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LAGUERRE CHARACTERIZATION OF SOME HYPERSURFACES

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Abstract. Let $x : M^{n-1} \to \mathbb{R}^n$ $(n \geq 4)$ be an umbilical free hypersurface with non-zero principal curvatures. Then x is associated with a Laguerre metric $\mathbf{g}$, a Laguerre tensor $\mathbf{L}$, a Laguerre form $\mathbf{C}$, and a Laguerre second fundamental form $\mathbf{B}$, which are invariants of x under Laguerre transformation group. We denote the Laguerre scalar curvature by R and the trace-free Laguerre tensor by $\tilde{\mathbf{L}} := \mathbf{L} - \frac{1}{n-1}tr(\mathbf{L})\mathbf{g}$. In this paper, we prove a local classification result under the assumption of parallel Laguerre form and an inequality of the type

$$\|\tilde{\mathbf{L}}\| \leq cR,$$

where $c = \frac{1}{(n-3)\sqrt{(n-2)(n-1)}}$ is appropriate real constant, depending on the dimension.

1. Introduction

Let $x : M^{n-1} \to \mathbb{R}^n$ be an umbilical free hypersurface with non-zero principal curvatures. Let $\xi : M \to S^{n-1}$ be its unit normal. Let $\{e_1, e_2, \ldots, e_{n-1}\}$ be the orthonormal basis for TM with respect to $dx \cdot dx$, consisting of unit principal vectors. Let $r_i = \frac{1}{k_i}$, $r = \frac{r_1+r_2+\cdots+r_{n-1}}{n-1}$ be the curvature radius and mean curvature radius of x respectively, where $k_i \neq 0$ is the principal curvature corresponding to e_i. We define $\rho = \sqrt{\sum_i (r_i - r)^2}$, $\tilde{E}_i = r_i e_i$, $1 \leq i \leq n-1$. Then $\mathbf{g} = \rho^2 d\xi \cdot d\xi$ is a Laguerre invariant metric, $\{\tilde{E}_1, \tilde{E}_2, \ldots, \tilde{E}_{n-1}\}$ is an orthonormal basis for $III = d\xi \cdot d\xi$. The normalized scalar curvature of Laguerre metric $\mathbf{g}$ will be denoted by R and is called the normalized Laguerre scalar curvature. Two basic Laguerre invariants of x, the Laguerre form $C = \sum_i C_i \omega_i$ and the Laguerre tensor $L = \sum_{ij} L_{ij} \omega_i \otimes \omega_j$, are defined by

$$C_i = -\rho^{-2}\Big(\tilde{E}_i(r) - \tilde{E}_i(\log \rho)(r_i - r)\Big), \tag{1}$$

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$$L_{ij} = \rho^{-2}\Big(Hess_{ij}(\log\rho) - \tilde{E}_i(\log\rho)\tilde{E}_j(\log\rho) + \frac{1}{2}(\|\nabla\log\rho\|^2 - 1)\delta_{ij}\Big), \tag{2}$$

where $(Hess_{ij})$ and ∇ are the Hessian-matrix and the gradient operator with respect to the third fundamental form $III = d\xi \cdot d\xi$.

Laguerre geometry of surfaces in $\mathbb{R}^3$ has been developed by Blaschke and his school (see [1]). Recently, there has been some renewed interest for the surface of R^3 in Laguerre geometry (see [2, 3, 4, 9]).

In [7], Li and Wang studied Laguerre differential geometry of oriented hypersurfaces in $\mathbb{R}^n$. For any umbilical-free hypersurface $x : M \to \mathbb{R}^n$ with non-zero principal curvatures, Li and Wang defined a Laguerre invariant metric $\mathbf{g}$, a Laguerre second fundamental form $\mathbf{B}$, a Laguerre form $\mathbf{C}$ and a Laguerre tensor $\mathbf{L}$ on M, and showed that $\{\mathbf{g}, \mathbf{B}\}$ is a complete Laguerre invariant system for hypersurfaces in $\mathbb{R}^n$ with $n \geq 4$. In the case $n = 3$, a complete Laguerre invariant system for surfaces in $\mathbb{R}^3$ is given by $\{\mathbf{g}, \mathbf{B}, \mathbf{L}\}$.

