

STABILITY ANALYSIS OF REGULARIZED VISCOUS VORTEX SHEETS

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ABSTRACT. A vortex sheet is susceptible to the Kelvin-Helmholtz instability, which leads to a singularity at finite time. The vortex blob model provided a regularization for the motion of vortex sheets in an inviscid fluid. In this paper, we consider the blob model for viscous vortex sheets and present a linear stability analysis for regularized sheets. We show that the diffusing viscous vortex sheet is unstable to small perturbations, regardless of the regularization, but the viscous sheet in the sharp limit becomes stable, when the regularization is applied. Both the regularization parameter and viscosity damp the growth rate of the sharp viscous vortex sheet for large wavenumbers, but the regularization parameter gives more significant effects than viscosity.

1. Introduction

A vortex sheet is an interface in an incompressible fluid across which the tangential velocity is discontinuous [3]. It serves as a simple model for a shear layer at a high Reynolds number. In a free shear flow, strong roll-ups evolve on the vortex sheet, which results in a small-scale structure and mixing of the fluid [10, 12, 18]. A variety of flows are described by a vortex sheet; for example, Rayleigh-Taylor instability, water waves and Hele-Shaw flows [4, 7].

The motion of vortex sheets suffers from the Kelvin-Helmholtz instability [3]. The small perturbation of a flat sheet proportional to $\exp(\lambda t + ik\Gamma)$, in an inviscid fluid, has the dispersion relation

$$(1) \quad \lambda(k) = \frac{k}{2},$$

where k represents the wavenumber of the solution, t is time and Γ is the circulation parameter. This relation indicates that short-wave disturbances grow spuriously and cause instability in the evolution of the sheet. As a result,

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the sheet develops a singularity at finite time [15]. Numerical computations break down near the singularity time.

Moore [15] first identified the singularity in a two-dimensional vortex sheet using asymptotic analysis. The curvature of the sheet was found to diverge at a finite time. Numerical studies by Krasny [11] and Shelley [19] support the asymptotic analysis. The singularity formation in axisymmetric, spherical and planar three-dimensional vortex sheets has been further studied by several authors [8, 16, 17]. Beyond the singularity time, the existence of weak solutions for the two-dimensional Euler equations for single-signed vortex sheet initial data was proved by Delort [5] and Majda [14]. The nature and complexity of the solution of the vortex sheet is indicated in the theory by Wu [24], which relates the vortex sheet to a chord-arc curve.

The singularity could be suppressed by giving a numerical smoothing or physical effects such as finite thickness [2] and surface tension [1, 9, 20]. Boundary integral methods, using a blob regularization, have been popularly used for the computation of the vortex sheet [12, 22]. In the blob-regularization, the singular integral kernel in the model is replaced by a convolution with a smoothing function, which damps the growth of high wavenumbers. Krasny [12] proposed a simple regularization model for the vortex sheet and demonstrated the long-time evolution of the interface. The Krasny model has been used for computations of various vortex sheet problems. Differences and similarities of vortex blob models are thoroughly studied in Sohn [22]. The convergence of blob models to a weak solution of the Euler equation for a vortex sheet in the zero limit of the regularization parameter was established by Liu and Xin [13].

Although extensive researches have been carried out for vortex sheets, most of the previous works focus on the model in an inviscid fluid. A fundamental weak point of the vortex sheet model is that it is only for the inviscid dynamics. Tryggvasson *et al.* [23] showed that the small-scale solution of the viscous evolution of a vortex sheet using Navier-Stokes simulations was significantly different from the result of the inviscid vortex sheet. For more realistic description of the flow, it is required to include viscous effects in the model, but only a few studied a model for a viscous vortex sheet.

Dhanak [6] presented a model for a diffusing viscous vortex sheet and showed that the diffusing vortex sheet under small perturbations is linearly unstable. However, many issues on the motion of the diffusing vortex sheet are still unexplored: whether the singularity forms on the interface, the structure of the singularity is similar to that of the inviscid sheet, if it forms, and a regularization stabilizes the motion of the sheet or not. Recently, the author [21] examines the limit of zero thickness of the viscous vortex sheet and finds that the curvature singularity also develops in the sharp viscous vortex sheet, but its appearance is delayed than the inviscid sheet. Since the singularity does not disappear only by giving viscosity to the sheet, a regularization is applied to the model, in order to calculate the evolution of the sheet past the singularity

time. The long-time evolution of the regularized sharp viscous vortex sheet is demonstrated in [21].

