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L^p-SOLUTIONS FOR REFLECTED BSDES WITH TIME DELAYED GENERATORS

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Abstract. In this paper, we establish the existence and uniqueness of the solution for a class of reflected backward stochastic differential equations with time delayed generator (RBSDEs with time delayed generator, in short) in the case when the terminal value and the obstacle process are L^p-integrable with $p \in]1,2[$ for a sufficiently small Lipschitz constant of the generator and the time horizon T.

1. Introduction

The theory of nonlinear backward stochastic differential equations (BSDEs, in short) was first introduced by Pardoux and Peng [20]. A solution to this equation, associated with the so-called generator f and the terminal value ξ, is a couple of adapted process (Y, Z) satisfying that

$$(1.1) \qquad Y(t) = \xi + \int_t^T f(s, Y(s), Z(s))\,\mathrm{d}s - \int_t^T Z(s)\,\mathrm{d}W(s), \quad 0 \le t \le T.$$

Nowadays, BSDEs have attracted researchers' great interest for their practical applications in mathematical finance ([10], [11]), stochastic control and stochastic games (see e.g. Hamadène [13]; Hamadène and Lepeltier ([14], [15])), providing the probabilistic representation for the solution of a large class of systems of semi-linear parabolic partial differential equations (see e.g. Pardoux [18], [19]; Pardoux and Peng [21]; Peng [22]).

Moreover, El Karoui et al. [9] introduced a special class of BSDEs which are called as reflected BSDEs (RBSDEs, in short), where the solution is forced to stay above a lower barrier. In details, a solution of such equations is a triple of

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processes (Y, Z, K) satisfying that
(1.2)

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s))\,\mathrm{d}s + K(T) - K(t) - \int_t^T Z(s)\,\mathrm{d}W(s),\ Y(t) \geq S(t),$$

where S, so-called the barrier or obstacle, is a given stochastic process. The role of the continuous non-decreasing process K is to push the state process upward with the minimal energy, in order to keep it above S; in this sense, it satisfies that $\int_0^T (Y(t) - S(t))\,\mathrm{d}K(t) = 0$. RBSDEs have been proven to be the powerful tools in mathematical finance (see e.g. [2], [11]), the mixed game problems (see e.g. [3], [16]), providing a probabilistic formula for the viscosity solution of an obstacle problem for a class of parabolic PDEs (see e.g. [4], [9], [17], [24]) and so on.

When considering the historic effect, Delong and Imkeller [7] introduced the following BSDE, called as BSDE with time delayed generator,

$$Y(t) = \xi + \int_t^T f(s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z(s)\,\mathrm{d}W(s), \quad 0 \leq t \leq T,$$

where the generator f at time s depends arbitrarily on the past values of a solution $(Y_s, Z_s) = (Y(s+u), Z(s+u))_{-T \leq u \leq 0}$. In Delong and Imkeller [7], the authors established the existence and uniqueness result of a solution for BSDEs with time delayed generators. For the applications of BSDEs with time delayed generators in insurance and finance, one can see [6]. Furthermore, in Delong and Imkeller [8], they proved the existence and uniqueness for BSDEs with time delayed generators driven by Brownian motions and Poisson random measures. Moreover, Reis et al. [23] extended the results of Delong and Imkeller ([7], [8]) in L^p-spaces. Very recently, Zhou and Ren [25] proved the existence and uniqueness of the solution for RBSDEs with time delayed generators.

With the appearance of BSDEs, many efforts have been done to derive the existence or uniqueness results under weaker assumptions than the ones of Pardoux and Peng [20] or El Karoui et al. [9]. Generally, many works focus on the weakness of the Lipschitz property of the generator. Very recently, some works have been done to derive the results on the existence and uniqueness of a solution for the standard BSDE (1.1) or RBSDE (1.2) in the case when the data belongs only to L^p for some $p \in]1, 2[$. For more details, we refer the readers to Chen [1], Fan and Jiang [12], Hamadène and Popier [17] and the references therein.

To our best knowledge, there is no result on the L^p-solutions to RBSDEs with time delayed generators. To close the gap, in this paper, we aim to derive the existence and uniqueness of the solution for RBSDEs with time delayed generator in the case when the terminal value and the obstacle process are L^p-integrable with $p \in]1, 2[$ for a sufficiently small Lipschitz constant of the generator and the time horizon T.

The paper is organized as follows. In Section 2, we propose some preliminaries and notations. The main results are given in Section 3.

2. Preliminaries and notations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a standard Brownian motion W. For a fixed real number $T > 0$, we consider the filtration $\mathbb{F} := (\mathcal{F}_t)_{0\leq t\leq T}$ which is generated by W and augmented by all $\mathbb{P}$-null sets. The filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfies the usual conditions.

We shall work with the following spaces. In the sequel, let $p \in]1, 2[$.

- Let $L^p(\mathbb{R})$ denote the space of real valued $\mathcal{F}_T$-measurable random variable $\xi : \Omega \to \mathbb{R}$ satisfying that
$$(\mathbb{E}[|\xi|^p])^{1/p} < \infty.$$
- For $\beta \geq 0$, $\mathbb{H}^p_\beta(\mathbb{R})$ denotes the space of all $\mathbb{F}$-predictable processes $Z : \Omega \times [0, T] \to \mathbb{R}$ such that
$$\|Z\|_{\mathbb{H}^p_\beta} = \left[\mathbb{E}\left(\int_0^T e^{\beta t}|Z(t)|^2 \mathrm{d}t\right)^{p/2}\right]^{1/p} < \infty.$$
- For $\beta \geq 0$, $\mathbb{S}^p_\beta(\mathbb{R})$ denotes the space of $\mathbb{F}$-adapted, product measurable processes $Y : \Omega \times [0, T] \to \mathbb{R}$ such that
$$\|Y\|_{\mathbb{S}^p_\beta} = \left[\mathbb{E}\left(\sup_{t\in[0,T]} e^{\beta t}|Y(t)|^p\right)\right]^{1/p} < \infty.$$
- Let $L^2_{-T}(\mathbb{R})$ denote the space of measurable functions $z : [-T, 0] \to \mathbb{R}$ satisfying that
$$\int_{-T}^0 |z(t)|^2 \mathrm{d}t < \infty.$$
- Let $\mathbb{S}^\infty_{-T}(\mathbb{R})$ denote the space of bounded, measurable functions $y : [-T, 0] \to \mathbb{R}$ satisfying that
$$\sup_{t\in[-T,0]} |y(t)|^2 < \infty.$$

As usual, by λ we denote Lebesgue measure on $([-T, 0], \mathcal{B}([-T, 0]))$, where $\mathcal{B}([-T, 0])$ stands for the Borel sets of $[-T, 0]$. In the sequel, let us simply write $\mathbb{S}^p_\beta(\mathbb{R}) \times \mathbb{H}^p_\beta(\mathbb{R})$ for $\mathbb{S}^p_\beta \times \mathbb{H}^p_\beta$.

Now, we give three objects. The first one is the terminal value ξ,

(i) $\xi \in L^p(\mathbb{R})$.