In [8], authors classified hypersurfaces with parallel Laguerre second fundamental form. Laguerre tensor is a codazzi tensor, which is another Laguerre invariant. An eigenvalue of Laguerre tensor $\mathbf{L}$ of x is called a Laguerre eigenvalue of x. If Laguerre eigenvalues of x are equal, i.e., $\mathbf{L} = \sum_{i,j}\lambda\delta_{ij}\omega_i \otimes \omega_j$, and Laguerre form is vanishing, then x is called Laguerre isotropic hypersurface. we define the trace-free Laguerre tensor $\tilde{\mathbf{L}} := \mathbf{L} - \frac{1}{n-1}tr(\mathbf{L})\mathbf{g}$. Authors classified hypersurfaces with vanishing Laguerre form $\mathbf{C}$ and vanishing trace-free Laguerre tensor $\tilde{\mathbf{L}}$ in [6].

In this paper, we prove the following local result:

Theorem 1.1. *Let $x : M^{n-1} \to \mathbb{R}^n$ $(n \geq 4)$ be an umbilical free hypersurface with non-zero principal curvatures. If its Laguerre form $\mathbf{C}$ is parallel and*

$$\|\tilde{\mathbf{L}}\| \leq \frac{R}{(n-3)\sqrt{(n-2)(n-1)}},$$

then R is constant, we have equality

$$\|\tilde{\mathbf{L}}\| = \frac{R}{(n-3)\sqrt{(n-2)(n-1)}}$$

and M^{n-1} is Laguerre equivalent to an open subset of one of the following hypersurfaces in $\mathbb{R}^n$:

(i) *the images of τ of the hypersurface $\tilde{x}$ in $\mathbb{R}^n_0$ with mean curvature radius $r = 0$ and $\rho =$constant, where for the definition of τ, please refer to* [6].

(ii) *the hypersurface $\tilde{x} : H^1 \times S^{n-2} \to \mathbb{R}^n$ by*

$$\tilde{x}(w, v, u) = \sqrt{\frac{(n-1)(n-3)}{R}}\Big(\frac{v}{w}, \frac{u}{w}(1 + w)\Big),$$

where $u : S^{n-2} \to \mathbb{R}^{n-1}$ and $(w, v) : H^1 \to \mathbb{R}^2_1$ are the canonical embeddings.

We organize the paper as follows. In Section 2 we give Lguerre invariants for hypersurfaces in $\mathbb{R}^n$. In Section 3, we make calculations for the example being characterized by our Theorem 1.1. Then we prove the Main Theorem in Section 4.

2. Laguerre geometry of hypersurfaces in $\mathbb{R}^n$

In this section we review the Laguerre invariants and structure equations for hypersurfaces in $\mathbb{R}^n$. For the detail we refer to [7].

Let $\mathbb{R}^{n+3}_2$ be the space $\mathbb{R}^{n+3}$ equipped with the inner product

$$\langle X, Y\rangle = -x_1y_1 + x_2y_2 + \cdots + x_{n+2}y_{n+2} - x_{n+3}y_{n+3}.$$

Let C^{n+2} be the light-cone in $\mathbb{R}^{n+3}$ given by $C^{n+2} = \{X \in \mathbb{R}^{n+3}_2 \,|\, \langle X, X\rangle = 0\}$. Let LG be the subgroup of orthogonal group $O(n+1, 2)$ on $\mathbb{R}^{n+3}_2$ given by

$$LG = \{T \in O(n+1, 2) \,|\, \varsigma T = \varsigma\},$$

where $\varsigma = (1, -1, \vec{0}, 0)$, where $\vec{0} \in \mathbb{R}^n$, is a light-like vector in $\mathbb{R}^{n+3}_2$.

