Bull. Korean Math. Soc. **53** (2016), No. 3, pp. 821–831
http://dx.doi.org/10.4134/BKMS.b150355
pISSN: 1015-8634 / eISSN: 2234-3016

HERMITIAN ALGEBRA ON GENERALIZED LEMNISCATES

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Abstract. A case study is added to our recent work on Quillen phenomenon. Pointwise positivity of polynomials on generalized lemniscates of the complex plane is related to sums of hermitian squares of rational functions, and via operator quantization, to essential subnormality.

1. Introduction and preliminaries

A generalized lemniscate is the level set of the norm of the resolvent of a complex matrix, localized at a non-zero vector. In analogy with classical lemniscates (i.e., level sets of the modulus of a polynomial) this class of algebraic curves exhibits special features well understood in the context of function theory, potential theory and fluid mechanics, see the references in [6]. The aim of the present note is to add a real algebra touch to these works. In analogy to the recently outlined example of ellipses [7], a subtle difference between sums of squares and hermitian sums of squares decompositions of positive polynomials persists on generalized lemniscates.

1.1. Generalized lemniscates

A *generalized lemniscate* is a real algebraic curve in $\mathbb{R}^2 = \mathbb{C}$ given by equation

$$\Gamma = \{z \in \mathbb{C};\ \|(A-z)^{-1}\xi\| = 1\},$$

where A is a $d \times d$ complex matrix and $\xi \in \mathbb{C}^d$ is a fixed vector. By abuse of terminology, the bounded open set

$$\Omega = \{z \in \mathbb{C};\ \|(A-z)^{-1}\xi\| > 1\} \cup \sigma(A),$$

where $\sigma(A)$ stands for the spectrum of A, is also called a generalized lemniscate. Without loss of generality we may assume that ξ is a cyclic vector for A, as the localized resolvent $(A-z)^{-1}\xi$ takes values in the A-cyclic subspace generated by the vector ξ.

Received May 12, 2015.
2010 *Mathematics Subject Classification.* 11E25, 47B20, 30C20.
Key words and phrases. generalized lemniscate, positive polynomial, hermitian square, quadrature domain, subnormal operator.

Generalized lemniscates admit a determinantal representation:

$$\begin{aligned}
&\det\begin{pmatrix} \xi\langle\cdot,\xi\rangle & A-z \\ A^*-\overline{z} & I \end{pmatrix} \\
&= \det\begin{pmatrix} \xi\langle\cdot,\xi\rangle-(A-z)(A^*-\overline{z}) & A-z \\ 0 & I \end{pmatrix} \\
&= \det[\xi\langle\cdot,\xi\rangle-(A-z)(A^*-\overline{z})] \\
&= |\det(A-z)|^2\det[(A-z)^{-1}\xi\langle\cdot,(A-z)^{-1}\xi\rangle-I] \\
&= -|\det(A-z)|^2[1-\|(A-z)^{-1}\xi\|^2].
\end{aligned}$$

Even more surprising is the sum of hermitian squares decomposition:

$$\|(A-z)^{-1}\xi\|^2 = \frac{|Q_{d-1}(z)|^2+|Q_{d-2}(z)|^2+\cdots+|Q_1(z)|^2+|Q_0(z)|^2}{|Q_d(z)|^2},$$

where $Q_d(z)$ is the minimal polynomial of A and $\deg Q_j = j$ for *every* $j \in \{0,1,\ldots,d\}$. As a matter of fact, the above hermitian sums of squares decomposition without gaps in any degree characterizes generalized lemniscates, see for a proof [2].

Thus, the defining equation of a generalized lemniscate takes the form

$$r(z,\overline{z}) = |Q_d(z)|^2 - \sum_{j=0}^{d-1}|Q_j(z)|^2, \tag{1}$$

where Q_j are arbitrary polynomials of exact degree $\deg Q_j = j,\ 0 \le j \le d$.

