_φ-integral $J_q^{an}(G)$ through direct calculations, but they are tedious.

By Theorems 3.4 and 4.1 we can easily obtain the following theorem.

Theorem 4.7. *If, in Theorem 4.4, the conditions* (iii) *and* (iii)$'$ *of Theorem 3.2 are replaced by* (14), *then conclusions of Theorem 4.4 hold true.*

By Theorems 3.8 and 4.1 we can also obtain the following theorem.

Theorem 4.8. *Let $r = 1$, $n \geq 2$, the assumptions be as given in Lemma 2.2, X_n be given by* (4) *and G be as given in Theorem 4.1. Let*

$$B_G(x) = f(x(t_0), \ldots, x(t_{n-1}))G(x)$$

for w_φ-a.e. $x \in C[0,t]$, where $f \in L_1(\mathbb{R}^n)$. Furthermore suppose that there exists a function Ψ on $\mathbb{C}_+ \times \mathbb{R}$ satisfying the conditions (i), (ii) *and* (iii) *of Theorem 3.2. Then for $\lambda \in \mathbb{C}_+$, the analytic operator-valued Wiener w_φ-integral $I_\lambda^{an}(B_G)$ exists as an element of $\mathcal{L}$ and is given by* (16) *with $r = 1$, where $E^{anw_\lambda}[F|X_n]$ is replaced by $E^{anw_\lambda}[G|X_n]$ which is as given in* (25). *In addition, suppose that for nonzero real q, Ψ can be extended to $(\mathbb{C}_+ \cup \{-iq\}) \times \mathbb{R}$ with the condition* (ii)$'$ *of Theorem 3.2. Then the analytic operator-valued Feynman w_φ-integral $J_q^{an}(B_G)$ exists as an element of $\mathcal{L}$ and it is given by the right hand side of* (16) *with $r = 1$, where λ and $E^{anw_\lambda}[F|X_n]$ are replaced by $-iq$ and $E^{anf_q}[G|X_n]$, respectively.*

Corollary 4.9. *Let $r = 1$, $n \geq 2$, X_n be given by* (4) *and B_G be as given in Theorem 4.8. Moreover let φ be normally distributed with mean* 0 *and variance α^2. Then for nonzero real q, the analytic operator-valued Feynman w_φ-integral $J_q^{an}(B_G)$ exists as an element of $\mathcal{L}$ and it is given by the right hand side of* (17) *with $r = 1$, where $E^{anf_q}[F|X_n]$ is replaced by $E^{anf_q}[G|X_n]$ which is as given in Theorem 4.1.*

The following theorem now follows from Theorems 3.10 and 4.1.

Theorem 4.10. *Let $r=1$, $n\ge 2$, the assumptions be as given in Lemma 2.2, X_n be given by* (4) *and G be as given in Theorem 4.1. Let*

$$D_G(x)=f(x(t_1),\dots,x(t_{n-1}))G(x)$$

for w_φ-a.e. $x\in C[0,t]$, where $f\in L_1(\mathbb{R}^{n-1})$. Furthermore suppose that there exists a function Ψ on $\mathbb{C}_+\times\mathbb{R}$ satisfying the conditions (i), (ii) *of Theorem 3.2 and* (14) *of Theorem 3.4. Then for $\lambda\in\mathbb{C}_+$, the analytic operator-valued Wiener w_φ-integral $I_\lambda^{an}(D_G)$ exists as an element of $\mathcal{L}$ and is given by* (16) *with $r=1$, where $E^{anw_\lambda}[F|X_n]$ and $f(\xi_0,\xi_1,\dots,\xi_{n-1})$ are replaced by $E^{anw_\lambda}[G|X_n]$ and $f(\xi_1,\dots,\xi_{n-1})$, respectively. In addition, suppose that for a nonzero real q, Ψ can be extended to $(\mathbb{C}_+\cup\{-iq\})\times\mathbb{R}$ with the condition* (ii)$'$ *of Theorem 3.2. Then the analytic operator-valued Feynman w_φ-integral $J_q^{an}(D_G)$ exists as an element of $\mathcal{L}$ and it is given by the expression of $I_\lambda^{an}(D_G)$, where λ is replaced by $-iq$.*

5. The conditional w_φ-integrals of cylinder functions and the operator-valued function space integrals

In this section, we investigate the conditional analytic Wiener and Feynman w_φ-integrals of cylinder functions and prove that the operator-valued function space integrals of those functions can be expressed by the conditional w_φ-integrals.

