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ON THE IRREDUCIBILITY OF SUM OF TWO RECIPROCAL POLYNOMIALS

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Abstract. For a certain kind of reciprocal polynomials $P(x), Q(x) \in \mathbb{Z}[x]$, their sums are considered. We demonstrate that the Mahler measure of polynomials plays a role to prove the irreducibility of the sums over the field of rationals.

1. Introduction and preliminaries

A polynomial $f(x) \in \mathbb{R}[x]$ satisfying $x^n f(x^{-1}) = f(x)$ (resp. $x^n f(x^{-1}) = -f(x)$) is said to be a *reciprocal* (resp. an *anti-reciprocal*) polynomial of *order* n. From the definition, it readily follows that $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{R}[x]$ is reciprocal (resp. anti-reciprocal) if and only if $a_k = a_{n-k}$ (resp. $a_k = -a_{n-k}$) for every $k = 0, 1, \ldots, n$. We note here that $\deg f \leq n$ and the equality holds if and only if $a_n \neq 0$. In the case of $\deg f < n$, some power of x divides $f(x)$.

The irreducibility of reciprocal polynomials with integer coefficients was investigated in the literature, e.g., by Dickson [2] and by Kleiman [3]. On the other hand, the present paper considers a sum of two (anti-)reciprocal polynomials, where this sum is not reciprocal. To be more precise, let $P(x), Q(x) \in \mathbb{Z}[x]$ be (anti-)reciprocal polynomials of different orders. And we suppose further that every zero of $Q(x)$ is, in a sense, close to the closed unit disk. Under a mild condition, this paper shows that for all $b \in \mathbb{Z}$ with sufficiently large $|b|$, a polynomial $P(x) + bQ(x)$ is irreducible over $\mathbb{Q}$. The Mahler measure of polynomials will play a pivotal role in the proof. As an application, we also produce an infinite number of irreducible polynomials over $\mathbb{Q}$. Moreover, the Chebyshev transform gives us an effective bound on $|b|$ for which $P(x) + bQ(x)$ is irreducible.

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Suppose that $g(x) = a_n x^n + \cdots + a_1 x + a_0 = a_n \prod_{i=1}^{n}(x - \alpha_i) \in \mathbb{Z}[x]$ with $a_n \neq 0$. Then the *Mahler measure* of g is a positive number defined by

$$M(g) := |a_n| \prod_{i=1}^{n} \max\{1, |\alpha_i|\}.$$

In particular, cyclotomic polynomials have 1 as their Mahler measures. Conversely, if $M(g) = 1$, then either $g(x)$ or $-g(x)$ is a product of cyclotomic polynomials and some power of x, which is known as Kronecker's theorem [4]. In the statement of the theorem below, we also use a modified Mahler measure

$$M'(g) := \prod_{i=1}^{n} \max\{1, |\alpha_i|\},$$

that is, the leading coefficient is neglected.

The famous problem on Mahler measures, posed by Lehmer [8], is whether or not 1 is an accumulation point of the set $\{M(g) : g \in \mathbb{Z}[x]\}$. He found that the Mahler measure of a polynomial

$$l(x) := x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1, \tag{1}$$

is $\tau_0 \approx 1.17628$, the unique real zero greater than 1. However, no other polynomials whose Mahler measures lie in the interval $(1, \tau_0)$ are known so far. In other words, τ_0 is the smallest *Salem number* that is ever known. See, e.g., [1].

The following result, owing to Smyth, reduces Lehmer's problem to the cases of reciprocal polynomials. An irreducible polynomial with a small Mahler measure should be reciprocal as $l(x)$ is in (1).

Proposition 1.1 ([10]). *Let $p(x) \in \mathbb{Z}[x]$ be irreducible over $\mathbb{Q}$ with $p(x) \neq x-1$ and let $\theta_0 \approx 1.32472$ be the unique real root of $x^3 - x - 1 = 0$. If $M(p) < \theta_0$, then $p(x)$ is a reciprocal polynomial.*

The number θ_0 in the proposition turned out to be the smallest *Pisot number* [9].