After passing to homogeneous coordinates one can check without difficulty that the map induced by

$$z \mapsto (Q_0(z), Q_1(z), \ldots, Q_d(z))$$

gives an embedding of projective space $\mathbb{P}^1(\mathbb{C})$ into $\mathbb{P}^d(\mathbb{C})$ and that the real algebraic curve Γ is the pull back of the unit sphere, in an affine chart of coordinates. This geometric observation illuminates some qualitative features of Schwarz' reflection with respect to Γ, see for details [2, 6].

1.2. Quadrature domains

Independent of the preceding matrix theoretic definition, one can construct a subclass of generalized lemniscates with potential theory techniques. More specifically, one proves that for every finite atomic positive measure in the plane

$$\mu = c_1\delta_{a_1} + c_2\delta_{a_2} + \cdots + c_d\delta_{a_d}$$

there exists at least one bounded open set Ω, containing the points $a_1,\ldots,a_d$, with the property

$$\int_\Omega f dx\, dy = \int f d\mu = c_1 f(a_1) + c_2 f(a_2) + \cdots + c_d f(a_d),$$

for *all* analytic functions $f(z)$ which are integrable in Ω. The set Ω is called a *quadrature domain.* We may allow points to merge in the above definition, in which case finitely many derivatives of the function f occur in the right hand side. An important fluid flow is modeled by the dynamics of a nested family of quadrature domains, in the case the coefficients c_j are linear functions of time. We refer to the survey [3] for details.

The boundary $\Gamma = \partial\Omega$ of a quadrature domain is a generalized lemniscate, with the nodes $\{a_1, \ldots, a_d\}$ coinciding with the eigenvalues of the cyclic matrix A, or equivalently, with the zeros of the minimal polynomial Q_d:

$$Q_d(z) = (z - a_1)(z - a_2) \cdots (z - a_d).$$

For a proof see [2].

Every simply connected quadrature domain is the image of the unit disk by a rational conformal mapping. Some low degree examples are contained in [3, 6] and in the last section below. Also, notice that a disjoint union of quadrature domains is also a quadrature domain. For instance a disjoint union of disks $D(a_j, r_j),\ 1 \leq j \leq d$, is a quadrature domain, although it is not immediate that there are polynomials $Q_j(z)$ of exact degree j, such that

$$\prod_{j=1}^{d} [|z - a_j|^2 - r_j^2] = |\prod_{j=1}^{d} (z - a_j)|^2 - \sum_{k=0}^{d-1} |Q_k(z)|^2.$$

Every connected open set in the complex plane can be approximated in the Hausdorff distance by a sequence of quadrature domains, [3]. Consequently, the non generic family of domains bounded by generalized lemniscates approximate any bounded open set in the plane.

1.3. Hermitian sums of squares on real algebraic curves

Let C be a real algebraic curve in $\mathbb{R}^2$. The question whether every non-negative polynomial on C admits a sum of squares representation was thoroughly studied and completely settled thanks to the works of C. Scheiderer [11] and D. Plaumann [4], see also the survey [12]. Conditions for a polynomial to be a sum of hermitian squares along the curve C are more restrictive and turn out to be related to a non-commutative functional calculus, in the classical spirit of F. Riesz original proof of the spectral theorem for unitary operators [10], and to a landmark observation of Quillen [9] referring to hermitian sums of squares on odd dimensional spheres.

A general framework for analyzing sums of hermitian squares on real algebraic varieties was recently proposed in [8]. We extract from there a few relevant facts for our note. Let C be a bounded real algebraic curve in $\mathbb{R}^2$ and let I_C denote the reduced ideal associated to C, regarded as a conjugation invariant ideal in $\mathbb{C}[z, \overline{z}]$ rather than in the algebra of real polynomials. A *hermitian square* is a polynomial of the form $|P(z)|^2$ with $P \in \mathbb{C}[z]$. Sums of hermitian squares form a convex cone, denoted Σ_h^2. In order to apply the abstract machinery of Positivstellensätze [5] it is important to remark that $\Sigma_h^2 + I_C$ is a semi ring, that is it contains the non-negative real numbers and it is closed under sums and products. A first result along these lines, proved in [8], asserts that every conjugation-invariant polynomial $p(z,\overline{z})$ which is strictly positive on C is a sum of hermitian squares modulo I_C if and only if there exists a real constant M so that $M-|z|^2 \in \Sigma_h^2 + I_C$.

