Bull. Korean Math. Soc. **53** (2016), No. 3, pp. 885–902
http://dx.doi.org/10.4134/BKMS.b150401
pISSN: 1015-8634 / eISSN: 2234-3016

HYPERSURFACES IN $\mathbb{S}^4$ THAT ARE OF L_k-2-TYPE

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Abstract. In this paper we begin the study of L_k-2-type hypersurfaces of a hypersphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ for $k \geq 1$. Let $\psi : M^3 \to \mathbb{S}^4$ be an orientable H_k-hypersurface, which is not an open portion of a hypersphere. Then M^3 is of L_k-2-type if and only if M^3 is a Clifford tori $\mathbb{S}^1(r_1)\times\mathbb{S}^2(r_2)$, $r_1^2 + r_2^2 = 1$, for appropriate radii, or a tube $T^r(V^2)$ of appropriate constant radius r around the Veronese embedding of the real projective plane $\mathbb{R}P^2(\sqrt{3})$.

1. Introduction

The theory of submanifolds of finite type were introduced by B. Y. Chen during the late 1970s, and the first results on this subject were collected in his books [12] and [13]. Although the first definition was given for a compact submanifold in the Euclidean space, Chen extended the concept to non-compact submanifolds in Euclidean $\mathbb{R}^m$ or pseudo-Euclidean spaces $\mathbb{R}^m_s$, [14]. An isometric immersion $\psi : M^n \to \mathbb{R}^m$ of a submanifold M^n (not necessarily compact) into $\mathbb{R}^m$ is said to be of finite type if it admits a finite spectral decomposition

$$\psi = a + \psi_1 + \cdots + \psi_q, \quad \Delta\psi_t = \lambda_t\psi_t,$$

for some natural number q, where λ_t are constants, a is a constant vector and ψ_t are non-constant vector functions. Otherwise, the immersion is said to be of infinite type.

A detailed survey of the results, up to 1996, on this subject was given by Chen in [17]. Since then, the study of finite type submanifolds, in particular, of biharmonic submanifolds, have received a growing attention with many progresses during last years. In a recent article [18], Chen provides a detailed account of recent development on problems and conjectures about finite type submanifolds.

Received May 27, 2015.

2010 *Mathematics Subject Classification.* 53C40, 53B25.

Key words and phrases. linearized operator L_k, L_k-finite-type hypersurface, higher order mean curvatures, Newton transformations.

This work has been partially supported by MINECO (Ministerio de Economía y Competitividad) and FEDER (Fondo Europeo de Desarrollo Regional), Project MTM2012-34037.

A special class of finite type submanifolds was introduced by O. J. Garay in [24]; he considered submanifolds of a Euclidean space whose position vector field satisfies $\Delta\psi = A\psi$, for some diagonal matrix A; in other words, each coordinate function of ψ is an eigenfunction of the Laplacian. Garay called such submanifolds coordinate finite type submanifolds. Later on, F. Dillen, J. Pas and L. Verstraelen observed in [22] that this condition is not coordinate invariant and proposed the study of submanifolds satisfying the condition $\Delta\psi = A\psi + b$, for some constant matrix A and some constant vector b. That condition has been deeply studied for submanifolds in Euclidean or pseudo-Euclidean spaces as well as in pseudo-Riemannian space forms (see for example [1], [2], [3], [20], [27], [38]).

It is well known that the Laplacian operator Δ can be seen as the first one of a sequence of n operators $L_0 = \Delta, L_1, \ldots, L_{n-1}$, where L_k stands for the linearized operator of the first variation of the $(k+1)$-th mean curvature arising from normal variations of the hypersurface (see, for instance, [39]). These operators are given by $L_k(f) = \operatorname{tr}(P_k \circ \nabla^2 f)$ for a smooth function f on M, where P_k denotes the k-th Newton transformation associated to the second fundamental form of the hypersurface and $\nabla^2 f$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f.

From this point of view, Kashani [28] introduced the notion of L_k-finite-type hypersurface in the Euclidean space. In general, a submanifold M^n in $\mathbb{R}^m$ is said to be of L_k-finite-type if the position vector $\psi : M^n \to \mathbb{R}^m$ of M^n into $\mathbb{R}^m$ admits the following finite spectral decomposition

$$\psi = a + \psi_1 + \cdots + \psi_q, \quad L_k\psi_t = \lambda_t\psi_t,$$

where a is a constant vector, λ_t are constants and ψ_t are non-constant $\mathbb{R}^m$-valued maps on M^n. If all λ_t's are mutually different, M^n is said to be of L_k-q-type, and if one of λ_t is zero M^n is said to be of L_k-null-q-type. Obviously, that definition is also valid for a pseudo-Riemannian submanifold M_t^n into the pseudo-Euclidean space $\mathbb{R}_s^m$.

Inspired by [22], Alías and Gürbuz initiated in [4] the study of hypersurfaces in Euclidean space satisfying the condition $L_k\psi = A\psi + b$, where $A \in \mathbb{R}^{(n+1)\times(n+1)}$ is a constant matrix and $b \in \mathbb{R}^{n+1}$ is a constant vector. This initial work has been extended to hypersurfaces in the hypersphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ ([5]), to hypersurfaces in Lorentzian space space forms ([30], [31]), and to hypersurfaces in pseudo-Riemannian space forms ([32], [33]). In particular, the results in these works can be used to characterize the coordinate L_k-finite-type hypersurfaces.

In [35] the authors, by using results of [4], show that k-minimal Euclidean hypersurfaces and open portions of hyperspheres are the only L_k-1-type hypersurfaces in $\mathbb{R}^{n+1}$. Next step is the study of L_k-2-type hypersurfaces in $\mathbb{R}^{n+1}$, and we find in [35] several results in this direction. In particular, the authors show that if M^n is a hypersurface with at most two distinct principal curvatures, then: (i) M^n is not of L_{n-1}-null-2-type (Theorem 3.5); (ii) M^n is of

L_k-null-2-type ($k \neq n-1$) if and only if M is locally isometric to a generalized cylinder (Theorems 3.11 and 3.12).

This paper begins the study of L_k-2-type hypersurfaces of hyperspheres $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$. The case $k = 0$ corresponds to the classical one, which has been well studied (see e.g. [6], [15], [19], [25], [26]), so we will concentrate in cases $k = 1$ and $k = 2$. After a section devoted to preliminaries and basic results we proceed, in the third section, to compute some formulae which is needed to present the examples. In Section 4 we present the main results, that we can collect in the following classification theorem (see Sections 2 and 3.1 for definitions and examples):

Main Theorem. *Let $\psi : M^3 \to \mathbb{S}^4$ be an orientable H_k-hypersurface, which is not an open portion of a hypersphere. Then M^3 is of L_k-2-type if and only if M^3 is a Clifford tori $\mathbb{S}^1(r_1) \times \mathbb{S}^2(r_2)$, $r_1^2 + r_2^2 = 1$, for appropriate radii, or a tube $T^r(V^2)$ of appropriate constant radius r around the Veronese embedding of the real projective plane $\mathbb{R}P^2(\sqrt{3})$.*