In this paper, we consider the blob-regularization of the diffusing viscous vortex sheet. The main purpose of this paper is to investigate the stability of the regularization model of the diffusing viscous vortex sheet, as well as the sharp viscous vortex sheet. We will show that the diffusing viscous vortex sheet is unstable under small perturbations, regardless of the regularization, but the sharp viscous sheet becomes stable, when the regularization is applied.

In Section 2, we describe the regularized viscous vortex sheet model. We present a linear stability analysis for the diffusing and the sharp viscous vortex sheets in Section 3, and results of the stability curve for the two vortex sheets in Section 4. Section 5 gives conclusions.

2. Viscous vortex sheet model

The vortex sheet is represented by $z(\Gamma, t) = x(\Gamma, t) + iy(\Gamma, t)$, $-\infty < \Gamma < \infty$, in complex notation, where Γ is the circulation parameter along the sheet. The motion of the inviscid vortex sheet is described by the Birkhoff-Rott equation [3],

$$(2) \quad \frac{\partial z^*}{\partial t}(\Gamma, t) = \frac{1}{2\pi i} \text{P.V.} \int_{-\infty}^{\infty} \frac{d\Gamma'}{z(\Gamma, t) - z(\Gamma', t)},$$

where P.V. represents the principal value integral, and the asterisk denotes the complex conjugate.

Dhanak [6] considered the motion of the centroid curve of a vortex layer, assuming that the instantaneous vorticity distribution $\omega(s, n, t)$ in the layer decays exponentially as

$$(3) \quad \omega(s, n, t) \sim \exp\left(-\frac{n}{H(s, t)}\right) \quad \text{as } n \rightarrow \pm\infty,$$

where s is the arc-length along the curve and n is the distance in the normal direction of the curve. In (3), $H(s, t)$ gives a measure of the thickness of the vortex layer. The layer is further assumed as thin, satisfying uniformly in s ,

$$(4) \quad \left| \frac{H(s, t)}{\rho(s, t)} \right| \leq \epsilon \ll 1,$$

where $\rho(s, t)$ is the radius of curvature. Under these assumptions, the centroid curve of the vortex layer is regarded as the diffusing vortex sheet in a viscous fluid. The evolution equation of the diffusing viscous vortex sheet is given by

$$(5) \quad \begin{aligned} \frac{\partial z^*}{\partial t}(\Gamma, t) &= \frac{1}{2\pi i} \text{P.V.} \int_{-\infty}^{\infty} \frac{d\Gamma'}{z(\Gamma, t) - z(\Gamma', t)} - i \frac{\partial}{\partial \Gamma} \left(\tau \gamma^3 \frac{\partial z^*}{\partial \Gamma} \right) \\ &\quad - \nu \gamma \frac{\partial \gamma}{\partial \Gamma} \frac{\partial z^*}{\partial \Gamma} + O(\epsilon^2), \end{aligned}$$

where ν denotes the kinematic viscosity. The vortex sheet strength γ is defined as

$$(6) \quad \gamma(\Gamma, t) = \frac{\partial \Gamma}{\partial s}(s, t),$$

and τ is defined as

$$(7) \quad \tau = \frac{1}{\gamma^2} \int_{-\infty}^{\infty} \Delta(\gamma - \Delta) dn, \quad \Delta(s, n, t) = \int_{-\infty}^n \omega(s, n', t) dn'.$$

Here, to $O(\epsilon^2)$, $\Delta(s, n, t)$ is the negative of the jump in the tangential velocity at s between position n and $n = -\infty$. Thus, τ can be written as the momentum thickness of the layer,

$$\tau = \int_{-\infty}^{\infty} \frac{(U_2 - u)(u - U_1)}{(U_1 - U_2)^2} dn,$$

where U_1 and U_2 are the free-stream tangential velocities at $y = \pm\infty$ respectively, and u is the local component of velocity tangential to the interface.