Let $x : M \to \mathbb{R}^n$ be an umbilical free hypersurface with non-zero principal curvatures. Let $\xi : M \to S^{n-1}$ be its unit normal. Let $\{e_1, e_2, \ldots, e_{n-1}\}$ be the orthonormal basis for TM with respect to $dx \cdot dx$, consisting of unit principal vectors. We write the structure equations of $x : M \to \mathbb{R}^n$ by

$$e_j(e_i(x)) = \sum_k \Gamma^k_{ij} e_k(x) + k_i\delta_{ij}\xi; e_i(\xi) = -k_ie_i(x),\ 1 \le i, j, k \le n-1,$$

where $k_i \neq 0$ is the principal curvature corresponding to e_i. Let

$$r_i = \frac{1}{k_i}, r = \frac{r_1 + r_2 + \cdots + r_{n-1}}{n-1}$$

be the curvature radius and mean curvature radius of x, respectively. We define Laguerre position vector of x by

$$Y = \rho(x \cdot \xi, -x \cdot \xi, \xi, 1) : M \to C^{n+2} \subset \mathbb{R}^{n+3}_2,$$

where $\rho = \sqrt{\sum_i (r_i - r)^2} > 0$.

Theorem 2.1. *Let $x, \tilde{x} : M \to \mathbb{R}^n$ be two umbilical oriented hypersurfaces with non-zero principal curvatures. Then x and $\tilde{x}$ are Laguerre equivalent if and only if there exists $T \in LG$ such that $\tilde{Y} = YT$.*

From the theorem we know that

$$\mathbf{g} = \langle dY, dY\rangle = \rho^2 d\xi \cdot d\xi = \rho^2 III$$

is a Laguerre invariant metric, where III is the third fundamental form of x. we call $\mathbf{g}$ the Laguerre metric of x. Let Δ be the Laplacian operator of $\mathbf{g}$, then we have

$$N = \frac{1}{n-1}\Delta Y + \frac{1}{2(n-1)^2}\langle \Delta Y, \Delta Y\rangle Y, \tag{3}$$

and

$$\eta = \Big(\frac{1}{2}(1 + |x|^2), \frac{1}{2}(1 - |x|^2), x, 0\Big) + r(x \cdot \xi, -x \cdot \xi, \xi, 1)$$

From (3) we get

$$\langle Y,Y\rangle=\langle N,N\rangle=0,\langle N,Y\rangle=-1,\langle\eta,\eta\rangle=0,\langle\eta,\varsigma\rangle=-1.$$

Let $\{E_1,E_2,\ldots,E_{n-1}\}$ be an orthonormal basis for $\mathbf{g}=\langle dY,dY\rangle$ with dual basis $\{\omega_1,\omega_2,\ldots,\omega_{n-1}\}$ and write $Y_i=E_i(Y)$, $1\le i\le n-1$. Then we have the following orthogonal decomposition,

$$R_2^{n+3}=Span\{Y,N\}\oplus Span\{Y_1,Y_2,\ldots,Y_{n-1}\}\oplus Span\{\eta,\varsigma\}.$$

We call $\{Y,N,Y_1,\ldots,Y_{n-1},\eta,\varsigma\}$ a Laguerre moving frame in $\mathbb{R}_2^{n+3}$ of x. By taking derivatives of this frame we obtain the following structure equations:

$$E_i(N)=\sum_j L_{ij}Y_j+C_i\varsigma, \tag{4}$$

$$E_j(Y_i)=L_{ij}Y+\delta_{ij}N+\sum_k\Gamma_{ij}^kY_k+B_{ij}\varsigma, \tag{5}$$

$$E_i(\eta)=-C_iY+\sum_j B_{ij}Y_j. \tag{6}$$

From these equations we obtain the following basic Laguerre invariants:

(i) The Laguerre metric $\mathbf{g}=\langle dY,dY\rangle$;

(ii) The Laguerre second fundamental form $\mathbf{B}=\sum_{ij}B_{ij}\omega_i\otimes\omega_j$;

(iii) The Laguerre tensor $\mathbf{L}=\sum_{ij}L_{ij}\omega_i\otimes\omega_j$;

(iv) The Laguerre form $\mathbf{C}=\sum_i C_i\omega_i$, where $L_{ij}=L_{ji}, B_{ij}=B_{ji}$.