Let $p(z) = \sum_{i=0}^{2n} a_i z^i \in \mathbb{R}[z]$ be a nonzero reciprocal polynomial of order $2n$, and suppose that the degree of $p(z)$ is, say, $n + k$. We observe that

$$\begin{aligned} p(z) = \sum_{i=0}^{2n} a_i z^n &= z^n \left[a_{n+k} \left(z^k + \frac{1}{z^k} \right) + \cdots + a_{n+1} \left(z + \frac{1}{z} \right) + a_n \right] \\ &= a_{n+k} z^n \prod_{i=1}^{k} \left(z + \frac{1}{z} - \alpha_i \right) = a_{n+k} z^{n-k} \prod_{i=1}^{k} (z^2 - \alpha_i z + 1) \end{aligned}$$

for some $\alpha_i \in \mathbb{C}$, $i = 1, \ldots, k$. Then the *Chebyshev transform* $\mathcal{T}$ of p is defined by

$$\mathcal{T} p(x) := a_{n+k} \prod_{i=1}^{k} (x - \alpha_i).$$

Note that

$$|p(e^{i\theta})| = |\mathcal{T}p(2\cos\theta)|.$$

We let T_n and U_n denote the nth *Chebyshev polynomials of the first* and *the second kinds* respectively, which are defined by

$$T_n(\cos\theta) = \cos n\theta, \quad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \; n = 0, 1, 2, \ldots.$$

For example, $T_2(x) = 2x^2 - 1$ and $U_3(x) = 8x^3 - 4x$. We also adopt the convention $U_{-1}(x) = 0$. The next propositions tell us how Chebyshev transforms and polynomials are interrelated.

Proposition 1.2 ([5, 7])**.** *Let* $v_{2n}(z) = z^{2n} + z^{2n-1} + \cdots + z + 1$ *and let* $w_{2n}(z) = z^{2n} + 1$. *Then*

$$\begin{aligned} \mathcal{T}v_{2n}(x) &= U_n\left(\frac{x}{2}\right) + U_{n-1}\left(\frac{x}{2}\right), \\ \mathcal{T}w_{2n}(x) &= 2T_n\left(\frac{x}{2}\right). \end{aligned}$$

Proposition 1.3 ([5, 7])**.** *Let* $\overline{v}_{2n}(z) = z^{2n} + z^{2n-2} + \cdots + z^2 + 1$ *and let* $\overline{w}_{2n}(z) = \frac{z^{2n+1}+1}{z+1}$. *Then*

$$\begin{aligned} \mathcal{T}\overline{v}_{2n}(x) &= U_n\left(\frac{x}{2}\right), \\ \mathcal{T}\overline{w}_{2n}(x) &= U_n\left(\frac{x}{2}\right) - U_{n-1}\left(\frac{x}{2}\right). \end{aligned}$$

2. Results and proofs

This section begins with the following variant of Rouché's theorem, whose proof is elementary but included for readers' comfortable reading.

Lemma 2.1. *Let* $f(x), g(x) \in \mathbb{C}[x]$ *and* $b \in \mathbb{C}$. *Suppose that* $\deg f = n$ *and* $\deg g = m$ *with* $m < n$, *and that* $g(x) = c\prod_{i=1}^{m}(x-\alpha_i)$. *Then some* m *(possibly multiple) zeros* $\beta_1(b), \beta_2(b), \ldots, \beta_m(b)$ *of* $f(x) + bg(x)$ *converge to each zero* $\alpha_1, \alpha_2, \ldots, \alpha_m$ *of* $g(x)$ *as* $|b| \to \infty$.

Proof. Suppose that $g(x) = c\prod_{i=1}^{m}(x-\alpha_i) = c\prod_{j=1}^{k}(x-\gamma_j)^{e_j}$ has distinct zeros $\gamma_1, \gamma_2, \ldots, \gamma_k$ with multiplicities $e_1, e_2, \ldots, e_k$ respectively. For $1 \le j \le k$, let C_j be a circle centered at γ_j with radius $\varepsilon_j > 0$. We assume that ε_j is small enough that no zeros of $g(x)$ other than γ_j lie inside nor on C_j. Set $\varepsilon := \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k\}$. Then all sufficiently large $|b|$ satisfy $|bg(x)| > |f(x)|$ on the circle C_j' centered at γ_j with radius ε for all $j = 1, 2, \ldots, k$. Rouché's theorem guarantees that $f(x) + bg(x)$ has e_j zeros inside C_j' as $g(x)$ does for every $j = 1, 2, \ldots, k$. $\square$

Now we consider a sum of two (anti-)reciprocal polynomials.