The second one is the "coefficient" f,
$$f : \Omega \times [0, T] \times \mathbb{S}^\infty_{-T}(\mathbb{R}) \times L^2_{-T}(\mathbb{R}) \to \mathbb{R},$$
which is a product measurable, $\mathbb{F}$-adapted map such that

(ii) $\mathbb{E}\left[\left(\int_0^T |f(t,0,0)|^2 \mathrm{d}t\right)^{p/2}\right] < \infty$,

(iii) for a probability measure α on $([-T,0],\mathcal{B}([-T,0]))$ and a positive constant L

$$|f(t,y_t,z_t) - f(t,\tilde{y}_t,\tilde{z}_t)|^2 \le L\Big(\int_{-T}^0 |y(t+u) - \tilde{y}(t+u)|^2 \alpha(\mathrm{d}u) + \int_{-T}^0 |z(t+u) - \tilde{z}(t+u)|^2 \alpha(\mathrm{d}u)\Big)$$

holds for $\mathbb{P}\times\lambda$-a.e. $(\omega,t) \in \Omega\times[0,T]$ and for any (y_t,z_t), $(\tilde{y}_t,\tilde{z}_t) \in \mathbb{S}^{\infty}_{-T}(\mathbb{R}) \times L^2_{-T}(\mathbb{R})$,

(iv) $f(t,\cdot,\cdot) = 0$ for $t < 0$.
The third one is the "obstacle" $\{S(t), 0 \le t \le T\}$, which is a continuous progressively measurable, $\mathbb{R}$-valued process satisfying that

(v) $S(T) \le \xi$ a.s., and $S^+ := S \vee 0 \in \mathbb{S}^p_\beta$.

Now, we propose the definition of the L^p-solution of the RBSDE with time delayed generator associated with the triple (f,ξ,S) which we consider throughout the paper.

Definition 2.1 (L^p-solution)**.** The L^p-solution of RBSDE with time delayed generator is a triple $\{(Y(t),Z(t),K(t)), 0 \le t \le T\}$ of $\mathcal{F}_t$-progressively measurable process taking values in $\mathbb{R}$, $\mathbb{R}$ and $\mathbb{R}_+$, respectively, and satisfying that

(1) $Z \in \mathbb{H}^p_\beta$, $Y \in \mathbb{S}^p_\beta$ and $K(T) \in L^p(\mathbb{R})$;
(2) $Y(t) = \xi + \int_t^T f(s,Y_s,Z_s)\mathrm{d}s + K(T) - K(t) - \int_t^T Z(s)\mathrm{d}W(s)$, $0 \le t \le T$;
(3) $Y(t) \ge S(t)$, $0 \le t \le T$;
(4) $K(t)$ is adapted, continuous and non-decreasing, $K_0 = 0$ and $\int_0^T \mathrm{e}^{\beta t}(Y(t) - S(t))\mathrm{d}K(t) = 0$ for some $\beta > 0$.

We remark that the generator f at time $s \in [0,T]$ depends on the past values of the solution denoted by $Y_s := (Y(s+u))_{-T\le u\le 0}$ and $Z_s := (Z(s+u))_{-T\le u\le 0}$. $f(t,0,0)$ in (ii) should be understood as a value of the generator $f(t,y_t,z_t)$ at $y(t+u) = z(t+u) = 0, -T \le u \le 0$.

Note that from (2) and (4), it follows that $\{Y(t), 0 \le t \le T\}$ is continuous. Intuitively, $\mathrm{d}K(t)/\mathrm{d}t$ represents the amount of "push upwards" that we add to $-(\mathrm{d}Y(t)/\mathrm{d}t)$, so that the constraint (3) is satisfied. Condition (4) says that the push is minimal, in the sense that we push only when the constraint is saturated, that is, when $Y(t) = S(t)$.

3. Main results

In this section, we give the priori estimates on the solution and the difference of two solutions for RBSDEs with time delayed generators, which play an important role in proving existence and uniqueness of the solution, which is the main result of this section and this article.

3.1. Priori estimates

We now give a priori estimate on the solution.

Proposition 3.1. *Let $\{(Y(t), Z(t), K(t)), 0 \le t \le T\}$ be a solution of the RBSDE with time delayed generator* (1)–(4)*. We assume moreover that*

$$\int_0^T \mathrm{e}^{\beta t}(Y(t) - S(t))^+ \mathrm{d}K(t) = 0.$$

If the Lipschitz constant L of the generator f and the time horizon T are small enough, then there exist two positive constants β and γ satisfying that

$$D_1 := \beta - \gamma > 0, \quad D_2 := 1 - \frac{2L\tilde{\alpha}}{\gamma} > 0,$$

and a positive constant $C = C(p, \beta, \gamma, \tilde{\alpha}, L, T)$ depending on $p, \beta, \gamma, \tilde{\alpha}$ (α as in (iii) *and $\tilde{\alpha} = \int_{-T}^0 \mathrm{e}^{-\beta u}\alpha(\mathrm{d}u)$), L and T such that*

$$\mathbb{E}\left(\sup_{0\le t\le T} \mathrm{e}^{\beta t}|Y(t)|^p + \Big(\int_0^T e^{\beta t}|Z(t)|^2 \mathrm{d}t\Big)^{p/2} + \mathrm{e}^{\beta T}|K(T)|^p\right)$$
$$\le C\mathbb{E}\left(\mathrm{e}^{\beta T}|\xi|^p + \Big(\int_0^T \mathrm{e}^{\beta t}|f(t,0,0)|^2 \mathrm{d}t\Big)^{p/2} + \sup_{0\le t\le T} \mathrm{e}^{\beta t}\big(S(t)^+\big)^p\right).$$

Proof. Step 1. We claim that

$$\mathbb{E}\Big(\int_0^T \mathrm{e}^{\beta s}|Z(s)|^2 \mathrm{d}s\Big)^{p/2} \le \tilde{C}\mathbb{E}\left[\Big(\int_0^T \mathrm{e}^{\beta s}|f(s,0,0)|^2 \mathrm{d}s\Big)^{p/2} + \Big(\sup_{0\le s\le T} \mathrm{e}^{\beta s}|Y(s)|^p\Big)\right], \tag{3.1}$$

where $\tilde{C} > 0$ depends on $p, \beta, \gamma, \tilde{\alpha}, L$ and T, being the linear function of $(LT)^{\frac{p}{2}}$. The estimate of (3.1) can be deduced as follows.

Let β be a positive real constant and for each integer k, we define

$$\tau_k = \inf\left\{t \in [0,T], \int_0^t \mathrm{e}^{\beta s}|Z(s)|^2 \mathrm{d}s \ge k\right\} \wedge T.$$