2. Preliminaries

In this section, we will recall basic formulae and notions about hypersurfaces in the unit hypersphere $\mathbb{S}^4$ centered at the origin of $\mathbb{R}^5$:

$$\mathbb{S}^4 = \Big\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \,\Big|\, \sum_{i=1}^{5} x_i^2 = 1\Big\}.$$

Let $\psi : M^3 \to \mathbb{S}^4 \subset \mathbb{R}^5$ be an isometric immersion of a connected orientable hypersurface M^3 with Gauss map N. We denote by ∇^0, $\overline{\nabla}$ and ∇ the Levi-Civita connections on $\mathbb{R}^5$, $\mathbb{S}^4$ and M^3, respectively. Then the Gauss and Weingarten formulae are given by

$$\nabla^0_X Y = \nabla_X Y + \langle SX, Y\rangle\, N - \langle X, Y\rangle\, \psi, \tag{1}$$

$$SX = -\overline{\nabla}_X N = -\nabla^0_X N, \tag{2}$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M^3)$, where $S : \mathfrak{X}(M^3) \longrightarrow \mathfrak{X}(M^3)$ stands for the shape operator (or Weingarten endomorphism) of M^3, with respect to the chosen orientation N.

As is well-known, for every point $p \in M^3$, S defines a linear self-adjoint endomorphism on the tangent space T_pM, and its eigenvalues $\kappa_1(p), \kappa_2(p)$ and $\kappa_3(p)$ are the principal curvatures of the hypersurface. The characteristic polynomial $Q_S(t)$ of S is defined by

$$Q_S(t) = \det(tI - S) = (t - \kappa_1)(t - \kappa_2)(t - \kappa_3) = t^3 + a_1 t^2 + a_2 t + a_3,$$

where the coefficients of $Q_S(t)$ are given by

$$a_1 = -(\kappa_1 + \kappa_2 + \kappa_3), \quad a_2 = \kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3, \quad a_3 = -\kappa_1\kappa_2\kappa_3.$$

These coefficients can be easily obtained, by making use of the Leverrier–Faddeev method (see [23, 29]), in terms of the traces of S^j, as follows:

$$a_k = -\frac{1}{k}\sum_{j=1}^{k} a_{k-j}\mathrm{tr}(S^j), \quad k = 1, 2, 3, \quad \text{with } a_0 = 1.$$

In particular, we obtain the following expressions:

$$a_1 = -\mathrm{tr}(S), \tag{3}$$

$$a_2 = -\frac{1}{2}(\mathrm{tr}(S^2) - \mathrm{tr}(S)^2), \tag{4}$$

$$a_3 = -\frac{1}{3}(\mathrm{tr}(S^3) - \frac{3}{2}\mathrm{tr}(S^2)\mathrm{tr}(S) + \frac{1}{2}\mathrm{tr}(S)^3). \tag{5}$$

The *k-th mean curvature* H_k or *mean curvature of order* k of M^3 is defined by

$$\tbinom{3}{k}H_k = (-1)^k a_k, \quad \text{with } H_0 = 1. \tag{6}$$

In particular, we have:

$$H_1 = -\frac{1}{3}a_1 = \frac{1}{3}\mathrm{tr}(S), \qquad H_2 = \frac{1}{3}a_2, \qquad H_3 = -a_3.$$

Observe that H_1 is nothing but the usual mean curvature H of M^3, which is one of the most important extrinsic curvatures of the hypersurface.

As usual, we say that M^3 is an H_k-hypersurface if its k-th mean curvature H_k is constant. If $H_{k+1} = 0$, then we say that M^3 is a k-minimal hypersurface; a 0-minimal hypersurface is nothing but a minimal hypersurface in the sphere.

2.1. The Newton transformations

The k-th Newton transformation of M is the operator $P_k : \mathfrak{X}(M^3) \to \mathfrak{X}(M^3)$ defined by

$$P_k = (-1)^k \sum_{j=0}^{k} a_{k-j} S^j.$$

In particular,

$$P_0 = I, \quad P_1 = 3HI - S, \quad P_2 = 3H_2 I - S \circ P_1, \quad P_3 = H_3 I - S \circ P_2. \tag{7}$$

Note that by Cayley-Hamilton theorem we have $P_3 = 0$. Let us recall that each $P_k(p)$ is also a self-adjoint linear operator on the tangent hyperplane T_pM which commutes with $S(p)$. Indeed, $S(p)$ and $P_k(p)$ can be simultaneously diagonalized: if $\{e_1, e_2, e_3\}$ are the eigenvectors of $S(p)$ corresponding to the eigenvalues $\kappa_1(p), \kappa_2(p), \kappa_3(p)$, respectively, then they are also the eigenvectors of $P_k(p)$ with corresponding eigenvalues given by

$$\mu^i_k(p) = \sum_{\substack{i_1 < \cdots < i_k \\ i_j \notin i}}^{3} \kappa_{i_1} \cdots \kappa_{i_k} \quad \text{for every } i = 1, 2, 3 \text{ and } k = 1, 2. \tag{8}$$

In particular,

$$\mu_1^1 = \kappa_2 + \kappa_3, \qquad \mu_1^2 = \kappa_1 + \kappa_3, \qquad \mu_1^3 = \kappa_1 + \kappa_2, \tag{9}$$

$$\mu_2^1 = \kappa_2\kappa_3, \qquad \mu_2^2 = \kappa_1\kappa_3, \qquad \mu_2^3 = \kappa_1\kappa_2. \tag{10}$$

We have the following properties of P_k (the proof is algebraic and straightforward).

Lemma 1. *The Newton transformations P_k, $k = 1, 2$, satisfy*:

- (a) $\operatorname{tr}(P_k) = c_k H_k$,
- (b) $\operatorname{tr}(S \circ P_k) = c_k H_{k+1}$,
- (c) $\operatorname{tr}(S^2 \circ P_1) = 3\big(3HH_2 - H_3\big)$,
- (d) $\operatorname{tr}(S^2 \circ P_2) = 3HH_3$,

where $c_1 = 6$ *and* $c_2 = 3$.

Now, we recall the notion of divergence of a vector field or an operator. According to [37, p. 86], for a tensor T the contraction of the new covariant slot in its covariant differential ∇T with one of its original slots is called a divergence of T. Hence the divergence of a vector field X is the differentiable function defined as the contraction of the operator ∇X, where $\nabla X(Y) := \nabla_Y X$, that is,

$$\operatorname{div}(X) = C(\nabla X) = \operatorname{tr}(\nabla X) = \sum_{i,j} g^{ij} \langle \nabla_{E_i} X, E_j \rangle ,$$

$\{E_i\}$ being any local frame of tangent vectors fields, where (g^{ij}) represents the inverse of the metric $(g_{ij}) = (\langle E_i, E_j \rangle)$. For an operator $T : \mathfrak{X}(M^3) \longrightarrow \mathfrak{X}(M^3)$ we have two divergences: one associated to the (1,1)-contraction C_1^1, and another associated to the metric contraction C_{12}; the first contraction produces a 1-form and the second contraction produces a vector field. We consider here the second one, so that the divergence of an operator T will be the vector field $\operatorname{div}(T) \in \mathfrak{X}(M^3)$ defined as

$$\operatorname{div}(T) = C_{12}(\nabla T) = \sum_{i,j} g^{ij} (\nabla_{E_i} T) E_j,$$

where $\nabla T(X, Y) = (\nabla_X T)Y = \nabla_X(TY) - T(\nabla_X Y)$.