Lemma 2.2. *Let $P(x)$ and $Q(x)$ be reciprocal or anti-reciprocal polynomials of orders p and q respectively. If both γ and γ^{-1} are zeros of $P(x)+Q(x)$, then either $P(\gamma)=Q(\gamma)=0$ or $\gamma^{q-p}=\pm 1$.*

Proof. For some positive integers $p \neq q$, we suppose one of the following. The double signs below should read in the same order.

(i) $x^p P(x^{-1}) = \pm P(x)$ and $x^q Q(x^{-1}) = \pm Q(x)$.
(ii) $x^p P(x^{-1}) = \pm P(x)$ and $x^q Q(x^{-1}) = \mp Q(x)$.

In each case, one deduces

(i) $P(\gamma) = -Q(\gamma) = \mp\gamma^q Q(\gamma^{-1}) = \pm\gamma^q P(\gamma^{-1}) = \gamma^{q-p}P(\gamma)$
$\Rightarrow P(\gamma)(\gamma^{q-p}-1)=0$,
(ii) $P(\gamma) = -Q(\gamma) = \pm\gamma^q Q(\gamma^{-1}) = \mp\gamma^q P(\gamma^{-1}) = -\gamma^{q-p}P(\gamma)$
$\Rightarrow P(\gamma)(\gamma^{q-p}+1)=0$. □

Every anti-reciprocal polynomial has 1 as its zero. Consequently, if both $P(x)$ and $Q(x)$ are anti-reciprocal, then $P(x)+Q(x)$ is divisible by $x-1$, and hence reducible. We exclude this trivial case in the next theorem. The Euclidean algorithm enables us to easily compute $\gcd(P(x),Q(x))$. If $\gcd(P(x), Q(x))$ is a nonconstant polynomial, then $P(x)+Q(x)$ is also trivially reducible.

Theorem 2.3. *For $i=1,2$, let each pair of $P_i(x)$ and $Q_i(x)$ be coprime polynomials with integer coefficients, and let $\deg P_i = \deg Q_i + 1 = n+1$. Suppose, for $i=1,2$, that $P_i(x)$ is monic, and that $M'(Q_i)$ is less than the smallest Pisot number θ_0. Assume that p and q are distinct positive integers.*

(a) *Let $x^p P_1(x^{-1}) = P_1(x)$ and $x^q Q_1(x^{-1}) = Q_1(x)$. If $\gcd(Q_1(x), x^{|q-p|}-1)=1$, then $P_1(x)+bQ_1(x)$ is irreducible over $\mathbb{Q}$ for all $b\in\mathbb{Z}$ with sufficiently large $|b|$.*
(b) *Let $x^p P_2(x^{-1}) = \pm P_2(x)$ and $x^q Q_2(x^{-1}) = \mp Q_2(x)$ with double signs in the same order. If $\gcd(Q_2(x), x^{|q-p|}+1)=1$, then $P_2(x)+bQ_2(x)$ is irreducible over $\mathbb{Q}$ for all $b\in\mathbb{Z}$ with sufficiently large $|b|$.*

In either case, any reciprocal irreducible factor of $P_i(x)+bQ_i(x)$ divides $x^{|q-p|}-1$ for (a) *and $x^{|q-p|}+1$ for* (b).

Proof. By Lemma 2.1, some n zeros of $f_i(x) := P_i(x)+bQ_i(x)$, $i=1,2$, tend to those of $Q_i(x)$ as $|b|$ increases. Suppose that $f_i(x)=g_i(x)h_i(x)$ for some nonconstant polynomials $g_i(x), h_i(x)\in\mathbb{Z}[x]$. Then they are all monic since $P_i(x)$ is. With no loss of generality, we may assume that $g_i(x)$ is irreducible, and that all the zeros of $g_i(x)$ are close to some of $Q_i(x)$. Because $M'(Q_i)<\theta_0$ and $g_i(x)$ is monic, the Mahler measure of $g_i(x)$ is eventually less than θ_0 as $|b|$ increases, and hence, $g_i(x)$ is reciprocal by Proposition 1.1. If γ is a zero of $g_i(x)$ and thus of $f_i(x)$, then so is γ^{-1}. Therefore, Lemma 2.2 implies $g_1(x)|x^{|q-p|}-1$ and $g_2(x)|x^{|q-p|}+1$. Note here that if $P_i(\gamma)=0$, then $Q_i(\gamma)=0$, and so $P_i(x)$ and $Q_i(x)$ are no more coprime. Now the hypotheses $\gcd(Q_1(x),x^{|q-p|}-1)=1$ and $\gcd(Q_2(x),x^{|q-p|}+1)=1$ lead us to contradictions for all $b\in\mathbb{Z}$ with sufficiently large $|b|$. □