By taking further derivatives of (4)-(6), we get the following relations between these invariants:

$$L_{ij,k}=L_{ik,j}; \tag{7}$$

$$C_{i,j}-C_{j,i}=\sum_k(B_{ik}L_{kj}-B_{kj}L_{ki}); \tag{8}$$

$$B_{ij,k}-B_{ik,j}=C_j\delta_{ik}-C_k\delta_{ij}; \tag{9}$$

$$R_{ijkl}=L_{jk}\delta_{il}+L_{il}\delta_{jk}-L_{ik}\delta_{jl}-L_{jl}\delta_{ik}, \tag{10}$$

where $\{L_{ij,k}\}$, $\{C_{i,j}\}$ and $\{B_{ij,k}\}$ are covariant derivatives of the tensors $\{L_{ij}, C_i, B_{ij}\}$ with respect to the Laguerre metric $\mathbf{g}$, respectively, and R_{ijkl} is the curvature tensor of $\mathbf{g}$. Moreover, we have the following identities (see [7]):

$$\sum_{i,j}(B_{ij})^2=1,\ \sum_i B_{ii}=0,\ \sum_i B_{ij,i}=(n-2)C_j, \tag{11}$$

$$\sum_i L_{ii}=-\frac{1}{2(n-1)}\langle\Delta Y,\Delta Y\rangle Y, \tag{12}$$

$$R_{ik}=-(n-3)L_{ik}-\Big(\sum_i L_{ii}\Big)\delta_{ik}, \tag{13}$$

$$R = -2(n-2)\sum_i L_{ii} = \frac{n-2}{(n-1)}\langle \Delta Y, \Delta Y\rangle Y \tag{14}$$

is the normalized scalar curvature.

In the case $n \geq 4$, we know from (11) and (14) that C_i and L_{ij} are completely determined by the Laguerre invariants $\{\mathbf{g}, \mathbf{B}\}$, thus we get:

Theorem 2.2. *Two umbilical free oriented hypersurfaces in $\mathbb{R}^n$ $(n > 3)$ with non-zero principal curvatures are Laguerre equivalent if and only if they have the same Laguerre metric $\mathbf{g}$ and Laguerre second fundamental form $\mathbf{B}$.*

In the case $n = 3$, a complete Laguerre invariant system for surfaces in $\mathbb{R}^3$ is given by $\{\mathbf{g}, \mathbf{B}, \mathbf{L}\}$.

We define $\tilde{E}_i = r_i e_i$, $1 \leq i \leq n-1$. Then $\{\tilde{E}_1, \tilde{E}_2, \ldots, \tilde{E}_{n-1}\}$ is an orthonormal basis for $III = d\xi \cdot d\xi$. Then $\{E_i = \rho^{-1}\tilde{E}_i \,|\, 1 \leq i \leq n-1\}$ is an orthonormal basis for the Laguerre metric $\mathbf{g}$. By direct calculations, we obtain the following local expressions:

$$\mathbf{g} = \sum_i (r_i - r)^2 III = \rho^2 III, \ B_{ij} = \rho^{-1}(r - r_i)\delta_{ij}. \tag{15}$$

3. Typical examples

In this section, for the purpose of proving Theorem 1.1, we will consider a umbilic-free hypersurface M in $\mathbb{R}^n$, and then calculate the Laguerre invariants for $x : H^1 \times S^{n-2}$ in $\mathbb{R}^n$.

Example 3.1. We denote by $H^1 = \{(w, v) \in \mathbb{R}^2_1 \mid -w^2 + v^2 = -1, w > 0\}$ the hyperbolic space embedded in the Minkowski space $\mathbb{R}^2_1$. We define $x : H^1 \times S^{n-2} \to \mathbb{R}^n$ by

$$x(w, v, u) = \Big(\frac{v}{w}, \frac{u}{w}(1 + w)\Big), \tag{16}$$

then x satisfies

$$\mathbf{C} \equiv 0, \ \nabla\mathbf{B} = 0, \tag{17}$$

$$R = (n-1)(n-3) = const, \tag{18}$$

$$\|\tilde{\mathbf{L}}\| = \sqrt{\frac{n-1}{n-2}}. \tag{19}$$

In fact: clearly x is a hypersurface with the unit normal field $\xi = (\frac{v}{w}, \frac{u}{w})$, and the first and the second fundamental forms of x are given by