The sequence $(\tau_k)_{k\ge 0}$ is of stationary type since the process Z belongs to $\mathbb{H}^p_\beta(\mathbb{R})$ and then $\int_0^T \mathrm{e}^{\beta s}|Z(s)|^2 \mathrm{d}s < \infty$, P-a.s.. The Itô formula yields that:

$$\begin{aligned} |Y(0)|^2 &+ \int_0^{\tau_k} \mathrm{e}^{\beta s}|Z(s)|^2 \mathrm{d}s \\ &= \mathrm{e}^{\beta \tau_k}|Y(\tau_k)|^2 + 2\int_0^{\tau_k} \mathrm{e}^{\beta s} Y(s) f(s, Y_s, Z_s) \mathrm{d}s \\ &\quad - \int_0^{\tau_k} \beta \mathrm{e}^{\beta s}|Y(s)|^2 \mathrm{d}s + 2\int_0^{\tau_k} \mathrm{e}^{\beta s} Y(s) \mathrm{d}K(s) \end{aligned} \tag{3.2}$$

$$-2\int_0^{\tau_k} \mathrm{e}^{\beta s} Y(s)Z(s)\mathrm{d}W(s).$$

By Young inequality for any $\gamma > 0$ and condition (iii), we have

(3.3)

$$\begin{aligned}
&|Y(0)|^2 + \int_0^{\tau_k} \mathrm{e}^{\beta s}|Z(s)|^2\mathrm{d}s \\
\le{}& \mathrm{e}^{\beta\tau_k}|Y(\tau_k)|^2 + (\gamma-\beta)\int_0^{\tau_k} \mathrm{e}^{\beta s}|Y(s)|^2\mathrm{d}s \\
&+ \frac{1}{\gamma}\int_0^{\tau_k} \mathrm{e}^{\beta s}|f(s,Y_s,Z_s)|^2\mathrm{d}s + 2\int_0^{\tau_k} \mathrm{e}^{\beta s}Y(s)\mathrm{d}K(s) \\
&- 2\int_0^{\tau_k} \mathrm{e}^{\beta s}Y(s)Z(s)\mathrm{d}W(s) \\
\le{}& \mathrm{e}^{\beta\tau_k}|Y(\tau_k)|^2 + \frac{2}{\gamma}\int_0^{\tau_k} \mathrm{e}^{\beta s}|f(s,Y_s,Z_s) - f(s,0,0)|^2\mathrm{d}s \\
&+ \frac{2}{\gamma}\int_0^{\tau_k} \mathrm{e}^{\beta s}|f(s,0,0)|^2\mathrm{d}s + (\gamma-\beta)\int_0^{\tau_k} \mathrm{e}^{\beta s}|Y(s)|^2\mathrm{d}s \\
&+ 2\int_0^{\tau_k} \mathrm{e}^{\beta s}Y(s)\mathrm{d}K(s) - 2\int_0^{\tau_k} \mathrm{e}^{\beta s}Y(s)Z(s)\mathrm{d}W(s) \\
\le{}& \mathrm{e}^{\beta\tau_k}|Y(\tau_k)|^2 + (\gamma-\beta)\int_0^{\tau_k} \mathrm{e}^{\beta s}|Y(s)|^2 + \frac{2}{\gamma}\int_0^{\tau_k} \mathrm{e}^{\beta s}|f(s,0,0)|^2\mathrm{d}s \\
&+ \frac{2L}{\gamma}\left[\int_0^{\tau_k}\Big(\int_{-T}^0 \mathrm{e}^{\beta s}|Y(s+u)|^2\alpha(\mathrm{d}u)\Big)\mathrm{d}s + \int_0^{\tau_k}\Big(\int_{-T}^0 \mathrm{e}^{\beta s}|Z(s+u)|^2\alpha(\mathrm{d}u)\Big)\mathrm{d}s\right] \\
&+ 2\int_0^{\tau_k} \mathrm{e}^{\beta s}Y(s)\mathrm{d}K(s) - 2\int_0^{\tau_k} \mathrm{e}^{\beta s}Y(s)Z(s)\mathrm{d}W(s).
\end{aligned}$$

By a change of integration order argument, for $\phi = Z$, we obtain

(3.4)

$$\begin{aligned}
&\int_0^{\tau_k}\left(\int_{-T}^0 \mathrm{e}^{\beta s}|\phi(s+u)|^2\alpha(\mathrm{d}u)\right)\mathrm{d}s \\
={}& \int_0^{\tau_k}\left(\int_{-T}^0 \mathrm{e}^{\beta(s+u)}\mathrm{e}^{-\beta u}I_{\{s+u\ge 0\}}|\phi(s+u)|^2\alpha(\mathrm{d}u)\right)\mathrm{d}s \\
={}& \int_{-T}^0\int_{u\vee 0}^{\tau_k+u} \mathrm{e}^{\beta r}\mathrm{e}^{-\beta u}I_{\{r\ge 0\}}|\phi(r)|^2\mathrm{d}r\alpha(\mathrm{d}u) \\
={}& \int_0^{\tau_k}\int_{r-\tau_k}^{r\wedge 0} \mathrm{e}^{\beta r}\mathrm{e}^{-\beta u}I_{\{r\ge 0\}}|\phi(r)|^2\alpha(\mathrm{d}u)\mathrm{d}r \\
\le{}& \int_0^{\tau_k} \mathrm{e}^{\beta r}|\phi(r)|^2\left(\int_{-\tau_k}^0 \mathrm{e}^{-\beta u}\alpha(\mathrm{d}u)\right)\mathrm{d}r \le \int_0^{\tau_k} \tilde{\alpha}\mathrm{e}^{\beta r}|\phi(r)|^2\mathrm{d}r,
\end{aligned}$$

with $\tilde{\alpha} = \int_{-T}^0 \mathrm{e}^{-\beta u}\alpha(\mathrm{d}u)$. Continuing the inequality (3.3) from above, we get

$$
\begin{aligned}
(3.5)\qquad & |Y(0)|^2 + \int_0^{\tau_k} \mathrm{e}^{\beta s}|Z(s)|^2\mathrm{d}s \\
&\leq \mathrm{e}^{\beta\tau_k}|Y(\tau_k)|^2 + (\gamma-\beta)\int_0^{\tau_k} \mathrm{e}^{\beta s}|Y(s)|^2\mathrm{d}s \\
&\quad + \frac{2L\tilde{\alpha}}{\gamma}\left(T\sup_{0\leq s\leq T}\mathrm{e}^{\beta s}|Y(s)|^2 + \int_0^{\tau_k}\mathrm{e}^{\beta s}|Z(s)|^2\mathrm{d}s\right) \\
&\quad + \frac{2}{\gamma}\int_0^{\tau_k}\mathrm{e}^{\beta s}|f(s,0,0)|^2\mathrm{d}s + 2\int_0^{\tau_k}\mathrm{e}^{\beta s}Y(s)\mathrm{d}K(s) \\
&\quad - 2\int_0^{\tau_k}\mathrm{e}^{\beta s}Y(s)Z(s)\mathrm{d}W(s).
\end{aligned}
$$

Reorganizing (3.5) we have

$$
\begin{aligned}
& |Y(0)|^2 + \left(1-\frac{2L\tilde{\alpha}}{\gamma}\right)\int_0^{\tau_k}\mathrm{e}^{\beta s}|Z(s)|^2\mathrm{d}s \\
&\leq \mathrm{e}^{\beta\tau_k}|Y(\tau_k)|^2 + (\gamma-\beta)\int_0^{\tau_k}\mathrm{e}^{\beta s}|Y(s)|^2\mathrm{d}s + \left(\frac{2L\tilde{\alpha}}{\gamma}T + \frac{1}{\varepsilon}\right)\sup_{0\leq s\leq T}\mathrm{e}^{\beta s}|Y(s)|^2 \\
&\quad + \frac{2}{\gamma}\int_0^{\tau_k}\mathrm{e}^{\beta s}|f(s,0,0)|^2\mathrm{d}s + \varepsilon\mathrm{e}^{\beta\tau_k}|K(\tau_k)|^2 - 2\int_0^{\tau_k}\mathrm{e}^{\beta s}Y(s)Z(s)\mathrm{d}W(s)
\end{aligned}
$$

for any $\varepsilon > 0$. From the equation

$$
K(\tau_k) = Y(0) - Y(\tau_k) - \int_0^{\tau_k} f(t,Y_t,Z_t)\mathrm{d}t + \int_0^{\tau_k} Z(t)\mathrm{d}W(t),
$$

there exists a constant $C_1 > 0$ (depending also on T) such that

$$
\begin{aligned}
\mathrm{e}^{\beta\tau_k}|K(\tau_k)|^2 \leq C_1\Bigg(&|Y(0)|^2 + |Y(\tau_k)|^2 + \int_0^{\tau_k}|f(s,0,0)|^2\mathrm{d}s + \sup_{0\leq s\leq T}|Y(s)|^2 \\
&+ \int_0^{\tau_k}|Z(s)|^2\mathrm{d}s + \left|\int_0^{\tau_k}Z(s)\mathrm{d}W(s)\right|^2\Bigg).
\end{aligned}
$$