In the following lemma we present two interesting properties of the Newton transformations (see Lemma 4 of [32] for details).

Lemma 2. *The Newton transformation P_k, for $k = 1, 2$, satisfies*:

a) $\operatorname{tr}(\nabla_X S \circ P_k) = \binom{3}{k+1} \langle \nabla H_{k+1}, X \rangle$.

b) $\operatorname{div}(P_k) = 0$.

Bearing in mind this lemma we obtain

$$\operatorname{div}(P_k(\nabla f)) = \operatorname{tr}\big(P_k \circ \nabla^2 f\big),$$

where $\nabla^2 f : \mathfrak{X}(M^3) \longrightarrow \mathfrak{X}(M^3)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f, given by

$$\big\langle \nabla^2 f(X), Y \big\rangle = \langle \nabla_X(\nabla f), Y\rangle, \qquad X, Y \in \mathfrak{X}(M^3).$$

Associated to each Newton transformation P_k, we can define the second-order linear differential operator $L_k : \mathcal{C}^\infty(M^3) \longrightarrow \mathcal{C}^\infty(M^3)$ by

$$L_k(f) = \operatorname{tr}\big(P_k \circ \nabla^2 f\big). \tag{11}$$

An interesting property of L_k is the following. For every couple of differentiable functions $f, g \in C^\infty(M^3)$ we have

$$\begin{aligned} L_k(fg) &= \operatorname{div}\big(P_k \circ \nabla(fg)\big) = \operatorname{div}\big(P_k \circ (g\nabla f + f\nabla g)\big) \\ &= gL_k(f) + fL_k(g) + 2\langle P_k(\nabla f), \nabla g\rangle . \end{aligned} \tag{12}$$

3. First formulas

We are going to compute L_k acting on the coordinate components of the immersion ψ, that is, a function given by $\langle \psi, e\rangle$, where $e \in \mathbb{R}^5$ is an arbitrary fixed vector.

A direct computation shows that

$$\nabla \langle \psi, e\rangle = e^\top = e - \langle N, e\rangle N - \langle \psi, e\rangle \psi, \tag{13}$$

where $e^\top \in \mathfrak{X}(M^3)$ denotes the tangential component of e. Taking covariant derivative in (13), and using that $\nabla^0_X e = 0$, jointly with the Gauss and Weingarten formulae, we obtain

$$\nabla_X \nabla \langle \psi, e\rangle = \nabla_X e^\top = \langle N, e\rangle SX - \langle \psi, e\rangle X \tag{14}$$

for every vector field $X \in \mathfrak{X}(M^3)$. Finally, by using (11) and Lemma 1, we find that

$$\begin{aligned} L_k \langle \psi, e\rangle &= \langle N, e\rangle \operatorname{tr}(S \circ P_k) - \langle \psi, e\rangle \operatorname{tr}(I \circ P_k) \\ &= c_k H_{k+1} \langle N, e\rangle - c_k H_k \langle \psi, e\rangle . \end{aligned} \tag{15}$$

This expression allows us to extend operator L_k to vector functions $F = (f_1, \dots, f_5)$, $f_i \in \mathcal{C}^\infty(M^3)$, as follows $L_k F := \big(L_k f_1, \dots, L_k f_5\big)$, and then $L_k\psi$ can be computed as

$$\begin{aligned} L_k\psi &= \big(L_k \langle \psi, e_1\rangle, \dots, L_k \langle \psi, e_5\rangle\big) \\ &= c_k H_{k+1}\big(\langle N, e_1\rangle, \dots, \langle N, e_5\rangle\big) - c_k H_k\big(\langle \psi, e_1\rangle, \dots, \langle \psi, e_5\rangle\big) \\ &= c_k H_{k+1} N - c_k H_k \psi, \end{aligned} \tag{16}$$

where $\{e_1, \dots, e_5\}$ stands for the standard orthonormal basis in $\mathbb{R}^5$.

Now, we need to compute $L_k N$, and to do that we are going to compute the operator L_k acting on the coordinate functions of the Gauss map N, that is, the functions $\langle N, e\rangle$ where $e \in \mathbb{R}^5$ is an arbitrary fixed vector. A straightforward computation yields

$$\nabla \langle N, e\rangle = -Se^\top.$$

From Weingarten formula and (14), we find that

$$\begin{aligned}\nabla_X\nabla\langle N,e\rangle &= -\nabla_X(Se^\top) = -(\nabla_X S)e^\top - S(\nabla_X e^\top)\\ &= -(\nabla_{e^\top}S)X - \langle N,e\rangle S^2X + \langle\psi,e\rangle SX\end{aligned}$$

for every tangent vector field X. This equation, jointly with (11), Lemmas 1 and 2, yields

$$\begin{aligned}L_k\langle N,e\rangle &= -\mathrm{tr}(\nabla_{e^\top}S\circ P_k) - \langle N,e\rangle\,\mathrm{tr}(S^2\circ P_k) + \langle\psi,e\rangle\,\mathrm{tr}(S\circ P_k)\\ &= -\tbinom{3}{k+1}\langle\nabla H_{k+1},e\rangle - \mathrm{tr}(S^2\circ P_k)\langle N,e\rangle + c_kH_{k+1}\langle\psi,e\rangle\,. \qquad (17)\end{aligned}$$

In other words,

$$L_kN = -\tbinom{3}{k+1}\nabla H_{k+1} - \mathrm{tr}(S^2\circ P_k)N + c_kH_{k+1}\psi. \qquad (18)$$

On the other hand, equations (12) and (15) lead to

$$\begin{aligned}L_k^2\langle\psi,e\rangle &= c_kH_{k+1}L_k\langle N,e\rangle + L_k(c_kH_{k+1})\langle N,e\rangle + 2c_k\big\langle P_k(\nabla H_{k+1}),\nabla\langle N,e\rangle\big\rangle\\ &\quad - c_kH_kL_k\langle\psi,e\rangle - L_k(c_kH_k)\langle\psi,e\rangle - 2c_k\big\langle P_k(\nabla H_k),\nabla\langle\psi,e\rangle\big\rangle,\end{aligned}$$

and by using again (15) and (17) we get

$$\begin{aligned}L_k^2\langle\psi,e\rangle = {}& -c_k\tbinom{3}{k+1}H_{k+1}\langle\nabla H_{k+1},e\rangle - 2c_k\langle(S\circ P_k)(\nabla H_{k+1}),e\rangle\\ & - 2c_k\langle P_k(\nabla H_k),e\rangle\\ & + \Big[c_kL_k(H_{k+1}) - \big(\mathrm{tr}(P_k\circ S^2) + c_kH_k\big)c_kH_{k+1}\Big]\langle N,e\rangle\\ & + \Big[c_k^2H_{k+1}^2 + c_k^2H_k^2 - c_kL_k(H_k)\Big]\langle\psi,e\rangle\,.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}L_k^2\psi = {}& -\frac{c_k}{2}\tbinom{3}{k+1}\nabla H_{k+1}^2 - 2c_k(S\circ P_k)(\nabla H_{k+1}) - 2c_kP_k(\nabla H_k)\\ & + \Big[c_kL_k(H_{k+1}) - \big(\mathrm{tr}(P_k\circ S^2) + c_kH_k\big)c_kH_{k+1}\Big]N\\ & + \Big[c_k^2H_{k+1}^2 + c_k^2H_k^2 - c_kL_k(H_k)\Big]\psi. \qquad (19)\end{aligned}$$