$$I = dx \cdot dx = \frac{1}{w^2}\{-dw \cdot dw + dv \cdot dv + (1 + w)^2 du \cdot du\},$$

$$II = -dx \cdot d\xi = -\frac{1}{w^2}\{-dw \cdot dw + dv \cdot dv + (1 + w) du \cdot du\},$$

respectively. Therefore x has two principal curvature

$$k_1 = -1,\ k_2 = \cdots = k_{n-1} = -\frac{1}{w+1}. \tag{20}$$

From (20) we see that

$$r = \frac{r_1 + r_2 + \cdots + r_{n-1}}{n-1} = -\frac{(n-2)w + (n-1)}{n-1},$$
$$\rho^2 = \sum_i (r_i - r)^2 = \frac{n-2}{n-1} w^2.$$

From (15) we get the Laguerre metric

$$\mathbf{g} = \frac{n-2}{n-1}(-dw^2 + dv \cdot dv + du \cdot du)$$

Therefore, $\mathbf{g} = g_1 + g_2$, where g_1, g_2 have constant sectional curvature $\frac{n-1}{n-2}$, $-\frac{n-1}{n-2}$ respectively. And the Laguerre second fundamental form is given, by using (15),

$$B_{ij} = b_i \delta_{ij},$$

$$b_1 = -\sqrt{\frac{n-2}{n-1}},\ b_2 = \cdots = b_{n-1} = \sqrt{\frac{1}{(n-1)(n-2)}}.$$

From (1) we get $C_i = 0, 1 \le i \le n-1$, that is (17).

Let $L_{ij} = a_i \delta_{ij}$, from (10) we get

$$a_1 = \frac{n-1}{2(n-2)},\ a_2 = \cdots = a_{n-1} = -\frac{n-1}{2(n-2)}.$$

Thus we have

$$tr\mathbf{L} = \sum_{i=1}^{n-1} a_i = -\frac{(n-1)(n-3)}{2(n-2)}$$

and $\tilde{L}_{ij} = L_{ij} - \frac{trL}{n-1}\delta_{ij} = \tilde{a}_i \delta_{ij}$ with

$$\tilde{a}_1 = 1, \tilde{a}_2 = \cdots = \tilde{a}_{n-1} = -\frac{1}{n-2}.$$

This gives

$$\|\tilde{\mathbf{L}}\|^2 = \sum_{i=1}^{n-1} \tilde{a}_i^2 = \frac{n-1}{n-2}.$$

On the other hand, from (14), we have

$$R = (n-1)(n-3).$$

4. The proof of the main theorem

We are going to calculate the Laplacian of the length of the Laguerre second fundamental form. By definition and (11) we have

$$0 = \frac{1}{2}\Delta\Big(\sum (B_{ij})^2\Big) = \sum (B_{ij,k})^2 + \sum B_{ij}B_{ij,kk}. \tag{21}$$

On the other hand, using (9) and Ricci identities, noting that the Laguerre form $\mathbf{C}$ is parallel, we obtain

$$B_{ij,kk} = B_{kk,ij} + B_{lk}R_{lijk} + B_{il}R_{lkjk}.$$

Form (10), (11) and the above equation, we easily obtain

$$B_{ij}B_{ij,kk} = -B_{ij}^2 L_{kk} - (n-1)B_{ij}B_{il}L_{lj}. \tag{22}$$

Inserting (21) into (22), we get the following lemma:

Lemma 4.1. *Let $x : M^{n-1} \to \mathbb{R}^n$ be an umbilical free hypersurface with non-zero principal curvatures. If the Laguerre form $\mathbf{C}$ of x is parallel, then*

$$\|\nabla\mathbf{B}\|^2 - (n-1)tr(\mathbf{B}^2\mathbf{L}) - tr(\mathbf{L}) = 0. \tag{23}$$

We state the following lemma which is needed in the proof of main theorem.