We now have the following theorem from (22), (27), (28), (29) and Theorem 3.3 of [6].

Theorem 5.1. *Let X_n be given by* (4) *with $r=1$ and $H(x)=F(x)f(\vec{v},x)$, where $f\in L_p(\mathbb{R}^\gamma)(1\le p\le\infty)$ and F is given by* (18)*. For $v\in L_2[0,t]$ let $c_j(\mathcal{P}^\perp v)$ be given by* (23)*, where v is replaced by $\mathcal{P}^\perp v$ for $j=1,\dots,\gamma$. Then for $\lambda\in\mathbb{C}_+$, $E^{anw_\lambda}[H|X_n](\vec{\xi}_n)$ exists for $\vec{\xi}_n\in\mathbb{R}^{n+1}$, and it is given by*

$$\begin{aligned}(30)\qquad & E^{anw_\lambda}[H|X_n](\vec{\xi}_n)\\ &=\left(\frac{\lambda}{2\pi}\right)^{\frac{\gamma}{2}}\int_{L_2[0,t]}\int_{\mathbb{R}^\gamma} f((\vec{v},[\vec{\xi}_n])+T_{A^T}(z))\exp\bigg\{i(v,[\vec{\xi}_n])\\ &\quad+\frac{1}{2\lambda}\bigg[\sum_{j=1}^{\gamma}[\lambda i z_j+c_j(\mathcal{P}^\perp v)]^2-\|\mathcal{P}^\perp v\|_2^2\bigg]\bigg\}dm_L^\gamma(z)d\sigma(v),\end{aligned}$$

where $z=(z_1,\dots,z_\gamma)$ and T_{A^T} is given by (22)*. In particular, if $p=1$, then for nonzero real q, $E^{anf_q}[H|X_n](\vec{\xi}_n)$ exists and it is given by the right hand side of* (30) *where λ is replaced by $-iq$. Furthermore, if $n=1$, then $E^{anw_\lambda}[H|X_1](\xi_0,\xi_1)$ is given by*

(31)

$$E^{anw_\lambda}[H|X_1](\xi_0,\xi_1)$$

$$= \left(\frac{\lambda}{2\pi}\right)^{\frac{\gamma}{2}} \int_{L_2[0,t]} \int_{\mathbb{R}^\gamma} f\bigg(\frac{\xi_1 - \xi_0}{t}\bigg(\int_0^t v_1(s)ds, \ldots, \int_0^t v_\gamma(s)ds\bigg) + \bigg(\sum_{j=1}^{\gamma} \alpha_{1j} z_j, \ldots, \sum_{j=1}^{\gamma} \alpha_{\gamma j} z_j\bigg)\bigg) \exp\bigg\{ i \frac{\xi_1 - \xi_0}{t} \int_0^t v(s)ds + \frac{1}{2\lambda}\bigg[\sum_{j=1}^{\gamma}\bigg[\lambda i z_j + \int_0^t v(s)e_j(s)ds - \frac{1}{t}\int_0^t v(s)ds \int_0^t e_j(s)ds\bigg]^2 - \int_0^t [v(s)]^2 ds + \frac{1}{t}\bigg[\int_0^t v(s)ds\bigg]^2 \bigg]\bigg\} dm_L^\gamma(z) d\sigma(v)$$

for $(\xi_0, \xi_1) \in \mathbb{R}^2$, *where the* $\alpha_{lj}s$ *are as given in* (21).

Using the same method as used in the proof of Corollary 4.2, we can prove the following corollary.