Now we suppose that M^3 is of L_k-2-type in $\mathbb{R}^5$, that is, its position vector ψ can be written as follows

$$\psi = a + \psi_1 + \psi_2, \qquad L_k\psi_1 = \lambda_1\psi_1, \qquad L_k\psi_2 = \lambda_2\psi_2,$$

where a is a constant vector in $\mathbb{R}^5$ and ψ_1,ψ_2 are $\mathbb{R}^5$-valued non-constant differentiable functions on M^3.

It is easy to see that $L_k\psi = \lambda_1\psi_1 + \lambda_2\psi_2$ and $L_k^2\psi = \lambda_1^2\psi_1 + \lambda_2^2\psi_2$, and thus

$$L_k^2\psi = (\lambda_1+\lambda_2)L_k\psi - \lambda_1\lambda_2(\psi - a).$$

By using (16) we get

$$\begin{aligned}L_k^2\psi &= \lambda_1\lambda_2a^\top + \big[(\lambda_1+\lambda_2)c_kH_{k+1} + \lambda_1\lambda_2\langle N,a\rangle\big]N\\ &\quad - \big[(\lambda_1+\lambda_2)c_kH_k + \lambda_1\lambda_2 - \lambda_1\lambda_2\langle\psi,a\rangle\big]\psi,\end{aligned}$$

that, jointly with (19), yields the following equations of L_k-2-type,

$$\lambda_1\lambda_2 a^\top = -\frac{c_k}{2}\binom{3}{k+1}\nabla H_{k+1}^2 - 2c_k(S\circ P_k)(\nabla H_{k+1}) - 2c_k P_k(\nabla H_k), \tag{20}$$

$$\lambda_1\lambda_2 \langle N, a\rangle = c_k L_k(H_{k+1}) - \big(\mathrm{tr}(S^2\circ P_k) + c_k H_k + \lambda_1 + \lambda_2\big)c_k H_{k+1}, \tag{21}$$

$$\lambda_1\lambda_2 \langle \psi, a\rangle = c_k^2 H_{k+1}^2 + (c_k H_k + \lambda_1)(c_k H_k + \lambda_2) - c_k L_k(H_k). \tag{22}$$

3.1. Examples of L_k-finite type hypersurfaces in $\mathbb{S}^4$

Example 1. Every k-minimal H_k-hypersurface in $\mathbb{S}^4$ is of L_k-1-type or L_k-null-1-type. In fact, from (16) we get that $L_k\psi = \lambda\psi$, with $\lambda = -c_k H_k$. If $H_k \neq 0$, then M^3 is of L_k-1-type, otherwise it is of L_k-null-1-type.

Example 2. Every totally umbilical (and not totally geodesic) hypersurface in $\mathbb{S}^4$ is of L_k-1-type. In fact, if M^3 is totally umbilical, then its shape operator S is given by $S = HI$, where H is a non-zero constant. Therefore, H_k and H_{k+1} are also nonzero constants. Since

$$\nabla^0_X(N + H\psi) = -SX + HX = 0 \quad \text{for all } X \in \mathfrak{X}(M^3),$$

we get that $N = C - H\psi$, where C is a constant vector. Bearing in mind (16) we find $L_k\psi = \lambda\psi + b$, where $\lambda = -c_k H^k(H^2+1) \neq 0$ and $b = c_k H^{k+1}C$. Then we can write

$$\psi = \psi_0 + \psi_1, \quad \psi_0 = -\frac{b}{\lambda} \quad \text{and} \quad \psi_1 = \psi + \frac{b}{\lambda},$$

where ψ_0 is constant and $L_k\psi_1 = \lambda\psi_1$. Therefore, M^3 is L_k-1-type in $\mathbb{R}^5$.

The following result shows that those hypersurfaces in $\mathbb{S}^4$ are the only spherical L_k-1-type hypersurfaces in $\mathbb{R}^5$.

Proposition 3. *k-minimal H_k-hypersurfaces in $\mathbb{S}^4$ and open portions of hyperspheres in $\mathbb{S}^4$ are the only L_k-1-type hypersurfaces in $\mathbb{S}^4$.*

Proof. Let M^3 be a L_k-1-type hypersurface in $\mathbb{S}^4$, then its position vector ψ can be put as $\psi = a + \psi_1$, where a is a constant vector and $L_k\psi_1 = \lambda\psi_1$. Hence we deduce $L_k\psi = A\psi + b$, with $A = \lambda I$ and $b = -\lambda a$. The result follows from Theorems 1.2 and 1.7 in [5]. □

Example 3. Clifford hypersurfaces or standard Riemannian products $M^3_{r_1,r_2} = \mathbb{S}^1(r_1)\times\mathbb{S}^2(r_2)$, $r_1^2+r_2^2 = 1$, are hypersurfaces of L_k-2-type in $\mathbb{R}^5$, for appropriate radii r_1 and r_2.

Given $0 < r < 1$, let $M^3(r) = \mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^2(r) \subset \mathbb{S}^4$. Observe that $M^3(r)$ is defined by the equation $M^3(r) = \{x \in \mathbb{S}^4 : x_3^2 + x_4^2 + x_5^2 = r^2\}$. In this case, the Gauss map on $M^3(r)$ is given by

$$N(x) = \left(\frac{-r}{\sqrt{1-r^2}}\,x_1, \frac{-r}{\sqrt{1-r^2}}\,x_2, \frac{\sqrt{1-r^2}}{r}\,x_3, \frac{\sqrt{1-r^2}}{r}\,x_4, \frac{\sqrt{1-r^2}}{r}\,x_5\right),$$

and its principal curvatures in $\mathbb{S}^4$ are

$$\kappa_1 = \frac{r}{\sqrt{1-r^2}} \quad \text{and} \quad \kappa_2 = \kappa_3 = -\frac{\sqrt{1-r^2}}{r}.$$

Hence we get

$$H_1 = \frac{3r^2-2}{3r\sqrt{1-r^2}}, \qquad H_2 = \frac{1-3r^2}{3r^2}, \qquad H_3 = \frac{\sqrt{1-r^2}}{r}.$$

If we put $\psi_1 = (x_1, x_2, 0, 0, 0)$ and $\psi_2 = (0, 0, x_3, x_4, x_5)$, then $\psi = \psi_1 + \psi_2$ and by using (16) we obtain:

a) $L_0\psi_1 = \lambda_1\psi_1$ and $L_0\psi_2 = \lambda_2\psi_2$, where $\lambda_1 = \frac{1}{r^2-1}$ and $\lambda_2 = -\frac{2}{r^2}$. Therefore, $M^3(r)$ is of L_0-2-type in $\mathbb{R}^5$ for $r^2 \neq \frac{2}{3}$.

b) $L_1\psi_1 = \lambda_1\psi_1$ and $L_1\psi_2 = \lambda_2\psi_2$, where $\lambda_1 = \frac{2}{r\sqrt{1-r^2}}$ and $\lambda_2 = \frac{2(1-2r^2)}{r^3\sqrt{1-r^2}}$. Therefore, $M^3(r)$ is of L_1-2-type in $\mathbb{R}^5$ for $r^2 \neq \frac{1}{3}$.

c) $L_2\psi_1 = \lambda_1\psi_1$ and $L_2\psi_2 = \lambda_2\psi_2$, where $\lambda_1 = -\frac{1}{r^2}$ and $\lambda_2 = \frac{2}{r^2}$. Therefore, $M^3(r)$ is of L_2-2-type in $\mathbb{R}^5$ for any r.