Lemma 4.2 (cf. [3]). *Let $a_1, \ldots, a_{n-1}$ and $b_1, \ldots, b_{n-1}$ be $2(n-1)$ real numbers satisfying $\sum_i a_i = 0$, $\sum_i b_i = 0$. Then*

$$\Big|\sum a_i b_i^2\Big| \le \frac{n-3}{\sqrt{(n-1)(n-2)}}\sqrt{\sum a_i^2}\sum b_i^2. \tag{24}$$

Moreover, if $\sum_i a_i^2 \neq 0$ and $\sum_i b_i^2 \neq 0$, then equality holds if and only if there are $(n-2)$ pairs of numbers (a_i, b_i) take the same value (a, b).

The proof of the main theorem. Define the free-trace tensor

$$\tilde{\mathbf{L}} := \mathbf{L} - \frac{1}{n-1}tr(\mathbf{L})\mathbf{g}.$$

Since the Laguerre form $\mathbf{C}$ of x is parallel, from (8) we have $\mathbf{BL} = \mathbf{LB}$. Hence, we can choose $\{E_i\}$ such that both, $\mathbf{B}$ and $\tilde{\mathbf{L}}$, are simultaneously diagonal, and therefore we can apply Lemma 4.2:

$$tr(\tilde{\mathbf{L}}\mathbf{B}^2) \le \frac{n-3}{\sqrt{(n-1)(n-2)}}\|\tilde{\mathbf{L}}\|\|\mathbf{B}\|^2. \tag{25}$$

Since the quantities on both side of (25) are invariant under orthogonal transformations, inequality (25) is independent of the choice of $\{E_i\}$. From (23) and (25) we have

$$0 \ge \|\nabla\mathbf{B}\|^2 - 2tr(\mathbf{L}) - \frac{(n-1)(n-3)}{\sqrt{(n-1)(n-2)}}\|\tilde{\mathbf{L}}\|\|\mathbf{B}\|^2. \tag{26}$$

Putting (11) and (13) into (26), we have

$$0 \geq \|\nabla \mathbf{B}\|^2 + \frac{1}{n-2}\Big(R - (n-3)\sqrt{(n-2)(n-1)}\|\tilde{\mathbf{L}}\|\Big). \tag{27}$$

The assumption of the theorem

$$\|\tilde{\mathbf{L}}\| \leq \frac{1}{(n-3)\sqrt{(n-2)(n-1)}}R,$$

and (26) imply

$$\nabla \mathbf{B} = 0,\ \|\tilde{\mathbf{L}}\| = \frac{1}{(n-3)\sqrt{(n-2)(n-1)}}R, \tag{28}$$

and we have equality in the inequality of (28). We consider the two cases:

Case(I): $\tilde{\mathbf{L}} = 0$.

If $\tilde{\mathbf{L}} = 0$, then from (28) and (14), we have

$$R = 0,\ \mathbf{L} = \frac{1}{n-1}tr(\mathbf{L})\mathbf{g} = 0. \tag{29}$$

This together with Theorem 1.1 in [6] implies that M is Laguerre equivalent to the images of τ of hypersurface $\tilde{x}$ in R^n_0 with mean curvature radius $r = 0$ and ρ =constant.

Case(II): $\tilde{\mathbf{L}} \neq 0$.

Now we assume that $\tilde{\mathbf{L}} \neq 0$. Since the Laguerre form is parallel, we can choose $\{E_i\}$ such that both, $\mathbf{B}$ and $\mathbf{L}$, are simultaneously diagonal. Let $\mu_1, \ldots, \mu_{n-1}$ and $\lambda_1, \ldots, \lambda_{n-1}$ are the eigenvalues of inequality holds, Lemma 4.2 gives

$$\mu_2, \ldots, \mu_{n-1} =: \mu,\ \lambda_2, \ldots, \lambda_{n-1} =: \lambda. \tag{30}$$

The relations $tr(\mathbf{B}) = 0$ and $\| \mathbf{B} \|^2 = 1$ imply

$$\mu_1 = -\sqrt{\frac{n-2}{n-1}},\ \mu = \sqrt{\frac{1}{(n-1)(n-2)}}. \tag{31}$$

We use the following convention on the ranges of indices:

$$1 \leq i, j, k, \ldots \leq (n-1),\ 2 \leq \alpha, \beta, \gamma, \ldots \leq (n-1). \tag{32}$$