In what follows, we deal with more concrete reciprocal polynomials via the Chebyshev transform. Though this consideration is far from being exhaustive, the reader may have an idea of how the Chebyshev transform works in this theme.

Theorem 2.4. *Let $P(x) \in \mathbb{Z}[x]$ be a reciprocal or anti-reciprocal monic polynomial of order and degree $n+1$. Let $Q(x) = x^n + x^{n-1} + \cdots + x^m$ for $0 \le m \le n$ but $m \ne 1$ if $P(x)$ is reciprocal. If $\gcd(P(x), Q(x)) = 1$, then $f(x) := P(x) + bQ(x)$ is irreducible over $\mathbb{Q}$ for all $b \in \mathbb{Z}$ with sufficiently large $|b|$.*

If $P(x)$ is reciprocal and if $m = 1$, then $P(x)$ and $Q(x)$ are reciprocal polynomials of the same order, and so $f(x)$ is itself a reciprocal polynomial. For the irreducibility of reciprocal polynomials, we refer to [2, 3].

Proof. Suppose that $f(x)$ is factored as a product of two nonconstant monic polynomials $g(x)$ and $h(x)$ in $\mathbb{Z}[x]$. By Lemma 2.1, some n zeros of $f(x)$ are close to the n zeros of $Q(x)$ for all $b \in \mathbb{Z}$ with sufficiently large $|b|$. Without loss of generality, we may assume that $g(x)$ is irreducible and that all the zeros of $g(x)$ are close to some zeros of $Q(x)$. Since all the zeros of $Q(x)$ are 0 and roots of unity, the Mahler measure of $g(x)$ is eventually less than the smallest Pisot number θ_0 as $|b| \to \infty$. Accordingly, Proposition 1.1 shows that $g(x)$ is a reciprocal polynomial for all sufficiently large $|b|$. That is, every zero γ of $g(x)$ satisfies $g(\gamma^{-1}) = 0$, and thus $f(\gamma) = f(\gamma^{-1}) = 0$. Note that $Q(x)$ is a reciprocal polynomial of order $n+m$. If $m = 1$ and $P(x)$ is anti-reciprocal, then Lemma 2.2 proves that $P(\gamma) = Q(\gamma) = 0$, which contradicts the fact that $P(x)$ and $Q(x)$ are relatively prime. If $m = 0$ or 2 and if $P(x)$ is reciprocal (resp. anti-reciprocal), then $\gamma = 1$ (resp. $\gamma = -1$). Note that $Q(1) \ne 0$ (resp. $Q(-1) \ne 0$). Otherwise $x - 1$ (resp. $x + 1$) is a common factor of $P(x)$ and $Q(x)$. The equation

$$P(1) + bQ(1) = 0 \quad (\text{resp. } P(-1) + bQ(-1) = 0) \tag{2}$$

eventually yields a contradiction as $|b|$ increases. Now we suppose $m \ge 3$.

By Lemma 2.2, $\gamma^{m-1} = 1$ if $P(x)$ is reciprocal, and $\gamma^{m-1} = -1$ if $P(x)$ is anti-reciprocal. Note that γ cannot be ± 1 by the similar reasoning involved in (2). Therefore, γ is equal to one of $\gamma_j := \cos\theta_j + i\sin\theta_j$, where, if $P(x)$ is reciprocal, $\theta_j = \frac{2j\pi}{m-1}$ for some $j = 1, 2, \ldots, m-2$, and where, if $P(x)$ is anti-reciprocal, $\theta_j = \frac{j\pi}{m-1}$ for some $j = 1, 2, \ldots, 2m-3$. We divide the rest of the proof into two cases.