Corollary 5.2. *Let the assumptions and notations be as given in Theorem 5.1.*

(1) *If* σ *is concentrated on* V, *then for* $\lambda \in \mathbb{C}_+$ *and* $\vec{\xi_n} = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$

$$E^{anw_\lambda}[H|X_n](\vec{\xi_n}) = \left(\frac{\lambda}{2\pi}\right)^{\frac{\gamma}{2}} \int_{L_2[0,t]} \int_{\mathbb{R}^\gamma} f((\vec{v}, [\vec{\xi_n}]) + T_{A^T}(z)) \exp\bigg\{ i \times \sum_{j=1}^{n} v(t_j)(\xi_j - \xi_{j-1}) - \frac{\lambda}{2}\|z\|_{\mathbb{R}^\gamma}^2 \bigg\} dm_L^\gamma(z) d\sigma(v)$$

and, for $n = 1$ *and for* $(\xi_0, \xi_1) \in \mathbb{R}^2$

$$E^{anw_\lambda}[H|X_1](\xi_0, \xi_1) = \left(\frac{\lambda}{2\pi}\right)^{\frac{\gamma}{2}} \int_{L_2[0,t]} \int_{\mathbb{R}^\gamma} f\bigg(\frac{\xi_1 - \xi_0}{t}\bigg(\int_0^t v_1(s)ds, \ldots, \int_0^t v_\gamma(s)ds\bigg) + \bigg(\sum_{j=1}^{\gamma} \alpha_{1j} z_j, \ldots, \sum_{j=1}^{\gamma} \alpha_{\gamma j} z_j\bigg)\bigg) \exp\bigg\{ iv(t)(\xi_1 - \xi_0) - \frac{\lambda}{2}\|z\|_{\mathbb{R}^\gamma}^2 \bigg\} dm_L^\gamma(z) d\sigma(v)$$

where $z = (z_1, \ldots, z_\gamma)$.

(2) *If* σ *is concentrated on* $V^\perp$, *then for* $\lambda \in \mathbb{C}_+$ *and for* $\vec{\xi_n} \in \mathbb{R}^{n+1}$,

$$E^{anw_\lambda}[H|X_n](\vec{\xi_n}) = \left(\frac{\lambda}{2\pi}\right)^{\frac{\gamma}{2}} \int_{L_2[0,t]} \int_{\mathbb{R}^\gamma} f((\vec{v}, [\vec{\xi_n}]) + T_{A^T}(z)) \exp\bigg\{ \frac{1}{2\lambda}\bigg[\sum_{j=1}^{\gamma} [\lambda i z_j + c_j(v)]^2 - \|v\|_2^2\bigg]\bigg\} dm_L^\gamma(z) d\sigma(v)$$

where $z = (z_1, \ldots, z_\gamma)$, *and for* $n = 1$ *and for* $(\xi_0, \xi_1) \in \mathbb{R}^2$,

$$E^{anw_\lambda}[H|X_1](\xi_0, \xi_1)$$

$$= \left(\frac{\lambda}{2\pi}\right)^{\frac{\gamma}{2}} \int_{L_2[0,t]} \int_{\mathbb{R}^\gamma} f\bigg(\frac{\xi_1-\xi_0}{t}\bigg(\int_0^t v_1(s)ds, \dots, \int_0^t v_\gamma(s)ds\bigg)$$
$$+ \bigg(\sum_{j=1}^{\gamma} \alpha_{1j} z_j, \dots, \sum_{j=1}^{\gamma} \alpha_{\gamma j} z_j\bigg)\bigg) \exp\bigg\{\frac{1}{2\lambda}\sum_{j=1}^{\gamma}\bigg[\lambda i z_j + \int_0^t v(s)$$
$$\times e_j(s)ds\bigg]^2 - \int_0^t [v(s)]^2 ds\bigg]\bigg\} dm_L^\gamma(z_1, \dots, z_\gamma) d\sigma(v).$$