Plugging the last inequality in the previous one to get

$$
\begin{aligned}
& (1-\varepsilon C_1)\,|Y(0)|^2 + \left(1-\frac{2L\tilde{\alpha}}{\gamma}\right)\int_0^{\tau_k}\mathrm{e}^{\beta s}|Z(s)|^2\mathrm{d}s - \varepsilon C_1\int_0^{\tau_k}|Z(s)|^2\mathrm{d}s \\
&\leq (\varepsilon C_1 + \mathrm{e}^{\beta\tau_k})|Y(\tau_k)|^2 + (\gamma-\beta)\int_0^{\tau_k}\mathrm{e}^{\beta s}|Y(s)|^2\mathrm{d}s \\
&\quad + \left(\frac{2L\tilde{\alpha}}{\gamma}T + \frac{1}{\varepsilon}\right)\sup_{0\leq s\leq T}\mathrm{e}^{\beta s}|Y(s)|^2 + \varepsilon C_1\sup_{0\leq s\leq T}|Y(s)|^2 \\
&\quad + \varepsilon C_1\left|\int_0^{\tau_k}Z(s)\mathrm{d}W(s)\right|^2 + \left(\frac{2}{\gamma}+\varepsilon C_1\right)\int_0^{\tau_k}\mathrm{e}^{\beta s}|f(s,0,0)|^2\mathrm{d}s
\end{aligned}
$$

$$+ 2\left|\int_0^{\tau_k} \mathrm{e}^{\beta s} Y(s)Z(s)\mathrm{d}W(s)\right|.$$

Choosing now ε small enough and γ such that $D_1 := \beta - \gamma > 0$, $D_2 := 1 - \frac{2L\tilde{\alpha}}{\gamma} > 0$, we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\int_0^{\tau_k} \mathrm{e}^{\beta s}|Z(s)|^2\mathrm{d}s\Big)^{p/2} \\
&\leq C_2\mathbb{E}\left[\Big(\int_0^{\tau_k} \mathrm{e}^{\beta s}|f(s,0,0)|^2\mathrm{d}s\Big)^{p/2} + \Big(\sup_{0\leq s\leq T} \mathrm{e}^{\beta s}|Y(s)|^p\Big)\right] \\
&\quad + C_2\mathbb{E}\left|\int_0^{\tau_k} \mathrm{e}^{\beta s}Y(s)Z(s)\mathrm{d}W(s)\right|^{p/2},
\end{aligned}$$

where C_2 depends on $p, \beta, \gamma, \tilde{\alpha}, L$ and T. If L is small enough, then C_2 can be chosen as a linear function of $(LT)^{\frac{p}{2}}$. Next thanks to the Burkholder-Davis-Gundy inequality (see e.g. Dellacherie and Meyer [5]), we have

$$\begin{aligned}
&\mathbb{E}\left|\int_0^{\tau_k} \mathrm{e}^{\beta s}Y(s)Z(s)\mathrm{d}W(s)\right|^{p/2} \\
&\leq C_3\mathbb{E}\left(\int_0^{\tau_k} \mathrm{e}^{\beta s}|Y(s)|^2|Z(s)|^2\mathrm{d}s\right)^{p/4} \\
&\leq C_3\mathbb{E}\left(\sup_{0\leq s\leq T} \mathrm{e}^{\beta s}|Y(s)|^2\right)^{p/4}\left(\int_0^{\tau_k} \mathrm{e}^{\beta s}|Z(s)|^2\mathrm{d}s\right)^{p/4} \\
&\leq \frac{C_3^2}{2\eta}\mathbb{E}\left(\sup_{0\leq s\leq T} \mathrm{e}^{\beta s}|Y(s)|^p\right) + \frac{\eta}{2}\mathbb{E}\Big(\int_0^{\tau_k} \mathrm{e}^{\beta s}|Z(s)|^2\mathrm{d}s\Big)^{p/2},
\end{aligned}$$

where η is an arbitrary positive real number and $C_3 > 0$ depends only on p. Finally substituting the last inequality in the previous one, choosing η small enough and using the Fatou lemma, we obtain that there exists a real constant $\tilde{C}$ depending on $p, \beta, \gamma, \tilde{\alpha}, L$ and T, being the linear function of $(LT)^{\frac{p}{2}}$ and such that

$$\begin{aligned}
&\mathbb{E}\Big(\int_0^{T} \mathrm{e}^{\beta s}|Z(s)|^2\mathrm{d}s\Big)^{p/2} \\
&\leq \tilde{C}\mathbb{E}\left[\Big(\int_0^{T} \mathrm{e}^{\beta s}|f(s,0,0)|^2\mathrm{d}s\Big)^{p/2} + \Big(\sup_{0\leq s\leq T} \mathrm{e}^{\beta s}|Y(s)|^p\Big)\right].
\end{aligned}$$

Step 2. We claim that

$$\begin{aligned}
(3.6)\quad &\mathbb{E}\left(\sup_{0\leq t\leq T} \mathrm{e}^{\beta t}|Y(t)|^p\right) \\
&\leq \hat{C}\mathbb{E}\left[\left(\mathrm{e}^{\beta T}|\xi|^p\right) + \Big(\int_0^{T} \mathrm{e}^{\beta s}|f(s,0,0)|^2\mathrm{d}s\Big)^{p/2} + \Big(\sup_{0\leq t\leq T} \mathrm{e}^{\beta t}\left(S(t)^+\right)^p\Big)\right]
\end{aligned}$$

holds for a positive constant $\hat{C}$ depending on $p, \beta, \gamma, \tilde{\alpha}, L$ and T. To prove (3.6), from ([17], Corollary 1), for any $\beta > 0$ and any $0 \le t \le T$ we have