Case 1: Let $n - m = 2a$ for some $a \in \mathbb{Z}$.
Since $Q(x) = x^m\left(x^{n-m} + x^{n-m-1} + \cdots + 1\right)$, it follows from Proposition 1.2 that $\mathcal{T}Q(x) = U_a\left(\frac{x}{2}\right) + U_{a-1}\left(\frac{x}{2}\right)$. So one derives that, for some j,

$$\begin{aligned} |P(\gamma_j)| &= |bQ(\gamma_j)| = |b\mathcal{T}Q(2\cos\theta_j)| \\ &= \frac{|b||\sin(a+1)\theta_j + \sin a\theta_j|}{|\sin\theta_j|} \end{aligned}$$

$$= \frac{|b||2\sin\frac{(2a+1)\theta_j}{2}\cos\frac{\theta_j}{2}|}{|\sin\theta_j|}$$
$$= \frac{|b||\sin\left(\frac{2a+1}{2}\right)\theta_j|}{|\sin\frac{\theta_j}{2}|}.$$

Note that $\sin\frac{\theta_j}{2} \neq 0$. And if $\sin\left(\frac{2a+1}{2}\right)\theta_j = 0$, then $P(\gamma_j) = Q(\gamma_j) = 0$, which implies $\gcd(P(x), Q(x)) \neq 1$. We eventually obtain a contradiction as $|b|$ increases.

Case 2: Let $n - m = 2a + 1$ for some $a \in \mathbb{Z}$.
Since $Q(-1) = 0$, we have $f(-1) \neq 0$. Define a polynomial $R(x)$ by

$$R(x) = \frac{Q(x)}{x+1} = \frac{x^m(x^{n-m} + \cdots + 1)}{x+1} = x^m(x^{2a} + x^{2a-2} + \cdots + x^2 + 1).$$

By Proposition 1.3, the Chebyshev transform of $R(x)$ is given by $\mathcal{T}R(x) = U_a\left(\frac{x}{2}\right)$. Hence, one finds that, for some j,

$$\left|\frac{P(\gamma_j)}{\gamma_j+1}\right| = \left|\frac{bQ(\gamma_j)}{\gamma_j+1}\right| = |bR(\gamma_j)| = |b\mathcal{T}R(2\cos\theta_j)| = |b|\left|\frac{\sin(a+1)\theta_j}{\sin\theta_j}\right|. \tag{3}$$

Here, $\sin\theta_j \neq 0$ since $\gamma_j \neq \pm 1$, and $\sin(a+1)\theta_j \neq 0$ since $\gcd(P(x), Q(x)) = 1$. As $|b|$ increases, the equation (3) is eventually violated. □

The next theorem does not exactly fit Theorem 2.3, but one sees that a similar argument as before also works in this setting.

Theorem 2.5. *Let $P(x) \in \mathbb{Z}[x]$ be a reciprocal or anti-reciprocal monic polynomial of order p, and $\deg P = n+1$. Let $Q_b(x) = b(x^n+1)+x^n+x^{n-1}+\cdots+1$. If $\gcd(x^n + 1, P(x) + x^n + x^{n-1} + \cdots + 1) = 1$, then $f(x) := P(x) + Q_b(x)$ is irreducible over $\mathbb{Q}$ for all $b \in \mathbb{Z}$ with sufficiently large $|b|$.*