(3) *If $v_l \in V^\perp$ for $l = 1, \dots, \gamma$, then for $\lambda \in \mathbb{C}_+$ and for $\vec{\xi}_n \in \mathbb{R}^{n+1}$,*

$$E^{anw_\lambda}[H|X_n](\vec{\xi}_n)$$
$$= \left(\frac{\lambda}{2\pi}\right)^{\frac{\gamma}{2}} \int_{L_2[0,t]} \int_{\mathbb{R}^\gamma} f(z_1, \dots, z_\gamma) \exp\bigg\{i(v, [\vec{\xi}_n]) + \frac{1}{2\lambda}\bigg[\sum_{j=1}^{\gamma}[\lambda i z_j$$
$$+ c_j(\mathcal{P}^\perp v)]^2 - \|\mathcal{P}^\perp v\|_2^2\bigg]\bigg\} dm_L^\gamma(z_1, \dots, z_\gamma) d\sigma(v)$$

and for $n = 1$ and for $(\xi_0, \xi_1) \in \mathbb{R}^2$,

$$E^{anw_\lambda}[H|X_1](\xi_0, \xi_1)$$
$$= \left(\frac{\lambda}{2\pi}\right)^{\frac{\gamma}{2}} \int_{L_2[0,t]} \int_{\mathbb{R}^\gamma} f(z_1, \dots, z_\gamma) \exp\bigg\{i\frac{\xi_1-\xi_0}{t}\int_0^t v(s)ds$$
$$+ \frac{1}{2\lambda}\bigg[\sum_{j=1}^{\gamma}\bigg[\lambda i z_j + \int_0^t v(s)v_j(s)ds - \frac{1}{t}\int_0^t v(s)ds \int_0^t v_j(s)ds\bigg]^2$$
$$- \int_0^t [v(s)]^2 ds + \frac{1}{t}\bigg[\int_0^t v(s)ds\bigg]^2\bigg]\bigg\} dm_L^\gamma(z_1, \dots, z_\gamma) d\sigma(v).$$

Letting $\sigma = \delta_0$ which is the Dirac measure concentrated at $0 \in L_2[0,t]$, we obtain the following corollary.

Corollary 5.3. *Let X_n be given by* (4) *with $r = 1$ and $H(x) = f(\vec{v}, x)$ where $f \in L_p(\mathbb{R}^\gamma)(1 \le p \le \infty)$. Then for $\lambda \in \mathbb{C}_+$, $E^{anw_\lambda}[H|X_n](\vec{\xi}_n)$ exists for $\vec{\xi}_n \in \mathbb{R}^{n+1}$ and it is given by*

$$E^{anw_\lambda}[H|X_n](\vec{\xi}_n) \tag{32}$$
$$= \left(\frac{\lambda}{2\pi}\right)^{\frac{\gamma}{2}} \int_{\mathbb{R}^\gamma} f((\vec{v}, [\vec{\xi}_n]) + T_{A^T}(z)) \exp\bigg\{-\frac{\lambda}{2}\|z\|_{\mathbb{R}^\gamma}^2\bigg\} dm_L^\gamma(z),$$

where T_{A^T} is given by (22)*. In particular, if $p = 1$, then for nonzero real q, $E^{anf_q}[H|X_n](\vec{\xi}_n)$ exists and it is given by the right hand side of* (32) *where λ is replaced by $-iq$. Furthermore, if $n = 1$, then $E^{anw_\lambda}[H|X_1](\xi_0, \xi_1)$ is given by*

$$E^{anw_\lambda}[H|X_1](\xi_0, \xi_1)$$

$$= \left(\frac{\lambda}{2\pi}\right)^{\frac{\gamma}{2}} \int_{\mathbb{R}^\gamma} f\bigg(\frac{\xi_1-\xi_0}{t}\bigg(\int_0^t v_1(s)ds,\ldots,\int_0^t v_\gamma(s)ds\bigg) + \bigg(\sum_{j=1}^{\gamma}\alpha_{1j}z_j,\ldots,\sum_{j=1}^{\gamma}\alpha_{\gamma j}z_j\bigg)\bigg)\exp\bigg\{-\frac{\lambda}{2}\|z\|^2_{\mathbb{R}^\gamma}\bigg\}dm_L^\gamma(z)$$

for $(\xi_0,\xi_1)\in\mathbb{R}^2$*, where* $z=(z_1,\ldots,z_\gamma)$ *and the* α_{lj}*s are as given in* (21)*.*