The simple example of the unit circle $\mathbb{T} = \{z \in \mathbb{C};\ z\overline{z}-1=0\}$ is relevant for our note. A well known lemma due to Fejér and Riesz states that for every non-negative polynomial $p(z,\overline{z})$ along the circle one can find polynomials $P \in \mathbb{C}[z],\ h \in \mathbb{C}[z,\overline{z}]$ with the property:

$$p(z,\overline{z}) = |P(z)|^2 + (1-|z|^2)h(z,\overline{z}), \qquad z \in \mathbb{C}.$$

From this innocent looking identity F. Riesz deduced the spectral theorem for unitary transformations, his main observation being that equation

$$U^*U = UU^* = I$$

satisfied by a linear bounded operator U acting on a Hilbert space implies

$$p(U,U^*) = P(U)^*P(U) \geq 0,$$

that is the functional calculus $p \mapsto p(U,U^*)$ is positive, in addition of being unital, linear and multiplicative. We can slightly relax the above reasoning and consider a linear bounded operator $V \in L(H)$ which only satisfies $V^*V = I$, that is V is a non necessarily surjective isometry. By adopting the convention (called *hereditaery functional calculus*) of putting the powers of V^* to the left of the powers of V we still obtain

$$p(V,V^*) = P(V)^*P(V) \geq 0,$$

with the consequence that there exists a positive operator valued measure E on $\mathbb{T}$ satisfying

$$p(V,V^*) = \int_{\mathbb{T}} p(z,\overline{z})E(dz).$$

In other terms there exists a larger Hilbert space K, a normal operator U on K, leaving the closed subspace H invariant with the property $Ux = Vx,\ x \in H$. That is V is a *subnormal* operator.

Returning to the general setting of a bounded real algebraic curve C, a second result of [8] states that condition $M - |z|^2 \in \Sigma_h^2 + I_C$ for some real constant M implies that every linear bounded Hilbert space operator T subject to the defining equations of C:

$$f \in I_C \quad \Rightarrow \quad f(T,T^*) = 0$$

is subnormal.

Higher dimensional analogs of the above results and a geometric interpretation of the algebraic/operator theoretic conditions referred to above are contained in [8].

2. Main result

We prove below that every positive polynomial on a generalized lemniscate is a sum of hermitian squares of rational functions, with denominators in a singly generated multiplicative semigroup.

Theorem 2.1. *Let* Γ *be a generalized lemniscate of equation*

$$|Q_d(z)|^2 = \sum_{j=0}^{d-1} |Q_j(z)|^2$$

with $Q_j \in \mathbb{C}[z],\ 0 \le j \le d$. *For every conjugate invariant polynomial* $f(z, \overline{z})$ *satisfying* $f > 0$ *on* Γ *there exists a non-negative integer* N *with the property* $|Q_d(z)|^{2N} f(z, \overline{z}) \in \Sigma_h^2$ *for all* $z \in \Gamma$.

Following the terminology adopted in [8], we say that a compact real algebraic variety $X \subseteq \mathbb{C}$ possesses *Quillen's property* if:

(Q) *every polynomial* $f(z, \overline{z})$ *which is positive on* X *satisfies* $f \in \Sigma_h^2 + I_X$,

where $I_X \subseteq \mathbb{C}[z, \overline{z}]$ *denotes the reduced ideal associated to* X.

Initially Quillen proved that the unit sphere in $\mathbb{C}^n$ satisfies this condition.