Proof. Suppose that $f(x) = g(x)h(x)$ for some nonconstant monic polynomials $g(x)$ and $h(x)$ in $\mathbb{Z}[x]$. By Lemma 2.1, some n zeros of $f(x)$ are close to the n zeros of $x^n + 1$ for sufficiently large $|b|$. Without loss of generality, we may assume that $g(x)$ is irreducible and that all the zeros of $g(x)$ are close to some zeros of $x^n + 1$. Hence, the Mahler measure of $g(x)$ is eventually less than θ_0, and so $g(x)$ is a reciprocal polynomial for all sufficiently large $|b|$. Let γ be a zero of $g(x)$, and hence of $f(x)$. One observes that $p \geq n + 1$. By Lemma 2.2, $\gamma^{p-n} = 1$ if $P(x)$ is reciprocal, and $\gamma^{p-n} = -1$ otherwise. As in the proof of Theorem 2.4, we have $\gamma = \gamma_j := \cos\theta_j + i\sin\theta_j$. Here, $\theta_j = \frac{2j\pi}{p-n}$ for some $j = 1, 2, \ldots, p - n - 1$ when $P(x)$ is reciprocal, and $\theta_j = \frac{j\pi}{p-n}$ for some $j = 1, 2, \ldots, 2(p-n)-1$ when $P(x)$ is anti-reciprocal. Note that $\gamma = 1$ cannot be a zero for all sufficiently large $|b|$ because $f(1) = P(1)+Q_b(1) = P(1)+2b+n+1$. For an odd n, in particular, $\gamma = -1$ is never a zero, either. Indeed, $f(-1) = 0$ implies $\gcd(x^n + 1, P(x) + x^n + x^{n-1} + \cdots + 1) \neq 1$.

Case 1: Let $n = 2a$ for some $a \in \mathbb{Z}$. Since $\mathcal{T}Q_b(x) = 2bT_a\left(\frac{x}{2}\right) + U_a\left(\frac{x}{2}\right) + U_{a-1}\left(\frac{x}{2}\right)$, one finds that

$$\begin{aligned}
|P(\gamma_j)| &= |Q_b(\gamma_j)| = |\mathcal{T}Q_b(2\cos\theta_j)| \\
&= \left|2b\cos a\theta_j + \frac{\sin(a+1)\theta_j + \sin a\theta_j}{\sin\theta_j}\right| \\
&= \left|2b\cos a\theta_j + \frac{\sin\frac{(2a+1)\theta_j}{2}}{\sin\frac{\theta_j}{2}}\right| \geq 2|b||\cos a\theta_j| - \left|\frac{\sin\frac{(2a+1)\theta_j}{2}}{\sin\frac{\theta_j}{2}}\right|, \qquad (4)
\end{aligned}$$

where $\sin\frac{\theta_j}{2}$ never vanishes. We also claim $\cos a\theta_j \neq 0$. If $\cos a\theta_j = 0$, then $\gamma_j^n + 1 = P(\gamma_j) + \gamma_j^n + \gamma_j^{n-1} + \cdots + 1 = 0$ that contradicts $\gcd(x^n + 1, P(x) + x^n + x^{n-1} + \cdots + 1) = 1$. The inequality (4) is violated for all sufficiently large $|b|$.

Case 2: $n = 2a + 1$ $(a \in \mathbb{Z})$. Define a polynomial $R(x)$ by

$$\begin{aligned}
R(x) = \frac{Q_b(x)}{x+1} &= b\frac{x^{2a+1}+1}{x+1} + \frac{x^{2a+1} + x^{2a} + \cdots + 1}{x+1} \\
&= b\frac{x^{2a+1}+1}{x+1} + x^{2a} + x^{2a-2} + \cdots + 1.
\end{aligned}$$

Applying Proposition 1.3, we have $\mathcal{T}R(x) = b\left(U_a\left(\frac{x}{2}\right) - U_{a-1}\left(\frac{x}{2}\right)\right) + U_a\left(\frac{x}{2}\right)$. So one deduces

$$\begin{aligned}
\left|\frac{P(\gamma_j)}{\gamma_j + 1}\right| &= |R(\gamma_j)| = |\mathcal{T}R(2\cos\theta_j)| \\
&= \left|b\frac{\sin(a+1)\theta_j - \sin a\theta_j}{\sin\theta_j} + \frac{\sin(a+1)\theta_j}{\sin\theta_j}\right| \\
&= \left|\frac{b\cos\frac{(2a+1)\theta_j}{2}}{\cos\frac{\theta_j}{2}} + \frac{\sin(a+1)\theta_j}{\sin\theta_j}\right| \\
&\geq |b|\left|\frac{\cos\frac{(2a+1)\theta_j}{2}}{\cos\frac{\theta_j}{2}}\right| - \left|\frac{\sin(a+1)\theta_j}{\sin\theta_j}\right|. \qquad (5)
\end{aligned}$$

Since $\gamma_j \neq \pm 1$, we observe $\cos\frac{\theta_j}{2} \neq 0$ and $\sin\theta_j \neq 0$. If $\cos\frac{(2a+1)\theta_j}{2} = 0$, then the minimal polynomial for γ_j divides both $x^n + 1$ and $P(x) + x^n + x^{n-1} + \cdots + 1$, which is absurd. Now the inequality (5) eventually yields a contradiction as $|b|$ increases. □

3. Examples

This section applies the theorems to obtain irreducible polynomials over $\mathbb{Q}$.