We may consider the viscous vortex sheet of zero thickness. If we assume that all the vorticity is concentrated in the sheet, ω in (7) is written as the delta function, i.e., $\omega(s, n, t) = \gamma(s, t)\delta(n)$. Then, $\Delta(s, n, t)$ becomes the Heaviside step function with the factor $\gamma(s, t)$, which in turn gives $\tau = 0$. Therefore, in the limit of $\tau \rightarrow 0$, (5) reduces to

$$(8) \quad \frac{\partial z^*(\Gamma, t)}{\partial t} = \frac{1}{2\pi i} \text{P.V.} \int_{-\infty}^{\infty} \frac{d\Gamma'}{z(\Gamma, t) - z(\Gamma', t)} - \nu\gamma \frac{\partial \gamma}{\partial \Gamma} \frac{\partial z^*}{\partial \Gamma}.$$

This equation describes the evolution of the sharp viscous vortex sheet. If viscosity is set to zero, we recover the Birkhoff-Rott equation.

The vortex blob model is to replace the singular integral kernel by a regularized one. Following Krasny [12], we apply the δ -parameter to the evolution equations. The regularization equation for the diffusing viscous vortex sheet is given by

$$(9) \quad \begin{aligned} \frac{\partial z^*(\Gamma, t)}{\partial t} = & \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{z^*(\Gamma, t) - z^*(\Gamma', t)}{|z(\Gamma, t) - z(\Gamma', t)|^2 + \delta^2} d\Gamma' \\ & - i \frac{\partial}{\partial \Gamma} \left(\tau \gamma^3 \frac{\partial z^*}{\partial \Gamma} \right) - \nu\gamma \frac{\partial \gamma}{\partial \Gamma} \frac{\partial z^*}{\partial \Gamma}. \end{aligned}$$

Similarly, the regularization equation for the sharp viscous vortex sheet is

$$(10) \quad \frac{\partial z^*(\Gamma, t)}{\partial t} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{z^*(\Gamma, t) - z^*(\Gamma', t)}{|z(\Gamma, t) - z(\Gamma', t)|^2 + \delta^2} d\Gamma' - \nu\gamma \frac{\partial \gamma}{\partial \Gamma} \frac{\partial z^*}{\partial \Gamma}.$$

Equations (9) and (10) are the main evolution equations we consider in this paper.

3. Linear stability analysis

In this section, we present the linear stability analysis of the regularization models for the viscous vortex sheets. The linear stability analysis of the unregularized viscous vortex sheets is given in [6, 21]. The dispersion relation for the diffusing viscous vortex sheet is

$$(11) \quad \lambda_d^0 = \frac{1}{2} \left[-\nu k^2 + \sqrt{\nu^2 k^4 + k^2 (1 - 2\tau k) (1 - 4\tau k)} \right].$$

The superscript 0 represents the unregularization ($\delta = 0$). In the limit of $\tau \rightarrow 0$ in (11), the sharp vortex sheet has the growth rate

$$(12) \quad \lambda_s^0 = \frac{1}{2} \left(-\nu k^2 + \sqrt{\nu^2 k^4 + k^2} \right).$$

Note that λ_d^0 grows with k^2 as $k \rightarrow \infty$, which is even more unstable than the growth rate of the inviscid vortex sheet (1). The growth rate λ_s^0 converges to $1/(4\nu)$ as $k \rightarrow \infty$. This implies that the amplitudes of high wavenumbers are bounded, not growing indefinitely. This bound is large for the fluid of small viscosity, and thus it would yield growths of short-wave disturbances in numerical calculations, causing instability in the evolution of the sheet.