$$\begin{aligned}
&\mathrm{e}^{\beta t}|Y(t)|^p + c(p)\int_t^T \mathrm{e}^{\beta s}|Y(s)|^{p-2}1_{Y(s)\neq 0}|Z(s)|^2\mathrm{d}s \\
&\le \mathrm{e}^{\beta T}|\xi|^p - \beta\int_t^T \mathrm{e}^{\beta s}|Y(s)|^p\mathrm{d}s + p\int_t^T \mathrm{e}^{\beta s}|Y(s)|^{p-1}\mathrm{sgn}(Y(s))f(s,Y_s,Z_s)\mathrm{d}s \\
&\quad + p\int_t^T \mathrm{e}^{\beta s}|Y(s)|^{p-1}\mathrm{sgn}(Y(s))\mathrm{d}K(s) \\
&\quad - p\int_t^T \mathrm{e}^{\beta s}|Y(s)|^{p-1}\mathrm{sgn}(Y(s))Z(s)\mathrm{d}W(s) \\
&\le \mathrm{e}^{\beta T}|\xi|^p - \beta\int_t^T \mathrm{e}^{\beta s}|Y(s)|^p\mathrm{d}s + p\int_t^T \mathrm{e}^{\beta s}|Y(s)|^{p-1}|f(s,Y_s,Z_s)-f(s,0,0)|\mathrm{d}s \\
&\quad + p\int_t^T \mathrm{e}^{\beta s}|Y(s)|^{p-1}|f(s,0,0)|\mathrm{d}s + p\int_t^T \mathrm{e}^{\beta s}|Y(s)|^{p-1}\mathrm{sgn}(Y(s))\mathrm{d}K(s) \\
&\quad - p\int_t^T \mathrm{e}^{\beta s}|Y(s)|^{p-1}\mathrm{sgn}(Y(s))Z(s)\mathrm{d}W(s),
\end{aligned}$$

where $c(p) = \frac{p(p-1)}{2}$ and $\mathrm{sgn}(y) := \frac{y}{|y|}1_{y\neq 0}$. By the Young inequality it holds that

$$\begin{aligned}
&p\int_t^T \mathrm{e}^{\beta s}|Y(s)|^{p-1}|f(s,0,0)|\mathrm{d}s \\
&\le (p-1)\vartheta^{\frac{p}{p-1}}\left(\sup_{t\le s\le T}|Y(s)|^p\right) + \vartheta^{-p}\Big(\int_t^T \mathrm{e}^{\beta s}|f(s,0,0)|\mathrm{d}s\Big)^p \\
&\le (p-1)\vartheta^{\frac{p}{p-1}}\left(\sup_{0\le s\le T}\mathrm{e}^{\beta s}|Y(s)|^p\right) + \vartheta^{-p}(C(T))^{\frac{p}{2}}\Big(\int_0^T \mathrm{e}^{\beta s}|f(s,0,0)|^2\mathrm{d}s\Big)^{p/2}
\end{aligned}$$

and

$$\begin{aligned}
&p\int_t^T \mathrm{e}^{\beta s}|Y(s)|^{p-1}|f(s,Y_s,Z_s)-f(s,0,0)|\mathrm{d}s \\
&\le (p-1)\vartheta^{\frac{p}{p-1}}\left(\sup_{t\le s\le T}|Y(s)|^p\right) + \vartheta^{-p}\Big(\int_t^T \mathrm{e}^{\beta s}|f(s,Y_s,Z_s)-f(s,0,0)|\mathrm{d}s\Big)^p,
\end{aligned}$$

for any $\vartheta > 0$, where $C(T) = \frac{2e^{\beta T}}{\beta}$. Since f is Lipschitz, we have

$$\begin{aligned}
&p\int_t^T \mathrm{e}^{\beta s}|Y(s)|^{p-1}|f(s,Y_s,Z_s)-f(s,0,0)|\mathrm{d}s \\
&\le (p-1)\vartheta^{\frac{p}{p-1}}(\sup_{t\le s\le T}|Y(s)|^p)
\end{aligned}$$

$$
\begin{aligned}
&+\vartheta^{-p}\left(C(T)L\tilde{\alpha}\Big(T\sup_{0\le s\le T}\mathrm{e}^{\beta s}|Y(s)|^2+\int_0^T\mathrm{e}^{\beta s}|Z(s)|^2\mathrm{d}s\Big)\right)^{p/2}\\
\le\ &\Big((p-1)\vartheta^{\frac{p}{p-1}}+\vartheta^{-p}(C(T)L\tilde{\alpha}T)^{p/2}\hat{C}_p\Big)\big(\sup_{0\le s\le T}\mathrm{e}^{\beta s}|Y(s)|^p\big)\\
&+\vartheta^{-p}(C(T)L\tilde{\alpha})^{p/2}\hat{C}_p\left(\int_0^T\mathrm{e}^{\beta s}|Z(s)|^2\mathrm{d}s\right)^{p/2},
\end{aligned}
$$

where $\hat{C}_p>0$ is a constant depending on p. Thus, we have

$$
\begin{aligned}
(3.7)\quad &\mathrm{e}^{\beta t}|Y(t)|^p+c(p)\int_t^T\mathrm{e}^{\beta s}|Y(s)|^{p-2}1_{Y(s)\neq 0}|Z(s)|^2\mathrm{d}s\\
\le\ &\mathrm{e}^{\beta T}|\xi|^p+\Big(2(p-1)\vartheta^{\frac{p}{p-1}}+\vartheta^{-p}(C(T)L\tilde{\alpha}T)^{p/2}\hat{C}_p\Big)\left(\sup_{0\le s\le T}\mathrm{e}^{\beta s}|Y(s)|^p\right)\\
&+\vartheta^{-p}(C(T)L\tilde{\alpha})^{p/2}\hat{C}_p\left(\int_0^T\mathrm{e}^{\beta s}|Z(s)|^2\mathrm{d}s\right)^{p/2}\\
&+\vartheta^{-p}(C(T))^{\frac{p}{2}}\Big(\int_0^T\mathrm{e}^{\beta s}|f(s,0,0)|^2\mathrm{d}s\Big)^{p/2}\\
&+p\int_t^T\mathrm{e}^{\beta s}|Y(s)|^{p-1}\mathrm{sgn}(Y(s))\mathrm{d}K(s)\\
&-p\int_t^T\mathrm{e}^{\beta s}|Y(s)|^{p-1}\mathrm{sgn}(Y(s))Z(s)\mathrm{d}W(s).
\end{aligned}
$$

Next let us deal with $\int_t^T\mathrm{e}^{\beta s}|Y(s)|^{p-1}\mathrm{sgn}(Y(s))\mathrm{d}K(s)$. Indeed, the hypothesis related to increments of K and $Y-S$ implies that $dK(s)=1_{\{Y(s)\le S(s)\}}dK(s)$, for any $s\le T$. Recall that here we just assume $\int_0^T\mathrm{e}^{\beta t}(Y(t)-S(t))^+\mathrm{d}K(t)=0$. Therefore we have:

$$
\begin{aligned}
&\int_t^T\mathrm{e}^{\beta s}|Y(s)|^{p-1}\mathrm{sgn}(Y(s))\mathrm{d}K(s)\\
=\ &\int_t^T\mathrm{e}^{\beta s}|Y(s)|^{p-1}\mathrm{sgn}(Y(s))1_{\{Y(s)\le S(s)\}}\mathrm{d}K(s)\\
\le\ &\int_t^T\mathrm{e}^{\beta s}\theta(S(s))\mathrm{d}K(s),
\end{aligned}
$$

where $\theta:x\in\mathbb{R}\mapsto\theta(x)=|x|^{p-1}\frac{x}{|x|}1_{x\neq 0}$, which is actually a non-decreasing function. It follows that

$$
\begin{aligned}
&\int_t^T\mathrm{e}^{\beta s}|Y(s)|^{p-1}\mathrm{sgn}(Y(s))\mathrm{d}K(s)\\
\le\ &\int_t^T\mathrm{e}^{\beta s}|S(s)|^{p-1}\mathrm{sgn}(S(s))\mathrm{d}K(s)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_t^T \mathrm{e}^{\beta s}(S^+(s))^{p-1}\mathrm{d}K(s)\\
&\leq \left(\sup_{0\leq s\leq T}(S^+(s))\right)^{p-1}\int_t^T \mathrm{e}^{\beta s}\mathrm{d}K(s)\\
&\leq \frac{p-1}{p}\frac{1}{\delta^{\frac{p-1}{p}}}\left(\sup_{0\leq s\leq T}(S^+(s))\right)^{p}+\frac{1}{p}\delta^p\left(\int_t^T \mathrm{e}^{\beta s}\mathrm{d}K(s)\right)^p
\end{aligned}
$$

for any $\delta > 0$.