Recall that a hypersurface M^n is called isoparametric if all the κ_i are constant functions; this is equivalent to say that all the H_i are constant functions. The classification problem of isoparametric hypersurfaces M^n in a sphere $\mathbb{S}^{n+1}$ is still open. However, it is known that the number g of distinct principal curvatures of isoparametric hypersurfaces is either $g = 1, 2, 3, 4$ or 6 (see [36]). Cartan classified these hypersurfaces when $g \leq 3$ (see e.g. [7, 8, 9]); Clifford hypersurfaces $\mathbb{S}^k(r_1) \times \mathbb{S}^{n-k}(r_2) \subset \mathbb{S}^{n+1}$, $r_1^2 + r_2^2 = 1$, constitute the case when $g = 2$. For $g = 3$, he showed that such hypersurfaces are tubes of constant radii around the Veronese embedding of the projective plane $\mathbb{F}P^2$ in $\mathbb{S}^{3m+1}$, where $m = 1, 2, 4$ or 8 is the dimension of the standard normed algebra $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or the Cayley algebra $\mathbb{O}$, respectively.

Proposition 4. *Let $\psi : M^3 \to \mathbb{S}^4$ be an orientable hypersurface, which is not an open portion of a hypersphere. If M^3 is an isoparametric hypersurface with nonzero H_{k+1}, then M^3 is a hypersurface of L_k-2-type.*

Proof. Let λ_1 and λ_2 be the solutions of the following system of equations:

$$\begin{aligned} \lambda_1 + \lambda_2 &= -\mathrm{tr}(S^2 \circ P_k) - c_k H_k, \\ \lambda_1\lambda_2 &= c_k H_k \mathrm{tr}(S^2 \circ P_k) - c_k^2 H_{k+1}^2. \end{aligned}$$

In other words, λ_1 and λ_2 are the roots of the quadratic equation $t^2 + bt + c = 0$, where $b = \mathrm{tr}(S^2 \circ P_k) + c_k H_k$ and $c = c_k H_k \mathrm{tr}(S^2 \circ P_k) - c_k^2 H_{k+1}^2$ are two constants. Since the discriminant of this equation is $b^2 - 4c = (\mathrm{tr}(S^2 \circ P_k) - c_k H_k)^2 + 4c_k^2 H_{k+1}^2 > 0$, we get $\lambda_1 \neq \lambda_2$.

Choose ψ_1 and ψ_2 as follows:

$$\psi_1 = \frac{1}{\lambda_2 - \lambda_1}\Big(- c_k H_{k+1} N + (c_k H_k + \lambda_2)\psi \Big),$$

$$\psi_2 = \frac{1}{\lambda_2 - \lambda_1}\Big(c_k H_{k+1} N - (c_k H_k + \lambda_1)\psi\Big),$$

where ψ is the position vector of M^3 in $\mathbb{R}^5$. It is evident that $\psi_1 + \psi_2 = \psi$. On the other hand ψ_1 and ψ_2 are non-constant $\mathbb{R}^5$-valued maps. In fact, if ψ_1 (or ψ_2) is a constant map we conclude that M^3 is totally umbilical in $\mathbb{S}^4$ and thus it is an open portion of a hypersphere, which is not possible. Moreover, by a straightforward calculation involving equations (16) and (18), we obtain $L_k\psi_1 = \lambda_1\psi_1$ and $L_k\psi_2 = \lambda_2\psi_2$, i.e., M^3 is of L_k-2-type. □

Example 4. Tubes of constant radius r around the Veronese embedding of the real projective plane $\mathbb{R}P^2$ are hypersurfaces in $\mathbb{S}^4$ of L_k-2-type for appropriate r.

Let (x, y, z) be the standard coordinates of $\mathbb{R}^3$ and $(u_1, \ldots, u_5)$ that of $\mathbb{R}^5$. The mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^5$ defined by

$$u_1 = \frac{yz}{\sqrt{3}}, \quad u_2 = \frac{xz}{\sqrt{3}}, \quad u_3 = \frac{xy}{\sqrt{3}}, \quad u_4 = \frac{x^2 - y^2}{2\sqrt{3}}, \quad u_5 = \frac{1}{6}(x^2 + y^2 - 2z^2),$$

gives rise to an isometric immersion of the 2-sphere $\mathbb{S}^2(\sqrt{3})$ of curvature $\frac{1}{3}$ into the unit sphere $\mathbb{S}^4$. This mapping defines an embedding $\tilde{\phi}$ of the real projective plane $\mathbb{R}P^2(\sqrt{3})$ into $\mathbb{S}^4$, known as the Veronese surface, which is the second standard immersion of the 2-sphere $\mathbb{S}^2(\sqrt{3})$.

Let us consider the tube $M^3(r) = T^r(V^2)$ with radius r over the Veronese surface V^2 in $\mathbb{S}^4$, $0 < r < \pi/3$, and consider $\psi : M^3(r) \to \mathbb{S}^4$ the standard isometric immersion. It follows from a direct computation that the principal curvatures of the tube in $\mathbb{S}^4$ are given by

$$\kappa_1 = \frac{\cot r - \sqrt{3}}{1 + \sqrt{3}\cot r}, \quad \kappa_2 = \frac{\cot r + \sqrt{3}}{1 - \sqrt{3}\cot r}, \quad \kappa_3 = \cot r.$$

Hence we get

$$H_1 = \frac{\cot r\,(3 - \cot^2 r)}{1 - 3\cot^2 r}, \qquad H_2 = \frac{3\cot^2 r - 1}{1 - 3\cot^2 r}, \qquad H_3 = \frac{\cot r\,(\cot^2 r - 3)}{1 - 3\cot^2 r}.$$

It is direct to verify from here and Theorem 4 that the tube $M^3(r)$ is of L_k-2-type in $\mathbb{R}^5$ (for appropriate radius r such that $H_{k+1} \neq 0$):

a) In the case $k = 0$, $M^3(r)$ is of L_0-2-type in $\mathbb{R}^5$ for $r \neq \frac{\pi}{6}$.
b) In the case $k = 1$, $M^3(r)$ is of L_1-2-type in $\mathbb{R}^5$ for any r.
c) In the case $k = 2$, $M^3(r)$ is of L_2-2-type in $\mathbb{R}^5$ for $r \neq \frac{\pi}{6}$.