We now examine the linear stability of the regularized viscous vortex sheets. The flat vortex sheet $x(\Gamma, t) = \Gamma$, $y(\Gamma, t) = 0$ is an equilibrium solution of (5). We consider the solution of small perturbations for the equilibrium under the linear approximation of the model. The governing equation for the diffusing vortex sheet is written as

$$(13a) \quad \frac{\partial x}{\partial t} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{y - y'}{r^2 + \delta^2} d\Gamma' - \frac{\partial}{\partial \Gamma} \left(\tau \gamma^3 \frac{\partial y}{\partial \Gamma} \right) - \nu \gamma \frac{\partial \gamma}{\partial \Gamma} \frac{\partial x}{\partial \Gamma},$$

$$(13b) \quad \frac{\partial y}{\partial t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{x - x'}{r^2 + \delta^2} d\Gamma' + \frac{\partial}{\partial \Gamma} \left(\tau \gamma^3 \frac{\partial x}{\partial \Gamma} \right) - \nu \gamma \frac{\partial \gamma}{\partial \Gamma} \frac{\partial y}{\partial \Gamma},$$

where $r = \sqrt{(x - x')^2 + (y - y')^2}$. Let us write the solution with small perturbations as

$$(14) \quad x(\Gamma, t) = \Gamma + \tilde{x}(\Gamma, t), \quad y(\Gamma, t) = \tilde{y}(\Gamma, t).$$

We assume the momentum thickness τ to be constant. The expression (14) is substituted into (13), and the linear terms in \tilde{x} and \tilde{y} are retained. The square and cubic terms of vortex sheet strength behaves as

$$\gamma^3 = 1 - 3 \frac{\partial \tilde{x}}{\partial \Gamma}, \quad \gamma^2 = 1 - 2 \frac{\partial \tilde{x}}{\partial \Gamma}.$$

Then, we obtain the linearized equations

$$(15a) \quad \frac{\partial \tilde{x}}{\partial t} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\tilde{y} - \tilde{y}'}{(\Gamma - \Gamma')^2 + \delta^2} d\Gamma' - \tau \frac{\partial^2 \tilde{y}}{\partial \Gamma^2} + \nu \frac{\partial^2 \tilde{x}}{\partial \Gamma^2},$$

$$(15b) \quad \frac{\partial \tilde{y}}{\partial t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\tilde{x} - \tilde{x}') \left[\frac{1}{(\Gamma - \Gamma')^2 + \delta^2} - 2 \frac{(\Gamma - \Gamma')^2}{((\Gamma - \Gamma')^2 + \delta^2)^2} \right] d\Gamma'$$

$$-2\tau \frac{\partial^2 \tilde{x}}{\partial \Gamma^2}.$$

We look for solutions of the form $\tilde{x} = X(t)e^{\lambda t + ik\Gamma}$ and $\tilde{y} = Y(t)e^{\lambda t + ik\Gamma}$. Substitution into (15) gives

$$X\lambda = -\frac{1}{2\pi} \frac{Y}{\delta} \int_{-\infty}^{\infty} \frac{1 - e^{ik\delta u}}{u^2 + 1} du + \tau k^2 Y - \nu k^2 X,$$

$$Y\lambda = \frac{1}{2\pi} \frac{X}{\delta} \int_{-\infty}^{\infty} \left[\frac{1 - e^{ik\delta u}}{u^2 + 1} - 2 \frac{(1 - e^{ik\delta u})u^2}{(u^2 + 1)^2} \right] du + 2\tau k^2 X.$$

The integrals can be evaluated by using the residue theorem:

$$(17a) \quad I_1 = \int_{-\infty}^{\infty} \frac{1 - e^{ik\delta u}}{u^2 + 1} du = \pi(1 - e^{-k\delta}),$$

$$(17b) \quad I_2 = \int_{-\infty}^{\infty} \frac{(1 - e^{ik\delta u})u^2}{(u^2 + 1)^2} du = \frac{1}{2}\pi[1 + (-1 + k\delta)e^{-k\delta}].$$

Then, the growth rate satisfies the equation,

$$(18) \quad \lambda^2 + \nu k^2 \lambda - \frac{1}{4}k \left[\frac{1}{\delta}(1 - e^{-k\delta}) - 2\tau k^2 \right] (e^{-k\delta} - 4\tau k) = 0.$$

The regularized diffusing vortex sheet has the solution of the growth rate

$$(19) \quad \lambda_d = \frac{1}{2} \left[-\nu k^2 + \sqrt{\nu^2 k^4 + k \left(\frac{1}{\delta}(1 - e^{-k\delta}) - 2\tau k^2 \right) (e^{-k\delta} - 4\tau k)} \right].$$

For $\tau > 0$, the growth rate increases asymptotically with

$$(20) \quad \lambda_d \sim \frac{1}{2}(-\nu + \sqrt{\nu^2 + 8\tau^2})k^2 \quad \text{as } k \rightarrow \infty.$$

Spurious short-wave disturbances thus would be amplified in numerical computations. Surprisingly, the regularization gives no influence on the asymptotic behavior of λ_d for large k . We conclude that the diffusing vortex sheet is unstable, regardless of the regularization.