Next we focus on the control of the term $\int_t^T \mathrm{e}^{\beta s}\mathrm{d}K(s)$. Using the predictable dual projection property (see e.g. Dellacherie and Meyer [5]), for all $0 \leq t \leq T$, we have

$$
\begin{aligned}
\mathbb{E}[(K(T)-K(t))^p] &= \mathbb{E}\left[\int_t^T p(K(T)-K(s))^{p-1}\mathrm{d}K(s)\right]\\
&= p\mathbb{E}\int_t^T \mathbb{E}[(K(T)-K(s))^{p-1}|\mathcal{F}_s]\mathrm{d}K(s)\\
&\leq p\mathbb{E}\int_t^T \mathbb{E}[(K(T)-K(s))|\mathcal{F}_s]^{p-1}\mathrm{d}K(s), \quad \text{since } p\in]1,2[.
\end{aligned}
$$

The last inequality holds true thanks to Jensen's conditional one. Recall now that

$$
K(T)-K(t) = Y(t)-\xi-\int_t^T f(s,Y_s,Z_s)\mathrm{d}s+\int_t^T Z(s)\mathrm{d}W(s),
$$

then,

$$
\begin{aligned}
&\mathbb{E}[(K(T)-K(t))^p]\\
&\leq p\mathbb{E}\int_t^T \mathbb{E}\left[Y(s)-\xi-\int_s^T f(u,Y_u,Z_u)\mathrm{d}u \mid \mathcal{F}_s\right]^{p-1}\mathrm{d}K(s)\\
&\leq p\mathbb{E}\int_t^T \mathbb{E}\left[2\sup_{t\leq u\leq T}|Y(u)|+\int_s^T |f(u,Y_u,Z_u)|\mathrm{d}u \mid \mathcal{F}_s\right]^{p-1}\mathrm{d}K(s)\\
&\leq \frac{1}{2}\mathbb{E}[(K(T)-K(t))^p]\\
&\quad + C_p\mathbb{E}\sup_{t\leq s\leq T}\left[\mathbb{E}\left(2\sup_{t\leq u\leq T}|Y(u)|+\int_t^T |f(u,Y_u,Z_u)|\mathrm{d}u \mid \mathcal{F}_s\right)\right]^p.
\end{aligned}
$$

The last inequality can be obtained by the same method as ([17], page 10). Thus, using the Doob maximal inequality we obtain

$$
\frac{1}{2}\mathbb{E}[(K(T)-K(t))^p]
$$

$$\le C_p \sup_{t\le s\le T}\mathbb{E}\left[\mathbb{E}\left(2\sup_{t\le u\le T}|Y(u)|+\int_t^T|f(u,Y_u,Z_u)|\mathrm{d}u\mid\mathcal{F}_s\right)\right]^p$$

$$\le \tilde{C}_p\mathbb{E}\left[\sup_{t\le u\le T}|Y(u)|^p+\Big(\int_t^T|f(u,Y_u,Z_u)|^2\mathrm{d}u\Big)^{p/2}\right]$$

$$\le C_4\mathbb{E}\left[\sup_{t\le u\le T}|Y(u)|^p+\Big(\int_t^T|f(u,0,0)|^2\mathrm{d}u\Big)^{p/2}\right.$$

$$\left.+(2L)^{p/2}\left(T\sup_{0\le u\le T}|Y(u)|^2+\int_0^T|Z(u)|^2\mathrm{d}u\right)^{p/2}\right]$$

$$\le C_5\mathbb{E}\left[\sup_{0\le u\le T}|Y(u)|^p+\Big(\int_t^T|f(u,0,0)|^2\mathrm{d}u\Big)^{p/2}+\Big(\int_0^T|Z(u)|^2\mathrm{d}u\Big)^{p/2}\right],$$

where $C_5=C_4+2^{\frac{3}{2}p-1}L^{\frac{p}{2}}\max(T^{\frac{p}{2}},1)$ and $C_4>0$ depends only on p. In fact, we have used the inequality $(a+b)^{\frac{p}{2}}\le 2^{p-1}\left(a^{\frac{p}{2}}+b^{\frac{p}{2}}\right)$ for $a>0,\ b>0$.

Then by Step 1 we have

$$\begin{aligned}&e^{p\beta T}\mathbb{E}[(K(T)-K(t))^p]\\&\le \bar{C}\mathbb{E}\left[\Big(\sup_{0\le t\le T}e^{\beta t}|Y(t)|\Big)^p+\Big(\int_0^T e^{\beta t}|f(t,0,0)|^2\mathrm{d}t\Big)^{p/2}\right]\end{aligned}\tag{3.8}$$

holds for a positive constant $\bar{C}$ depending on $p,\beta,\gamma,\tilde{\alpha},L$ and T. If L and T are small enough, then $\bar{C}$ can be chosen as the linear function of $L^pT^{\frac{p}{2}}$ and $(LT)^{\frac{p}{2}}$. Now the local martingale

$$\left\{\int_0^t e^{\beta s}|Y(s)|^{p-1}\mathrm{sgn}(Y(s))Z(s)\mathrm{d}W(s)\right\}_{t\le T}$$

is actually a martingale, therefore taking expectation in (3.7) and taking into account of (3.8) and Step 1 to obtain:

(3.9)

$$\begin{aligned}&c(p)\mathbb{E}\int_t^T e^{\beta s}|Y(s)|^{p-2}1_{Y(s)\neq 0}|Z(s)|^2\mathrm{d}s\\
\le&\Big(2(p-1)\vartheta^{\frac{p}{p-1}}+(1+\tilde{C}T^{-\frac{p}{2}}))\vartheta^{-p}(C(T)L\tilde{\alpha}T)^{p/2}\hat{C}_p+\bar{C}\delta^p\Big)\\
&\times\mathbb{E}\left(\sup_{0\le t\le T}e^{\beta t}|Y(t)|^p\right)+\mathbb{E}e^{\beta T}|\xi|^p\\
&+\Big(\vartheta^{-p}(C(T)L\tilde{\alpha})^{p/2}\tilde{C}\hat{C}_p+\vartheta^{-p}(C(T))^{\frac{p}{2}}+\bar{C}\delta^p\Big)\mathbb{E}\Big(\int_0^T e^{\beta s}|f(s,0,0)|^2\mathrm{d}s\Big)^{p/2}\\
&+\frac{p-1}{\delta^{\frac{p-1}{p}}}\mathbb{E}\left(\sup_{0\le s\le T}(S^+(s))\right)^p.\end{aligned}$$

Next going back to (3.7) taking the supremum and the expectation we get after taking into account of (3.8) and Step 1

(3.10)