Proof of Theorem 2.1. We add one more complex variable and will work in the quotient algebra $A = \mathbb{C}[z, w, \overline{z}, \overline{w}]/I$ where I is the conjugation invariant ideal generated by the two equations

$$|Q_d(z)|^2 - \sum_{j=0}^{d-1} |Q_j(z)|^2 = 0 \qquad wQ_d(z) - 1 = 0. \tag{2}$$

Since

$$1 = \sum_{j=0}^{d-1} |wQ_j(z)|^2 \pmod{I},$$

we infer that all monomials $w, wz, \ldots, wz^{d-1}$ are "bounded", that is there is a positive real constant M satisfying

$$M - |wz^j|^2 \in I + \Sigma_h^2, \quad 0 \le j \le d-1.$$

On the other hand, the identity

$$|z|^2 = \sum_{j=0}^{d-1} |wzQ_j(z)|^2 \pmod{I},$$

proves that $M - |z|^2 \in I + \Sigma_h^2$ for a sufficiently large M. We used here the observation that a sum of two "bounded" elements is bounded, as a consequence of the parallelogram identity, see for instance [8]. In other terms the convex cone (and semiring) $I + \Sigma_h^2$ is archimedean in $\mathbb{C}[z, w, \overline{z}, \overline{w}]$.

Let $f(z, \overline{z})$ be a positive polynomial on Γ. As a polynomial in the ensemble of all variables, f remains positive on the real curve Δ in $\mathbb{C}^2$ given by equations

(2). Note that Δ is compact, as $|Q_d(z)| \geq |Q_0(z)| > 0, \ \ z \in \Gamma$. According to the sum of hermitian squares decomposition theorem in [8], we find finitely many polynomials $S_k \in \mathbb{C}[z,w], \ 1 \leq k \leq m$, with the property

$$f(z,\overline{z}) = \sum_{k=1}^{m} |S_k(z,w)|^2 \quad (\mathrm{mod}\ I).$$

In view of the identity $wQ_d(z) = 1$, by multiplying the above decomposition with a sufficiently high power of $|Q_d(z)|^2$ one obtains the conclusion in the statement. □

We will show in the next section that denominators are necessary, that is, in general, not every positive polynomial on Γ is a sum of hermitian squares. Among the necessary conditions implied by Quillen's property in one complex dimension is the subnormality of any linear bounded operator annihilated via hereditary functional calculus by the respective real ideal, see [8] and the example of isometries alluded above. In the context of generalized lemniscates, with lurking denominators around, a slight variation of this "quantized" Quillen condition holds.

Let $K(H) \subseteq L(H)$ denote the ideal of compact operators, and let $\pi : L(H) \longrightarrow L(H)/K(H)$ be the canonical quotient map onto Calkin algebra $L(H)/K(H)$. A linear operator $T \in L(H)$ is called *essentially subnormal* if $\pi(T)$ is subnormal, that is the hereditary functional calculus

$$f \mapsto \pi f(T,T^*), \quad f \in \mathbb{C}[z,\overline{z}],$$

is positive on a compact set $F \subseteq \mathbb{C}$:

$$f|_F > 0 \quad \Rightarrow \pi f(T,T^*) \geq 0,$$

where the latter non-negativity is in the C^*-algebra $L(H)/K(H)$. A rather recent theorem due to N. Feldman asserts that $T \in L(H)$ is essentially subnormal if and only if it is the restriction to a closed invariant subspace of a normal plus compact operator [1].

Corollary 2.2. *Let $T \in L(H)$ be a finitely cyclic linear bounded operator on a Hilbert space satisfying $Q_d(T)^*Q_d(T) = \sum_{j=0}^{d-1} Q_j(T)^*Q_j(T)$ where $Q_j \in \mathbb{C}[z], \ 0 \leq j \leq d-1$. Then T is essentially subnormal and its normal plus compact extension has essential spectrum on the generalized lemniscate of equation $|Q_d(z)|^2 = \sum_{j=0}^{d-1} |Q_j(z)|^2$.*

Proof. Assume $T \in L(H)$ satisfies the algebraic equation in the statement. Then $Q_d(T)^*Q_d(T) \geq \delta I$, where $\delta = |Q_0(z)|^2 > 0$ is constant. That is, the linear operator $Q_d(T)$ is invertible to the left. Let S denote its left inverse $SQ_d(T) = I$. In particular the range $Q_d(T)H$ is closed. We show next that $Q_d(T)H$ is a finite codimension subspace of H.