In the limit of $\tau \rightarrow 0$ in (19), the regularized sharp vortex sheet has the growth rate

$$(21) \quad \lambda_s = \frac{1}{2} \left[-\nu k^2 + \sqrt{\nu^2 k^4 + \frac{1}{\delta}k(1 - e^{-k\delta})e^{-k\delta}} \right].$$

We find that the growth rate λ_s decreases as viscosity ν is increased. It also decreases as the parameter δ is increased. The growth rate decays to zero with

$$(22) \quad \lambda_s \sim \frac{1}{4\delta\nu k} e^{-k\delta} \quad \text{as } k \rightarrow \infty.$$

This indicates the stability of the sheet motion. Furthermore, it shows different effects of ν and δ on the growth rate. The increase of ν gives an algebraic decrease of the growth rate, whereas the increase of δ yields an exponential

decrease. This explains the fact that varying the δ -parameter produces more significant effects on the evolution of the sheet than varying ν does, which was demonstrated in the numerical results for various Reynolds numbers and δ -values in [21].

Taking $\nu = 0$ in (21), we have the growth rate of the regularized inviscid vortex sheet,

$$(23) \quad \lambda_s(\nu = 0) = \frac{1}{2} \sqrt{\frac{1}{\delta} k(1 - e^{-k\delta})e^{-k\delta}}.$$

The growth rate of the regularized inviscid vortex sheet decays to zero with

$$(24) \quad \lambda_s \sim \frac{1}{2\sqrt{\delta}} \sqrt{k} e^{-\frac{1}{2}k\delta} \quad \text{as } k \rightarrow \infty.$$

4. Stability curves

We present the results of the stability curves for the regularized viscous vortex sheets. Figure 1 shows the growth rate of the regularized diffusing vortex sheet for several values of τ . The values of viscosity and the regularization parameter are set to $\nu = 0.001$ and $\delta = 0.2$. The growth rate of the sharp viscous vortex sheet ($\tau = 0$) is also given for comparison. In Fig. 1, the growth rates with $\tau > 0$ increase indefinitely for large k , and the curve of the larger value of τ increases faster, as indicated in (20). Therefore, the sheet would be more unstable for a larger diffusion width. In Fig. 1, there are some stable modes in the intermediate ranges of k whose values of λ are imaginary.

Figure 2 shows the growth rate of the sharp viscous vortex sheet for several values of δ . Viscosity is set to $\nu = 0.001$. The growth rate of the unregularized vortex sheet is also given. (Remind that λ_s^0 has a constant limit for large k .) In Fig. 2, the growth rates of the regularized vortex sheet are damped for large wavenumbers. The growth rate of the sheet has the smaller maximum and decays faster for the larger value of δ . Figure 3 plots the growth rate of the sharp viscous vortex sheet for varying ν . The regularization parameter is set to $\delta = 0.2$. The growth rate of the sheet has smaller maximum and decays faster for larger viscosity. The results of Figs. 2 and 3 clearly shows that the increase of the regularization parameter δ gives larger damping effects on the growth rate than the increase of ν does.

5. Conclusions

We have presented the linear stability analysis for the regularized viscous vortex sheets. The blob model provides sufficient regularization for the motion of the sharp viscous vortex sheet, but it does not work to damp the fast growth of high modes in the diffusing vortex sheet. It is found that both the δ -parameter and viscosity damp the growth rate of large wavenumbers in the