Since $\nabla \mathbf{B} = 0$, we have

$$0 = B_{1\alpha,k}\omega_k = dB_{1\alpha} + B_{1k}\omega_{k\alpha} + B_{k\alpha}\omega_{k1} = (\mu_1 - \mu)\omega_{1\alpha}. \tag{33}$$

As x is umbilic-free, we have

$$\omega_{1\alpha} = 0, \tag{34}$$

this gives

$$-\frac{1}{2}R_{1\alpha ij}\omega_i \wedge \omega_j = d\omega_{1\alpha} - \omega_{1i} \wedge \omega_{i\alpha} = 0. \tag{35}$$

Form the Gauss equation (2.8) we have

$$0 = R_{1\alpha 1\alpha} = -\lambda_1 - \lambda. \tag{36}$$

That is

$$\lambda_1 = -\lambda. \tag{37}$$

We are going to show that both λ_1 and λ are constant. In fact, noting that $L_{ij} = 0$ for $i \neq j$, from (34) we get

$$L_{1\alpha,k}\omega_k = dL_{1\alpha} + L_{1k}\omega_{k\alpha} + L_{k\alpha}\omega_{k1} = 0. \tag{38}$$

In particular, we have

$$L_{\alpha\alpha,1} = L_{\alpha 1,\alpha} = L_{1\alpha,\alpha} = 0,\ L_{11,\alpha} = L_{1\alpha,1}. \tag{39}$$

(38) and (39) give

$$L_{11,1} = -L_{\alpha\alpha,1}. \tag{40}$$

Hence

$$d\lambda = 0. \tag{41}$$

From this and (37) we see that both λ_1 and λ are constant. From (34), it followings that the two distributions, defined by $\omega_1 = 0$ and $\omega_2 = \cdots = \omega_{n-1} = 0$, are both integrable and thus give a local decomposition of M. Then every point of M has a neighborhood U which is a Riemannian product $V_1 \times V_2$, where V_1 and V_2 are simply connected, with $\dim V_1 = 1$ and $\dim V_2 = n-2$. Since $n \geq 4$, the sectional curvature of V_2 is given by

$$R_{\alpha\beta\alpha\beta} = -2\lambda. \tag{42}$$

V_2 is a manifold with constant curvature. From (28) and (37) we see that

$$\lambda = -\frac{R}{2(n-2)(n-3)}. \tag{43}$$

Hence

$$B_{11} = -\sqrt{\frac{n-2}{n-1}},\ B_{\alpha\alpha} = \sqrt{\frac{1}{(n-1)(n-2)}}. \tag{44}$$

$$L_{11} = \frac{R}{2(n-2)(n-3)},\ L_{\alpha\alpha} = -\frac{R}{2(n-2)(n-3)} \tag{45}$$

Now we compare with Example 3.1 and then consider the following example: the hypersurface $\tilde{x} : H^1 \times S^{n-2} \to \mathbb{R}^n$ by

$$\tilde{x}(w,v,u) = \sqrt{\frac{(n-1)(n-3)}{R}}\Big(\frac{v}{w}, \frac{u}{w}(1+w)\Big),$$

where $u : S^{n-2} \to \mathbb{R}^{n-1}$ and $(w,v) : H^1 \to \mathbb{R}^2_1$ are the canonical embeddings.

We get that the laguerre metric $\tilde{\mathbf{g}}$ of $\tilde{x}$

$$\tilde{\mathbf{g}} = \frac{(n-2)(n-3)}{R}\Big(-dw^2 + dv\cdot dv + du\cdot du\Big) = \tilde{g}_1 + \tilde{g}_2,$$

where $\tilde{g}_1 = \frac{(n-2)(n-3)}{R}\Big(-dw^2 + dv \cdot dv\Big)$ and $\tilde{g}_2 = \frac{(n-2)(n-3)}{R}\Big(du \cdot du\Big)$.

We know that $x : M^{n-1} \to \mathbb{R}^n$ and $\tilde{x} : H^1 \times S^{n-2} \to \mathbb{R}^n$ have the same Laguerre invariants. Thus from Theorem 2.2 x and $\tilde{x}$ are locally Laguerre equivalent. □

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