4. Main results

Hasanis and Vlachos [26] showed that if a hypersurface $M^n \subset \mathbb{S}^{n+1}$ is of 2-type (i.e., of L_0-2-type), then it has nonzero constant mean curvature and constant scalar curvature. If the number of distinct principal curvatures is less than 4 and M^n is closed, Chang [11] (see also [10, 21]) proved that these conditions imply that the hypersurface is isoparametric. In particular, we have that a 2-type closed hypersurface M^3 in the sphere $\mathbb{S}^4$ has to be isoparametric. But we know that $M^3 \subset \mathbb{S}^4$ is an isoparametric hypersurface if and only if (i) M^3 is a round hypersphere $\mathbb{S}^3(r)$, $0 < r \leq 1$; (ii) M^3 is a Clifford tori $\mathbb{S}^1(r_1) \times \mathbb{S}^2(r_2)$, $r_1^2 + r_2^2 = 1$; or (iii) M^3 is a tube $T^r(V^2)$ of constant radius r around the Veronese embedding of the real projective plane $\mathbb{R}P^2$.

Hasanis and Vlachos [26] also obtain a converse: if a hypersurface $M^n \subset \mathbb{S}^{n+1}$, which is not an open portion of a hypersphere, has nonzero constant mean curvature and constant scalar curvature, then it is of 2-type. Bearing in mind [26] and [11], and the classification of isoparametric hypersurfaces $M^3 \subset \mathbb{S}^4$, one has the following (see [16]).

Theorem 5. *Let $\psi : M^3 \to \mathbb{S}^4$ be a closed orientable hypersurface, which is not an open portion of a hypersphere. Then M^3 is of 2-type if and only if M^3 is a Clifford tori $\mathbb{S}^1(r_1) \times \mathbb{S}^2(r_2)$, $r_1^2 + r_2^2 = 1$ and $r_2^2 \neq \frac{2}{3}$, or a tube $T^r(V^2)$ of constant radius $r \neq \frac{\pi}{6}$ around the Veronese embedding of the real projective plane $\mathbb{R}P^2(\sqrt{3})$.*

Our goal is to prove similar results for operators L_1 and L_2.

Theorem 6. *Let $\psi : M^3 \to \mathbb{S}^4$ be an orientable H_2-hypersurface. If M^3 is of L_2-2-type, then the Gauss-Kronecker curvature H_3 is a nonzero constant.*

Proof. Let $\{E_1, E_2, E_3\}$ be a local orthonormal frame of principal directions of S such that $SE_i = \kappa_i E_i$ for every $i = 1, 2, 3$, and consider the open set

$$\mathcal{U}_3 = \Big\{ p \in M^3 \mid \nabla H_3^2(p) \neq 0 \Big\}.$$

Our goal is to show that $\mathcal{U}_3$ is empty. Otherwise, since we are assuming that M^3 is L_2-2-type and H_2 is constant, then by taking covariant derivative in (22) we have

$$\lambda_1 \lambda_2 a^\top = 9 \nabla H_3^2,$$

and using this in (20) we obtain

$$(S \circ P_2)(\nabla H_3^2) = -\frac{7}{2} H_3 \nabla H_3^2 \quad \text{on } \mathcal{U}_3. \tag{23}$$

Since $P_3 = 0$ then $S \circ P_2 = H_3 I$ and so

$$(S \circ P_2)(\nabla H_3^2) = H_3 \nabla H_3^2,$$

that jointly with (23) implies $H_3 \nabla H_3^2 = 0$ on $\mathcal{U}_3$, which is not possible. □

We want to extend last theorem for the operator L_1; next theorem is an intermediate step.

Theorem 7. *Let M^3 be an orientable H_k-hypersurface of $\mathbb{S}^4$, which is not an open portion of a hypersphere, and consider the following conditions:*

- a) *H_{k+1} is a nonzero constant.*
- b) *$\mathrm{tr}(S^2 \circ P_k)$ is constant.*
- c) *M^3 is of L_k-2-type.*

Then any two conditions imply the third one.

Proof. First, we show that conditions a) and b) imply condition c). From Lemma 1 we obtain that M^3 is an isoparametric hypersurface, and then the claim follows from Proposition 4.

Secondly, we show that conditions a) and c) imply condition b). By taking covariant differentiation in equation (21), and bearing in mind (22), we find

$$c_k H_{k+1} X(\mathrm{tr}(S^2 \circ P_k)) = -\lambda_1\lambda_2 X(\langle N, a\rangle) = \lambda_1\lambda_2 \left\langle a^\top, SX\right\rangle = 0,$$

that is, $\mathrm{tr}(S^2 \circ P_k)$ is constant on M^3.

Finally, we show that conditions b) and c) imply condition a). In the case $k = 2$, the proof follows directly from Theorem 6. To prove the claim in the case $k = 1$, let us consider the open set

$$\mathcal{U}_2 = \{p \in M^3 \mid \nabla H_2^2(p) \neq 0\}.$$

Our goal is to show that $\mathcal{U}_2$ is empty. Since H is constant, by taking covariant derivative in (22) we obtain that $\lambda_1\lambda_2 a^\top = 36\nabla H_2^2$. Using this in (20) we get

$$(S \circ P_1)(\nabla H_2^2) = -\frac{15}{2} H_2 \nabla H_2^2 \quad \text{on } \mathcal{U}_2, \tag{24}$$

that jointly with equation (7) leads to $P_2(\nabla H_2^2) = \frac{21}{2} H_2 \nabla H_2^2$. Now, by applying the operator S on both sides, we have

$$(S \circ P_2)(\nabla H_2^2) = \frac{21}{2} H_2 S(\nabla H_2^2). \tag{25}$$

Since $P_3 = 0$ we get $S \circ P_2 = H_3 I$, and then

$$(S \circ P_2)(\nabla H_2^2) = H_3 \nabla H_2^2,$$

that jointly with (25) implies

$$S(\nabla H_2^2) = \frac{2H_3}{21H_2} \nabla H_2^2.$$

Without loss of generality, let us assume that E_1 is parallel to ∇H_2^2, i.e. the principal curvature $\kappa_1 = \frac{2H_3}{21H_2}$. Then we have

$$(S \circ P_1)(\nabla H_2^2) = \kappa_1 \mu_1^1 \nabla H_2^2 = \frac{2H_3}{21H_2}\left(3H - \frac{2H_3}{21H_2}\right)\nabla H_2^2,$$

that jointly with (24) yields the following equation,

$$6615\, H_2^3 + 252\, H H_2 H_3 - 8H_3^2 = 0.$$

From Lemma 1 we have that $3H_3 = 9HH_2 - \mathrm{tr}(S \circ P_1)$, and then last equation can be rewritten as follows

$$6615\, H_2^3 + 684 H^2\, H_2^2 - 68 H \mathrm{tr}(S^2 \circ P_1)\, H_2 - \tfrac{8}{9} \mathrm{tr}(S^2 \circ P_1) = 0.$$

In other words, H_2 is a root of a polynomial with constant coefficients, and so it is constant. □

An interesting consequence is the following.