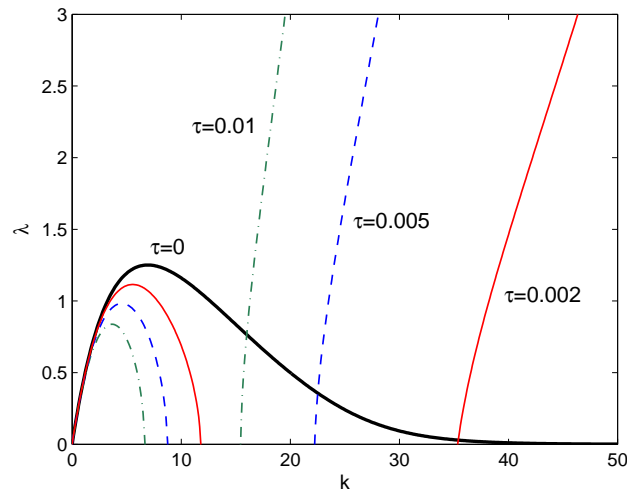


FIGURE 1. Growth rate of the diffusing vortex sheet for several values of τ . The values of viscosity and the regularization parameter are set to $\nu = 0.001$ and $\delta = 0.2$. The growth rate of the sharp viscous vortex sheet ($\tau = 0$) is also given for comparison.

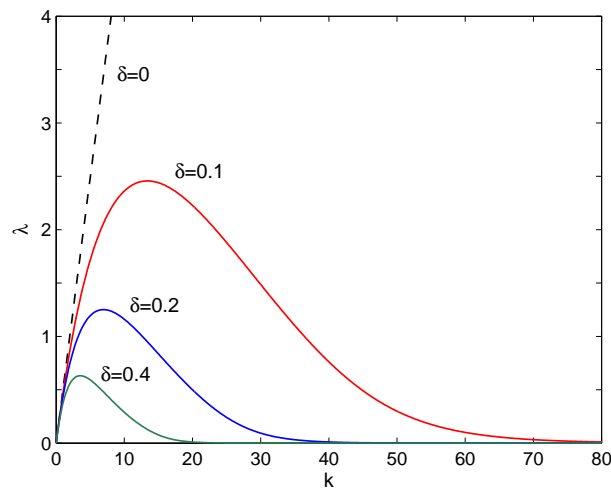


FIGURE 2. Growth rate of the sharp viscous vortex sheet for several values of δ . Viscosity is set to $\nu = 0.001$. The growth rate of the unregularized vortex sheet ($\delta = 0$) is also given.

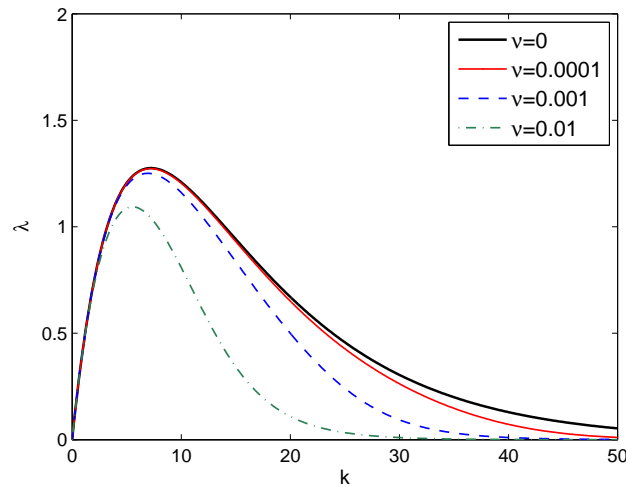


FIGURE 3. Growth rate of the sharp viscous vortex sheet for varying ν . The regularization parameter is set to $\delta = 0.2$.

sharp viscous vortex sheet, but the δ -parameter gives more significant effects than viscosity.

Our study provides better understanding on the modelling of the Kelvin-Helmholtz instability and highlights the fundamental difference between the vortex sheet model and the shear layer model. The key difference of the two models is that the vortex sheet model is based on the approach allowing discontinuity in the tangential component of the velocity on the interface, even when viscous diffusion is considered, while in the shear layer model, a continuous transition layer of physical variables is given to the interface. Our result indicates that numerical computations may become unstable as long as the interface has discontinuity on the tangential velocity, and explains why numerical simulations using Navier-Stokes equations usually fail when the diffusion layer is very sharp. We conclude that the Kelvin-Helmholtz instability of velocity discontinuity is intrinsically unstable, and inclusion of viscous diffusion would not prevent the growth of disturbances of high wavenumbers.

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