Theorem 5.4. *If, in Theorem 4.4, G is replaced by* $H(p=1)$ *which is as given in Theorem 5.1, then the conclusions of Theorem 4.4 hold true, where* $E^{anw_\lambda}[H|X_n]$ *and* $E^{anf_q}[H|X_1]$ *are given by* (30) *and* (31)*, respectively, replacing* λ *by* $-iq$*.*

Proof. For $\lambda\in\mathbb{C}_+$, for $\vec{\xi_n}\in\mathbb{R}^{n+1}$ and for $v\in L_2[0,t]$,

$$\bigg|\exp\bigg\{i(v,[\vec{\xi_n}]) + \frac{1}{2\lambda}\bigg[\sum_{j=1}^{\gamma}[\lambda i z_j + c_j(\mathcal{P}^\perp v)]^2 - \|\mathcal{P}^\perp v\|_2^2\bigg]\bigg\}\bigg|$$
$$= \exp\bigg\{-\frac{\operatorname{Re}\lambda}{2}\sum_{j=1}^{\gamma}z_j^2 - \frac{\operatorname{Re}\lambda}{2|\lambda|^2}\bigg[\|\mathcal{P}^\perp v\|_2^2 - \sum_{j=1}^{\gamma}\langle\mathcal{P}^\perp v, e_j\rangle_2^2\bigg]\bigg\} \le 1$$

by (23) and the Bessel's inequality so that

$$|E^{anw_\lambda}[H|X_n](\vec{\xi_n})| \le \|\sigma\|\left(\frac{|\lambda|}{2\pi}\right)^{\frac{\gamma}{2}}\int_{\mathbb{R}^\gamma}|f((\vec{v},[\vec{\xi_n}])+T_{A^T}(z))|dm_L^\gamma(z)$$

by Theorem 5.1. Let Ω be a bounded subset of $\mathbb{C}_+$ and take $M_\Omega>0$ such that $|\lambda|\le M_\Omega$ for all $\lambda\in\Omega$. Then for $\lambda\in\Omega$ and for $\vec{\xi_n}\in\mathbb{R}^{n+1}$,

$$|E^{anw_\lambda}[H|X_n](\vec{\xi_n})| \le |\det((A^T)^{-1})|\,\|f\|_1\|\sigma\|\left(\frac{M_\Omega}{2\pi}\right)^{\frac{\gamma}{2}} \tag{33}$$

by the change of variable theorem. The theorem now follows from Theorems 3.2 and 5.1. □

Corollary 5.5. *If, in Corollary 4.5, G is replaced by H which is as given in Theorem 5.4, then the conclusion of the corollary holds true, where* $E^{anw_\lambda}[H|X_n]$ *is given by* (30)*.*

By Theorems 3.4, 5.1 and (33) we can easily obtain the following theorem.

Theorem 5.6. *If, in Theorem 5.4, the conditions* (iii) *and* (iii)′ *of Theorem 3.2 are replaced by* (14)*, then conclusions of Theorem 5.4 hold true.*

By Theorems 3.8, 5.1 and (33), we can also obtain the following theorem.

Theorem 5.7. *If, in Theorem 4.8, G is replaced by H which is as given in Theorem 5.4, then the conclusions of Theorem 4.8 hold true, where* $E^{anw_\lambda}[H|X_n]$ *and* $E^{anf_q}[H|X_n]$ *are as given in Theorem 5.1.*

Corollary 5.8. *If we replace G in Corollary 4.9 by H which is as given in Theorem 5.4, then the conclusion of the corollary holds true, where $E^{anf_q}[H|X_n]$ is as given in Theorem 5.1.*

The following theorem now follows from Theorems 3.10, 5.1 and (33).

Theorem 5.9. *If, in Theorem 4.10, G is replaced by H which is as given in Theorem 5.4, then the conclusions of Theorem 4.10 hold true, where $E^{anw_\lambda}[H|X_n]$ and $E^{anf_q}[H|X_n]$ are as given in Theorem 5.1.*

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