Let $\mathbf{a} = (a_0, a_1, \ldots, a_d) \in \mathbb{R}^{d+1}$, and let σ be one of permutations on $\{0, 1, \ldots d\}$ satisfying $a_{\sigma(0)} \leq a_{\sigma(1)} \leq \cdots \leq a_{\sigma(d)}$. We define

$$\underline{m}(\mathbf{a}) := a_{\sigma(\lfloor d/2 \rfloor)} \quad \text{and} \quad \overline{m}(\mathbf{a}) := a_{\sigma(\lceil d/2 \rceil)}.$$

One observes that $\underline{m}(\mathbf{a}) = \overline{m}(\mathbf{a})$ if d is even. A real function $H(y) := \sum_{j=0}^{d} |y - a_j|$ assumes its minimum whenever y belongs to the closed interval or the singleton $[a_{\sigma(\lfloor d/2 \rfloor)}, a_{\sigma(\lceil d/2 \rceil)}]$. We set $L(\mathbf{a}) := \min_{y \in \mathbb{R}} \sum_{j=0}^{d} |y - a_j|$. In the first example demonstrating Theorem 2.3, we utilize the next proposition.

Proposition 3.1 ([6]). *Let $f(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0 \in \mathbb{R}[x]$ be a reciprocal polynomial with $a_d > 0$, and let $\mathbf{a} = (a_0, a_1, \ldots, a_d)$ and $\mathbf{a}' = (a_1, a_2 \ldots, a_{d-1})$. If one of the following conditions holds, then all the zeros of f lie on the unit circle*:

(a) $\overline{m}(\mathbf{a}) \geq L(\mathbf{a})$,
(b) $f(1) \geq 0$ *and* $2a_d \geq L(\mathbf{a}') + \underline{m}(\mathbf{a}')$.

Example 1. Let $P(x) = x^5 - 3x^4 + 2x^3 - 3x^2 + x$ and $Q(x) = 2x^4 + x^3 + x + 2$. Then $x^6 P(x^{-1}) = P(x)$, $x^4 Q(x^{-1}) = Q(x)$, and $\gcd(P(x), Q(x)) = 1$ with $\gcd(Q(x), x^2 - 1) = 1$. Moreover, $M'(Q) = 1$ by Proposition 3.1. We compute the maximum and minimum moduli of $P(\theta_0^{1/4} e^{it})$ and $Q(\theta_0^{1/4} e^{it})$, respectively:

$$\max_{|z| = \theta_0^{1/4}} |P(z)| = 12.390649 \cdots, \qquad \min_{|z| = \theta_0^{1/4}} |Q(z)| = 0.750626 \cdots.$$

Hence, if $|b| \geq 17$, then $|bQ(z)| > |P(z)|$ on the circle $|z| = \theta_0^{1/4}$. Now Rouché's theorem guarantees that $P(x) + bQ(x)$ and $Q(x)$ have the same number of zeros inside this circle. i.e., $P(x) + bQ(x)$ has four zeros inside the circle $|z| = \theta_0^{1/4}$. If $P(x) + bQ(x)$ is reducible, then it has a monic reciprocal factor $g(x)$ dividing $x^2 - 1$. In other words,

$$P(1) + bQ(1) = -2 + 6b = 0 \quad \text{or} \quad P(-1) + bQ(-1) = -10 + 2b.$$

But these equations cannot be true for $|b| \geq 17$. By additional irreducibility checks for $-16 \leq b \leq 16$, we can state the following.

> *$P(x) + bQ(x)$, $b \in \mathbb{Z}$ is irreducible over $\mathbb{Q}$ if and only if $b \neq 0, 5$. If $b = 0$, then $P(x) + 0Q(x) = x(x^2 + 1)(x^2 - 3x + 1)$. And if $b = 5$, then $P(x) + 5Q(x) = (x + 1)(x^4 + 6x^3 + x^2 - 4x + 10)$.*

The next example follows the proof of Theorem 2.4.