Let $\xi_1, \dots, \xi_m \in H$ be a finite system of cyclic vectors for T, that is the linear span of the vectors $T^k\xi_j, \ k \geq 0, 1 \leq j \leq m$ is dense in H. Denote the finite dimensional span $H_1 = \bigvee_{k<d,1\leq j\leq m} T^k\xi_j$ and remark that $H_1 + Q_d(T)H$ is dense in H. But $H_1 + Q_d(T)H$ is a closed subspace, that is it coincides with H. Consequently $\mathrm{codim} Q_d(T)H < \infty$ and $Q_d(T)S$ is a bounded linear projection with $\mathrm{rank}(I - Q_d(T)S) < \infty$.

Assume that the polynomial $f(z,\overline{z})$ satisfies $f > 0$ on the lemniscate Γ, keeping the notation of the previous proof. According to Theorem 2.1, a decomposition of the form

$$|Q_d(z)|^{2N} f(z,\overline{z}) = \sum_{j=1}^{n} |R_j(z)|^2 + (|Q_d(z)|^2 - \sum_{j=0}^{d-1} |Q_j(z)|^2) g(z,\overline{z})$$

holds, where $R_j \in \mathbb{C}[z]$, $1 \leq j \leq m$, and $g \in \mathbb{C}[z,\overline{z}]$. Passing to the hereditary calculus for T we obtain

$$Q_d(T)^* f(T,T^*) Q_d(T) = \sum_{j=1}^{n} R_j(T)^* R_j(T) \geq 0,$$

and consequently

$$S^* Q_d(T)^* f(T,T^*) Q_d(T) S = \sum_{j=1}^{n} S^* R_j(T)^* R_j(T) S \geq 0.$$

But $\pi Q_d(T)S$ equals the identity I in the Calkin algebra, therefore

$$f(\pi(T), \pi(T)^*) = \pi f(T,T^*) \geq 0,$$

that is the operator T is essentially subnormal. According to the main theorem in [1], there exists a normal plus compact extension of T with essential spectrum on the curve Γ. □

Let $N + K$ denote the normal plus compact extension of the operator T appearing in the previous statement. The essential spectrum of N coincides with that of $N + K$ and lies on the generalized lemniscate Γ. That means that the spectra of both N and $N + K$ consist of Γ union with a part of the bounded connected components of $\mathbb{C} \setminus \Gamma$, union with some discrete set in $\mathbb{C} \setminus \Gamma$, of finite spectral multiplicity.

On the other hand, let λ denote an element in the approximate point spectrum of T, that is $\lim(T - \lambda)f_n = 0$ for a sequence of unit vectors $f_n \in H$. For every polynomial $g(z)$ we infer $\lim(g(T) - g(\lambda))f_n = 0$, whence $\lim \|g(T)f_n\|^2 = |g(\lambda)|^2$. Thus the quantized equation of the generalized lemniscate $Q_d(T)^*Q_d(T) = \sum_{j=0}^{d-1} Q_j(T)^*Q_j(T)$ implies $\lambda \in \Gamma$. In particular one finds that the topological boundary of the spectrum of T is contained in the real algebraic curve Γ. Then the spectrum of T consists of Γ union a part of the bounded connected components of $\mathbb{C} \setminus \Gamma$.

3. Examples

3.1. Two circles

We consider two symmetric circles of equations

$$|z \pm 1|^2 - 1 = 0.$$

The generalized lemniscate Γ of equation

$$q(z, \overline{z}) = (|z-1|^2 - 1)(|z+1|^2 - 1) = |z^2 - 1|^2 - 2|z|^2 - 1$$

passes through the point $z = 0$ and is invariant under the involution $z \mapsto -z$.