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le T} \mathrm{e}^{\beta t}|Y(t)|^p\right) + c(p)\mathbb{E}\int_t^T \mathrm{e}^{\beta s}|Y(s)|^{p-2}1_{Y(s)\neq 0}|Z(s)|^2\mathrm{d}s \\
&\le 2\Big(2(p-1)\vartheta^{\frac{p}{p-1}} + (1+\tilde{C}T^{-\frac{p}{2}})\vartheta^{-p}(C(T)L\tilde{\alpha}T)^{p/2}\hat{C}_p + \bar{C}\delta^p\Big) \\
&\quad \times \mathbb{E}(\sup_{0\le t\le T} \mathrm{e}^{\beta t}|Y(t)|^p) + 2\mathbb{E}\mathrm{e}^{\beta T}|\xi|^p \\
&\quad + 2\Big(\vartheta^{-p}(C(T)L\tilde{\alpha})^{p/2}\tilde{C}\hat{C}_p + \vartheta^{-p}(C(T))^{\frac{p}{2}} + \bar{C}\delta^p\Big)\mathbb{E}\Big(\int_0^T \mathrm{e}^{\beta s}|f(s,0,0)|^2\mathrm{d}s\Big)^{p/2} \\
&\quad + \frac{2(p-1)}{\delta^{\frac{p-1}{p}}}\mathbb{E}\left(\sup_{0\le s\le T}(S^+(s))\right)^p \\
&\quad + p\mathbb{E}\sup_{0\le t\le T}\left|\int_t^T \mathrm{e}^{\beta s}|Y(s)|^{p-1}\mathrm{sgn}(Y(s))Z(s)\mathrm{d}W(s)\right|.
\end{aligned}$$

Next using the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_t^T \mathrm{e}^{\beta s}|Y(s)|^{p-1}\mathrm{sgn}(Y(s))Z(s)\mathrm{d}W(s)\right|\right] \\
&\le 2\mathbb{E}\left(\int_0^T \mathrm{e}^{\beta s}|Y(t)|^{2(p-1)}1_{Y(s)\neq 0}|Z(s)|^2\mathrm{d}s\right)^{1/2} \\
&\le 2\mathbb{E}\left[\left(\sup_{0\le t\le T}\mathrm{e}^{\beta t/2}|Y(t)|^{p/2}\right)\left(\int_0^T \mathrm{e}^{\beta s}|Y(s)|^{p-2}1_{Y(s)\neq 0}|Z(s)|^2\mathrm{d}s\right)\right] \\
&\le \rho\mathbb{E}\left(\sup_{0\le t\le T}\mathrm{e}^{\beta t}|Y(t)|^p\right) + \frac{1}{\rho}\mathbb{E}\left(\int_0^T \mathrm{e}^{\beta s}|Y(s)|^{p-2}1_{Y(s)\neq 0}|Z(s)|^2\mathrm{d}s\right),
\end{aligned}$$

where $\rho > 0$ is a constant. Plugging this inequality in (3.10), we obtain

(3.11)

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le T}\mathrm{e}^{\beta t}|Y(t)|^p\right) \\
&\le 2\mathbb{E}\mathrm{e}^{\beta T}|\xi|^p + \frac{p}{\rho}\mathbb{E}\left(\int_0^T \mathrm{e}^{\beta s}|Y(s)|^{p-2}1_{Y(s)\neq 0}|Z(s)|^2\mathrm{d}s\right) \\
&\quad + 2\Big(2(p-1)\vartheta^{\frac{p}{p-1}} + (1+\tilde{C}T^{-\frac{p}{2}})\vartheta^{-p}(C(T)L\tilde{\alpha}T)^{p/2}\hat{C}_p + \bar{C}\delta^p + p\rho\Big) \\
&\quad \times \mathbb{E}\left(\sup_{0\le t\le T}\mathrm{e}^{\beta t}|Y(t)|^p\right)
\end{aligned}$$

$$
\begin{aligned}
&+2\Big(\vartheta^{-p}(C(T)L\tilde{\alpha})^{p/2}\tilde{C}\hat{C}_p+\vartheta^{-p}(C(T))^{\frac{p}{2}}+\bar{C}\delta^p\Big)\\
&\times\mathbb{E}\Big(\int_0^T \mathrm{e}^{\beta s}|f(s,0,0)|^2\mathrm{d}s\Big)^{p/2}\\
&+\frac{2(p-1)}{\delta^{\frac{p-1}{p}}}\mathbb{E}\left(\sup_{0\le s\le T}(S^+(s))\right)^p\\
\le{}&(2+\frac{p}{c(p)\rho})\mathbb{E}\mathrm{e}^{\beta T}|\xi|^p+(2+\frac{p}{c(p)\rho})\frac{p-1}{\delta^{\frac{p-1}{p}}}\mathbb{E}\left(\sup_{0\le s\le T}(S^+(s))\right)^p\\
&+\Big\{(2+\frac{p}{c(p)\rho})\Big(2(p-1)\vartheta^{\frac{p}{p-1}}+(1+\tilde{C}T^{-\frac{p}{2}})\vartheta^{-p}(C(T)L\tilde{\alpha}T)^{p/2}\hat{C}_p+\bar{C}\delta^p\Big)\\
&+p\rho\Big\}\mathbb{E}(\sup_{0\le t\le T}\mathrm{e}^{\beta t}|Y(t)|^p)\\
&+(2+\frac{p}{c(p)\rho})\Big(\vartheta^{-p}(C(T)L\tilde{\alpha})^{p/2}\tilde{C}\hat{C}_p+\vartheta^{-p}(C(T))^{\frac{p}{2}}+\bar{C}\delta^p\Big)\\
&\times\mathbb{E}\Big(\int_0^T \mathrm{e}^{\beta s}|f(s,0,0)|^2\mathrm{d}s\Big)^{p/2}.
\end{aligned}
$$

The second inequality comes from (3.9). Finally it is enough to choose $\rho=\frac{1}{2p}$, $\vartheta=T^{\frac{1}{2}}$ and δ and T small enough to obtain (3.6). From the above inequalities (3.11), (3.6) and (3.8), we obtain that there exists a positive constant $C=C(p,\beta,\gamma,\tilde{\alpha},L,T)$ depending on $p,\beta,\gamma,\tilde{\alpha},L$ and T such that

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\mathrm{e}^{\beta t}|Y(t)|^p+\Big(\int_0^T \mathrm{e}^{\beta t}|Z(t)|^2\mathrm{d}t\Big)^{p/2}+\mathrm{e}^{\beta T}|K(T)|^p\right]\\
\le{}&C\mathbb{E}\left[\mathrm{e}^{\beta T}|\xi|^p+\Big(\int_0^T \mathrm{e}^{\beta t}|f(t,0,0)|^2\mathrm{d}t\Big)^{p/2}+\sup_{0\le t\le T}\mathrm{e}^{\beta t}\big(S(t)^+\big)^p\right].
\end{aligned}
$$

□

We can now estimate the variation in the solution induced by a variation in the data.