Example 2. Let $P(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$ and $Q(x) = x^9 + x^8 + x^7 + x^6 + x^5$. Then one notes that $x^{10} P(x^{-1}) = P(x)$ and $\gcd(P(x), Q(x)) = 1$. The maximum and minimum moduli of $P(\theta_0^{1/9} e^{it})$ and $Q(\theta_0^{1/9} e^{it})$ are calculated as

$$\max_{|z| = \theta_0^{1/9}} |P(z)| = 8.059197 \cdots, \qquad \min_{|z| = \theta_0^{1/9}} |Q(z)| = 0.165449 \cdots.$$

Let $|b| \geq 49$. Then we find that $|bQ(z)| > |P(z)|$ on the circle $|z| = \theta_0^{1/9}$, and thus $P(x) + bQ(x)$ has nine zeros inside this circle. If $P(x) + bQ(x)$ is reducible, then, for some $j = 1, 2, 3$,

$$|P(\gamma_j)| = |b| \left| \frac{\sin \frac{5\theta_j}{2}}{\sin \frac{\theta_j}{2}} \right|,$$

where $\gamma_j = e^{i\theta_j}$ and $\theta_j = j\pi/2$. But one verifies that

$$\left| \frac{P(\gamma_j) \sin \frac{\theta_j}{2}}{\sin \frac{5\theta_j}{2}} \right| = 1 \text{ or } 3,$$

which is a contradiction. More irreducibility checks for $-48 \leq b \leq 48$ enable us to show the following.

> *$P(x) + bQ(x)$, $b \in \mathbb{Z}$ is irreducible over $\mathbb{Q}$ if and only if $b \neq 1, -3$. We also have $P(x)+1Q(x) = (x+1)(x^9+x^8-x^3+1)$ and $P(x)-3Q(x) = (x^2+1)(x^8-2x^7-4x^6-2x^5-2x^3-x^2+x+1)$.*

We follow the proof of Theorem 2.5 in the last example.

Example 3. Let $P(x) = x^{11} + 2x^{10} + 3x^9 - x^8 + x^7 - 3x^6 - 2x^5 - x^4$ and $Q_b(x) = b(x^{10} + 1) + x^{10} + x^9 + \cdots + 1$. Then one observes that $x^{15}P(x^{-1}) = -P(x)$ and $\gcd(x^{10} + 1, P(x) + x^{10} + x^9 + \cdots + 1) = 1$. In the similar fashion as above, we appeal to the following computations:

$$\max_{|z|=\theta_0^{1/10}} |P(z) + z^{10} + z^9 + \cdots + 1| = 14.263090\cdots,$$

$$\min_{|z|=\theta_0^{1/10}} |z^{10} + 1| = 0.324717\cdots.$$

Accordingly, for $|b| \geq 44$, we deduce that $P(x) + Q_b(x)$ has ten zeros inside the circle $|z| = \theta_0^{1/10}$. Suppose that $P(x) + Q_b(x)$ is reducible. Then, for some $j = 1, 2, \ldots, 9$,

$$|P(\gamma_j)| \geq 2|b||\cos 5\theta_j| - \left| \frac{\sin \frac{11\theta_j}{2}}{\sin \frac{\theta_j}{2}} \right|,$$

where $\gamma_j = e^{i\theta_j}$ and $\theta_j = j\pi/5$. But a computation shows that

$$\left(|P(\gamma_j)| + \left| \frac{\sin \frac{11\theta_j}{2}}{\sin \frac{\theta_j}{2}} \right| \right) /(2|\cos 5\theta_j|) < 6,$$

which contradicts $|b| \geq 44$. Together with irreducibility checks for $-43 \leq b \leq 43$, we conclude the following.

> *$P(x)+Q_b(x)$, $b \in \mathbb{Z}$ is irreducible over $\mathbb{Q}$ if and only if $b \neq -1$. In the case of $b = -1$, the factorization is given by $P(x) + Q_{-1}(x) = x(x^{10} + 2x^9 + 4x^8 + 2x^6 - 2x^5 - x^4 + x^2 + x + 1)$.*

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