We prove that not every positive polynomial on Γ is a sum of hermitian squares along the curve Γ. That would imply, according to [8], that every linear bounded operator T, satisfying equation $q(T, T^*) = 0$ is subnormal.

Let $T = S + 1 - bp$, where b is a real constant, $S \in L(\ell^2(\mathbb{N}))$ is the unilateral shift and $p = e_0\langle \cdot, e_0 \rangle$ is the orthogonal projection on the first coordinate, so that

$$Se_0 = e_1, \quad S^*e_0 = 0, \quad pS = 0.$$

Above e_1 denotes the second coordinate vector. We compute

$$(T^* - 1)(T - 1) - 1 = (S^* - bp)(S - bp) - 1 = b^2 p.$$

Then

$$(T^* + 1)[(T^* - 1)(T - 1) - 1](T + 1) = (S^* + 2 - bp)b^2 p(S + 2 - bp) = b^2(2 - b)^2 p,$$

therefore

$$q(T, T^*) = b^2[(2 - b)^2 - 1]p,$$

so, choosing $b = 3$ we obtain $q(T, T^*) = 0$.

On the other hand, the operator T is not subnormal, because the commutator

$$[T^*, T] = [S^* - bp, S - bp] = I + b^2 p - SS^* + bpS^* + bSp - b^2 p = p + bpS^* + bSp$$

is not positive semi-definite. Indeed,

$$[T^*, T]e_0 = e_0 + be_1, \quad [T^*, T]e_1 = be_0,$$

hence

$$\langle [T^*, T]e_1, e_1 \rangle = 0$$

and this would contradict

$$\langle [T^*, T]e_1, e_0 \rangle = b.$$

3.2. Simply connected quadrature domains

Every simply connected and connected quadrature domain is the conformal image of the unit disk by a rational map $h : \mathbb{D} \longrightarrow \Omega$. Regarded as a map of the Riemann sphere to itself, h is a branched cover with d points (counting multiplicity) in each fibre. The quadrature nodes of Ω are then

$$a_i = h(\frac{1}{\overline{\lambda_i}}), \quad 1 \leq i \leq d,$$

where $\lambda_i \in \mathbb{C} \cup \infty, \ 1 \leq i \leq d$, are the poles of h, repeated in the labeling in case of higher multiplicities.

Our main result states that every positive polynomial on $\partial\Omega = h(\partial\mathbb{D})$ is equal to a sum of moduli squares of rational functions with poles at $a_1, \dots, a_d$, modulo the ideal generated by the irreducible defining equation of $\partial\Omega$. For the quoted facts concerning quadrature domains see the survey [3].

For example, consider the conformal mapping of the unit disk $z = w^2 + bw$, where $b \geq 2$. The image domain is a limaçon defined by equation

$$|z|^4 - (2 + b^2)|z|^2 - b^2 z - b^2 \overline{z} + 1 - b^2 = 0.$$

The only quadrature node is at $z = 0$ and has double multiplicity.

Similarly, a quadrature domain of order two, with two distinct nodes is a lemniscate of equation

$$h(z, \overline{z}) = (|z - a|^2 - b^2)(|z + a|^2 - b^2) - c^2 = 0,$$

where $a, b, c \geq 0$ are real parameters satisfying $b \geq a > 0, c^2 = (a^2 - b^2)^2$, so that:

$$h(z, \overline{z}) = |z|^4 - 2b|z|^2 - a^2(z^2 + \overline{z}^2).$$

Note that, in spite the fact h is an irreducible polynomial, the real zero locus is disconnected, containing a continuum and the special point $z = 0$. See for details [2, 3].

3.3. Ellipse

The note [7] contains a detailed account of Quillen's phenomenon on planar ellipses. We showed there that on an ellipse of non-zero eccentricity, there are positive polynomials which cannot be written as a sum of hermitian squares, although every linear bounded operator satisfying the equation of the ellipse is subnormal. We prove below that every positive polynomial on an ellipse is a sum of hermitian squares of rational functions.