Theorem 8. *Let* $\psi : M^3 \to \mathbb{S}^4$ *be an orientable* H_2*-hypersurface. If* M *is of* L_2*-2-type, then* M^3 *is an isoparametric hypersurface.*

Proof. From Theorem 6 we get that H_3 is a non-zero constant, and then Theorem 7 yields that $\mathrm{tr}(S^2 \circ P_2)$ is constant. Now we use Lemma 1(d) to deduce that the mean curvature H is constant, and this concludes the proof. □

Another consequence is the following. Let M^3 be an isoparametric hypersurface, which is not an open portion of a hypersphere, satisfying $H_{k+1} \neq 0$. From Theorem 7 we get M^3 is of L_k-2-type. Then the following result, that extends Theorem 5, is clear.

Theorem 9. *Let* $\psi : M^3 \to \mathbb{S}^4$ *be an orientable* H_2*-hypersurface, which is not an open portion of a hypersphere. Then* M^3 *is of* L_2*-2-type if and only if* M^3 *is a Clifford tori* $\mathbb{S}^1(r_1) \times \mathbb{S}^2(r_2)$, $r_1^2 + r_2^2 = 1$, *or a tube* $T^r(V^2)$ *of constant radius* $r \neq \frac{\pi}{6}$ *around the Veronese embedding of the real projective plane* $\mathbb{R}P^2(\sqrt{3})$.

We now state our main result.

Theorem 10. *Let* $\psi : M^3 \to \mathbb{S}^4$ *be an orientable* H_k*-hypersurface. If* M *is of* L_k*-2-type, then* H_{k+1} *is a nonzero constant.*

Proof. Case $k = 0$ is shown in [26, Theorem 2.1] and case $k = 2$ has been proved in Theorem 6, so we can assume $k = 1$. Let us consider $\{E_1, E_2, E_3\}$ a local orthonormal frame of principal directions of S such that $SE_i = \kappa_i E_i$ for every $i = 1, 2, 3$. Let us define the open set

$$\mathcal{U}_2 = \{p \in M^3 \mid \nabla H_2^2(p) \neq 0\},$$

our goal is to show that $\mathcal{U}_2$ is empty. Since we are assuming that M^3 is L_1-2-type and H is constant, then equation (22) leads to

$$\lambda_1 \lambda_2 a^\top = 36 \nabla H_2^2. \tag{26}$$

Using this equation in (20) we have that $(S \circ P_1)(\nabla H_2^2) = -\frac{15}{2} H_2 \nabla H_2^2$ on $\mathcal{U}_2$, and substituting this into (7) we obtain

$$P_2(\nabla H_2^2) = \frac{21}{2} H_2 \nabla H_2^2 \quad \text{on } \mathcal{U}_2. \tag{27}$$

The vector field ∇H_2^2 can be written as $\nabla H_2^2 = E_1(H_2^2)E_1 + E_2(H_2^2)E_2 + E_3(H_2^2)E_3$, and then

$$P_2(\nabla H_2^2) = \sum_{i=1}^{3} E_i(H_2^2) \mu_2^i E_i.$$

Therefore equation (27) is equivalent to

$$E_i(H_2^2)\left(\mu_2^i - \frac{21}{2} H_2\right) = 0 \quad \text{on } \mathcal{U}_2$$

for every $i = 1, 2, 3$. An immediate and important consequence of this equation is that $E_i(H_2^2) = 0$ for some i. Otherwise, we deduce that

$$\operatorname{tr}(P_2) = \sum_{i=1}^{3} \mu_2^i = \frac{63}{2} H_2,$$

that jointly with Lemma 1 leads to $H_2 = 0$ on $\mathcal{U}_2$, which is a contradiction.

From that consequence, and without loss of generality, we have to analyze the following two possible cases.

Case 1: $E_1(H_2^2) \neq 0$, $E_2(H_2^2) \neq 0$ and $E_3(H_2^2) = 0$.

As $\mu_2^1 = \mu_2^2 = \frac{21}{2} H_2$ then $(\kappa_1 - \kappa_2)\kappa_3 = 0$, and therefore $\kappa_1 = \kappa_2$. Observe that $\kappa_i \neq 0$ for all i, otherwise $H_2 = 0$. It is easy to see that

$$\kappa_2\kappa_3 = \mu_2^1 = \frac{21}{2} H_2 = \frac{7}{2}(\kappa_2^2 + 2\kappa_2\kappa_3),$$

and so $7\kappa_2 + 12\kappa_3 = 0$. On the other hand, we know that $3H = 2\kappa_2 + \kappa_3$ and then we get κ_2 and κ_3 are constants. So H_2 is also constant, which can not be possible.

Case 2: $E_1(H_2^2) \neq 0$, $E_2(H_2^2) = 0$ and $E_3(H_2^2) = 0$.

We know that $3H_2 = \kappa_1\mu_1^1 + \mu_2^1$ and $\mu_2^1 = \frac{21}{2} H_2$, then we have

$$H_2 = \frac{2}{15}\left(\kappa_1^2 - 3H\kappa_1\right) \qquad \text{and} \qquad H_2^2 = p(\kappa_1), \tag{28}$$

where $p(x) = \left(\frac{2}{15}\right)^2 \left(x^4 - 6Hx^3 + 9H^2x^2\right)$. Observe that $H \neq 0$; otherwise, $\kappa_2 + \kappa_3 = -\kappa_1$ and from (28) we get $\kappa_2\kappa_3 = \frac{7}{5}\kappa_1^2$. Then κ_2 and κ_3 are the roots of the equation $t^2 + \kappa_1 t + \frac{7}{5}\kappa_1^2 = 0$, but this is not possible since the discriminant of this equation is negative.

We claim that

$$E_1(H_2^2) = p'(\kappa_1) E_1(\kappa_1), \tag{29}$$

$$\lambda_1\lambda_2 \langle \psi, a \rangle = 36 p(\kappa_1) + A_0, \tag{30}$$

$$\lambda_1\lambda_2 \langle N, a \rangle = q(\kappa_1) + B_0, \tag{31}$$

where $q(x) = -\left(\frac{4}{5}\right)^2 \left(\frac{4}{5}x^5 - \frac{9H}{2}x^4 + 6H^2x^3\right)$, and A_0, B_0 are two constants. First, (29) and (30) follow directly from (28) and (22), respectively. On the other hand, bearing in mind (26) we find that

$$\begin{aligned} X(\lambda_1\lambda_2 \langle N, a \rangle) &= -\lambda_1\lambda_2 \left\langle Sa^\top, X \right\rangle = -36\kappa_1 \left\langle \nabla H_2^2, X \right\rangle \\ &= -36\kappa_1 X(H_2^2) = X\big(q(\kappa_1)\big) \end{aligned}$$

for any tangent vector field X, and this implies equation (31).