Let $\alpha \in (0, 1/2)$ be a parameter and let $E \subseteq \mathbb{C}$ denote the ellipse of equation

$$|z|^2 + \alpha z^2 + \alpha \overline{z}^2 - 1 = 0.$$

In real coordinates $z = x + iy$:

$$(1 + 2\alpha)x^2 + (1 - 2\alpha)y^2 = 1.$$

We repeat the proof of the main result above combined with an observation from [7]. To be more precise, choose $M > 1$ a sufficiently large constant, so that the equation of the ellipse can be arranged as

$$|M - \frac{\alpha}{M} z^2|^2 = M^2 - 1 + |z|^2 + \frac{\alpha^2}{M^2}|z^2|^2.$$

Although this is not a generalized lemniscate, the proof of Theorem 2.1 applies verbatim, with the conclusion that every positive polynomial on the ellipse E can be written, along E, as a sum of moduli square of rational functions with denominators powers of $z^2 - \frac{M^2}{\alpha}$.

Notice that the inversion of the ellipse in a circle is a quadrature domain. Specifically, choose $R > 0$ large enough so that the ellipse is contained in the open disk $D(0, R)$. Denote $\zeta = \frac{R^2}{\overline{z}}$, so that the equation of the ellipse becomes

$$\frac{R^4}{|\zeta|^2} + \frac{\alpha R^4}{\overline{\zeta}^2} - \frac{\alpha R^4}{\zeta^2} - 1 = 0,$$

or

$$|\zeta|^4 - R^4|\zeta|^2 - \alpha R^4(\zeta^2 + \overline{\zeta}^2) = 0,$$

which is exactly the equation of a quadrature domain of order two, with two distinct nodes at $w = \pm\sqrt{\alpha}R^2$ and boundary Γ. The sum of hermitian squares structure of positive polynomials $f(\zeta, \overline{\zeta})$ on $\Gamma \cup \{0\}$ inferred from Theorem 2.1 does not immediately imply a similar conclusion via inversion, due to the special inner point condition $f(0, 0) > 0$ which becomes too restrictive (at infinity) for the polynomial $|z|^{2N} f(\frac{R^2}{\overline{z}}, \frac{R^2}{z})$.

References

[1] N. Feldman, *Essentially subnormal operators*, Proc. Amer. Math. Soc. **127** (1999), no. 4, 1171–1181.

[2] B. Gustafsson and M. Putinar, *Linear analysis of quadrature domains. II*, Israel J. Math. **119** (2000), 187–216.

[3] B. Gustafsson and H. S. Shapiro, *What is a quadrature domain?*, Quadrature domains and their applications, 1–25, Oper. Theory Adv. Appl. **156**, Birkhäuser, Basel, 2005.

[4] D. Plaumann, *Sums of squares on reducible real curves*, Math. Z. **265** (2010), no. 4, 777–797.

[5] A. Prestel and Ch. N. Delzell, *Positive Polynomials*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2001.

[6] M. Putinar, *Notes on generalized lemniscates*, Operator theory, systems theory and scattering theory: multidimensional generalizations, pp. 243–266, Oper. Theory Adv. Appl. **157**, Birkhäuser, Basel, 2005.

[7] M. Putinar and C. Scheiderer, *Hermitian algebra on the ellipse*, Illinois J. Math. **56** (2012), no. 1, 213–220.

[8] ———, *Quillen property of real algebraic varieties*, Muenster J. Math., to appear.

[9] D. G. Quillen, *On the representation of hermitian forms as sums of squares*, Invent. Math. **5** (1968), 237–242.

[10] F. Riesz and B. Szőkefalvi-Nagy, *Functional Analysis*, Dover Publ., New York, 1990.

[11] C. Scheiderer, *Sums of squares on real algebraic curves*, Math. Z. **245** (2003), no. 4, 725–760.

[12] ———, *Positivity and sums of squares*: *A guide to recent results*, In: Emerging Applications of Algebraic Geometry, pp. 271–324. IMA Vol. Math. Appl. **149**, Springer, New York, 2009.

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