Now, by taking covariant differentiation in (26) in the direction of an arbitrary tangent vector field X, we have

$$\begin{aligned} \lambda_1\lambda_2 \nabla_X a^\top &= 36 \nabla_X \nabla H_2^2 = 36 \nabla_X \big(E_1(H_2^2) E_1\big) \\ &= 36 X\big(E_1(H_2^2)\big) E_1 + 36 E_1(H_2^2) \nabla_X E_1, \end{aligned}$$

that jointly with (14) yields

$$36E_1(H_2^2)\nabla_X E_1 = -36X\big(E_1(H_2^2)\big)E_1 + \lambda_1\lambda_2\big(\langle N,a\rangle SX - \langle \psi,a\rangle X\big), \tag{32}$$

or equivalently

$$36E_1(H_2^2)\langle \nabla_X E_1, E_i\rangle = -36X\big(E_1(H_2^2)\big)\delta_{1i} + \lambda_1\lambda_2\big(\langle N,a\rangle \kappa_i - \langle \psi,a\rangle\big)\langle X, E_i\rangle \tag{33}$$

for $i=1,2,3$. If we take $X=E_1$, then (33) reduces to the following equations

$$\begin{aligned} 36E_1(E_1(H_2^2)) &= \lambda_1\lambda_2\big(\langle N,a\rangle \kappa_1 + \langle \psi,a\rangle\big),\\ E_1(H_2^2)\langle \nabla_{E_1}E_1, E_i\rangle &= 0, \qquad i=2,3. \end{aligned}$$

From the last equation we conclude that $\nabla_{E_1}E_1 = 0$, that is, the integral curves of E_1 on $\mathcal{U}_2$ are geodesics of M^3.

Let X be a tangent vector field orthogonal to E_1. Then equation (33) for $i=1$ leads to $X(E_1(H_2^2))=0$ and thus (32) yields

$$36E_1(H_2^2)\nabla_X E_1 = \lambda_1\lambda_2\big(\langle N,a\rangle SX - \langle \psi,a\rangle X\big), \qquad \forall\ X\perp E_1. \tag{34}$$

From the Codazzi equation $(\nabla_{E_j}S)E_1 = (\nabla_{E_1}S)E_j$, we get

$$E_1(\kappa_j) = (\kappa_1-\kappa_j)\left\langle \nabla_{E_j}E_1, E_j\right\rangle, \quad j=2,3,$$

that jointly with (34) for $X=E_j$ yields

$$\begin{aligned} 36E_1(H_2^2)E_1(\kappa_j) &= (\kappa_1-\kappa_j)\big[\lambda_1\lambda_2\langle N,a\rangle \kappa_j - \lambda_1\lambda_2\langle \psi,a\rangle\big]\\ &= -\lambda_1\lambda_2\langle N,a\rangle \kappa_j^2 + \lambda_1\lambda_2\langle N,a\rangle \kappa_1\kappa_j + \lambda_1\lambda_2\langle \psi,a\rangle \kappa_j\\ &\quad - \lambda_1\lambda_2\langle \psi,a\rangle \kappa_1. \end{aligned}$$

Last equation implies

$$\begin{aligned} 36E_1(H_2^2)\sum_{j=2}^{3}E_1(\kappa_j) &= -\lambda_1\lambda_2\langle N,a\rangle \sum_{j=2}^{3}\kappa_j^2 + \lambda_1\lambda_2\langle N,a\rangle \kappa_1\sum_{j=2}^{3}\kappa_j\\ &\quad + \lambda_1\lambda_2\langle \psi,a\rangle \sum_{j=2}^{3}\kappa_j - 2\lambda_1\lambda_2\langle \psi,a\rangle \kappa_1, \end{aligned}$$

that is,

$$\begin{aligned} 36E_1(H_2^2)E_1(3H-\kappa_1) &= -\lambda_1\lambda_2\langle N,a\rangle (\operatorname{tr}(S^2)-\kappa_1^2) + \lambda_1\lambda_2\langle N,a\rangle \kappa_1(3H-\kappa_1)\\ &\quad + \lambda_1\lambda_2\langle \psi,a\rangle (3H-\kappa_1) - 2\lambda_1\lambda_2\langle \psi,a\rangle \kappa_1. \end{aligned}$$

By using (28) and (29), last equation can be written as

$$\begin{aligned} 36p'(\kappa_1)\big[E_1(\kappa_1)\big]^2 = &-\frac{1}{5}\lambda_1\lambda_2\langle N,a\rangle (4\kappa_1^2+3H\kappa_1-45H^2)\\ &+ 3\lambda_1\lambda_2\langle \psi,a\rangle (\kappa_1-H). \end{aligned} \tag{35}$$

A direct computation shows

$$\begin{aligned} 36^2\big[p'(\kappa_1)E_1(\kappa_1)\big]^2 &= 36^2\big[E_1(H_2^2)\big]^2 = 36^2\left\langle \nabla H_2^2, \nabla H_2^2\right\rangle = \lambda_1^2\lambda_2^2|a^\top|^2\\ &= \lambda_1^2\lambda_2^2|a|^2 - (\lambda_1\lambda_2\langle N,a\rangle)^2 - (\lambda_1\lambda_2\langle \psi,a\rangle)^2. \end{aligned} \tag{36}$$

From equations (35) and (37), and taking into account (30) and (31), we find a polynomial $T(x)$ with constant coefficients given by

$$\begin{aligned} T(x) = {} & \big[q(x)+B_0\big]^2 + \big[36p(x)+A_0\big]^2 \\ & - \tfrac{36}{5}\big[q(x)+B_0\big](4x+15H)(x-3H)p'(x) \\ & + 108\big[36p(x)+A_0\big](x-H)p'(x) - \lambda_1^2\lambda_2^2|a|^2, \end{aligned} \tag{37}$$

and satisfying $T(\kappa_1)=0$. Therefore, κ_1 is locally constant on $\mathcal{U}_2$, and so is H_2, which is a contradiction with the definition de $\mathcal{U}_2$. This finishes the proof. ☐

An interesting consequence is the following result, similar to Theorem 8.

Theorem 11. *Let $\psi : M^3 \to \mathbb{S}^4$ be an orientable H-hypersurface. If M^3 is of L_1-2-type, then M^3 is an isoparametric hypersurface.*

Proof. From Theorem 10 we get that H_2 is a non-zero constant, and then Theorem 7 yields that $\operatorname{tr}(S^2 \circ P_1)$ is constant. Now we use Lemma 1(c) to deduce that the Gauss-Kronecker curvature H_3 is constant, and this concludes the proof. ☐

Bearing in mind Theorems 7 and 11, and the classification of isoparametric hypersurfaces M^3 in the sphere $\mathbb{S}^4$, the following result, that extends Theorems 5 and 9, is clear.

Theorem 12. *Let $\psi : M^3 \to \mathbb{S}^4$ be an orientable H-hypersurface, which is not an open portion of a hypersphere. Then M^3 is of L_1-2-type if and only if M^3 is a Clifford tori $\mathbb{S}^1(r_1)\times\mathbb{S}^2(r_2)$, $r_1^2+r_2^2=1$ and $r_2^2 \neq \frac{1}{3}$, or a tube $T^r(V^2)$ of constant radius r around the Veronese embedding of the real projective plane $\mathbb{R}P^2(\sqrt{3})$.*

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