Proposition 3.2. *Let* (ξ,f,S) *and* (ξ',f',S') *be two triplets satisfying the above assumptions* (i)–(v). *Suppose that* (Y,Z,K) *is a solution of the RBSDE with time delayed generator* (ξ,f,S) *and* (Y',Z',K') *is a solution of the RBSDE with time delayed generator* (ξ',f',S'). *Define*

$$
\begin{aligned}
&\triangle\xi=\xi-\xi',\quad \triangle f=f-f',\quad \triangle S=S-S';\\
&\triangle Y=Y-Y',\quad \triangle Z=Z-Z',\quad \triangle K=K-K'.
\end{aligned}
$$

If the Lpicshitz constant L *of the generator* f *and the time horizon* T *are small enough, then there exist two positive constants* β *and* γ *satisfying that*

$$
D_1:=\beta-\gamma>0,\quad D_2:=1-\frac{2L\tilde{\alpha}}{\gamma}>0
$$

and a positive constant $D = D(p,\beta,\gamma,\tilde{\alpha},L,T)$ depending on $p,\beta,\gamma,\tilde{\alpha},L$ and T such that

$$\mathbb{E}\left(\sup_{0\le t\le T} \mathrm{e}^{\beta t}|\triangle Y(t)|^p + \Big(\int_0^T \mathrm{e}^{\beta t}|\triangle Z(t)|^2\mathrm{d}t\Big)^{\frac{p}{2}}\right) \tag{3.12}$$
$$\le D\mathbb{E}\left(\mathrm{e}^{\beta T}|\triangle\xi|^2 + \Big(\int_0^T \mathrm{e}^{\beta t}|\triangle f(t,Y_t,Z_t)|^2\mathrm{d}t\Big)^{\frac{p}{2}}\right)$$
$$+ D\left[\mathbb{E}\left(\sup_{0\le t\le T}\mathrm{e}^{\beta t}|\triangle S(t)|^p\right)\right]^{\frac{p-1}{p}} \Psi_T^{1/p},$$

where

$$\Psi_T = \mathbb{E}\Bigg[\mathrm{e}^{\beta T}|\xi|^p + \Big(\int_0^T \mathrm{e}^{\beta t}|f(t,0,0)|^2\mathrm{d}t\Big)^{\frac{p}{2}} + \sup_{0\le t\le T}\mathrm{e}^{\beta t}\left(S(t)^+\right)^p$$
$$+\mathrm{e}^{\beta T}|\xi'|^p + \Big(\int_0^T \mathrm{e}^{\beta t}|f'(t,0,0)|^2\mathrm{d}t\Big)^{\frac{p}{2}} + \sup_{0\le t\le T}\mathrm{e}^{\beta t}\left(S(t)'^+\right)^p\Bigg].$$

Proof. The computations are similar to those in the previous proof, so we shall only sketch the arguments.

Step 1. We claim that

$$\mathbb{E}\Big(\int_0^T \mathrm{e}^{\beta s}|\triangle Z(s)|^2\mathrm{d}s\Big)^{p/2} \le \tilde{D}\mathbb{E}\left[\mathrm{e}^{\beta T}\triangle|\xi|^p + \Big(\int_0^T \mathrm{e}^{\beta s}|\triangle f(s,Y_s,Z_s)|^2\mathrm{d}s\Big)^{p/2}\right.$$
$$\left. +\Big(\sup_{0\le s\le T}\mathrm{e}^{\beta s}|\triangle Y(s)|^p\Big)\right], \tag{3.13}$$

where $\tilde{D} > 0$ depends on $p,\beta,\gamma,\tilde{\alpha},L$ and T, being the linear function of $T^{\frac{p}{2}}$, which can be deduced as follows. For each integer $n \ge 1$ let us set:

$$\tau_n = \inf\left\{t\in[0,T], \int_0^t \mathrm{e}^{\beta s}|\triangle Z(s)|^2\mathrm{d}s\right\}\wedge T.$$

Similar to (3.2), (3.3), (3.4) and (3.5), applying the Itô formula to $\mathrm{e}^{\beta t}|\triangle Y(t)|^2$ and reorganizing the inequalities, we get

$$|\triangle Y(0)|^2 + \left(1-\frac{L\tilde{\alpha}}{\gamma}\right)\int_0^{\tau_n}\mathrm{e}^{\beta s}|\triangle Z(s)|^2\mathrm{d}s$$
$$\le \mathrm{e}^{\beta\tau_n}|\triangle Y(\tau_n)|^2 + (\gamma-\beta)\int_0^{\tau_n}\mathrm{e}^{\beta s}|\triangle Y(s)|^2\mathrm{d}s$$
$$+\Big(\frac{L\tilde{\alpha}T}{\gamma}+\frac{1}{\upsilon}\Big)\sup_{0\le t\le T}\mathrm{e}^{\beta t}|\triangle Y(t)|^2$$
$$+2\int_0^{\tau_n}\mathrm{e}^{\beta s}|\triangle f(s,Y_s,Z_s)\triangle Y(s)|\mathrm{d}s + \upsilon \mathrm{e}^{\beta\tau_n}|\triangle K(\tau_n)|^2$$
$$-2\int_0^{\tau_n}\mathrm{e}^{\beta s}\triangle Y(s)\triangle Z(s)\mathrm{d}W(s),$$

for any $\gamma > 0$, $\upsilon > 0$, where we have used (3.4) for $\phi = \triangle Z$. We have

$$\begin{aligned} &2\int_0^{\tau_n} \mathrm{e}^{\beta s}|\triangle f(s,Y_s,Z_s)\triangle Y(s)|\mathrm{d}s \\ &\leq \tilde{\lambda} \sup_{0\leq t\leq T} \mathrm{e}^{\beta t}|\triangle Y(t)|^2 + \frac{1}{\tilde{\lambda}}\int_0^{\tau_n} \mathrm{e}^{\beta s}|\triangle f(s,Y_s,Z_s)|^2\mathrm{d}s \end{aligned}$$

for any $\tilde{\lambda} > 0$. Using the same discussion as Proposition 3.1, we deduce that

$$\begin{aligned} \mathrm{e}^{\beta\tau_n}|\triangle K(\tau_n)|^2 \leq C_6 \Bigg(&|\triangle Y(0)|^2 + |\triangle Y(\tau_n)|^2 + \int_0^{\tau_n} |\triangle f(s,Y_s,Z_s)|^2\mathrm{d}s \\ &+ \sup_{0\leq t\leq T} |\triangle Y(t)|^2 + \int_0^{\tau_n} |\triangle Z(s)|^2\mathrm{d}s + \left|\int_0^{\tau_n} \triangle Z(s)\mathrm{d}W(s)\right|^2 \Bigg), \end{aligned}$$

where $C_6 > 0$. Plugging the above two inequalities in the previous one, we get

$$\begin{aligned} &(1-\upsilon C_6)\,|\triangle Y(0)|^2 + \left(1 - \frac{L\tilde{\alpha}}{\gamma}\right)\int_0^{\tau_n} \mathrm{e}^{\beta s}|\triangle Z(s)|^2\mathrm{d}s - \upsilon C_6\int_0^{\tau_n} |\triangle Z(s)|^2\mathrm{d}s \\ &\leq (\mathrm{e}^{\beta\tau_n} + \upsilon C_6)|\triangle Y(\tau_n)|^2 + \left(\frac{L\tilde{\alpha}}{\gamma}T + \frac{1}{\upsilon} + \tilde{\lambda}\right) \sup_{0\leq t\leq T} \mathrm{e}^{\beta t}|\triangle Y(t)|^2 \\ &\quad + \upsilon C_6 \sup_{0\leq t\leq T} |\triangle Y(t)|^2 + \upsilon C_6\left|\int_0^{\tau_n} \triangle Z(s)\mathrm{d}W(s)\right|^2 \\ &\quad + \left(\frac{1}{\tilde{\lambda}} + \upsilon C_6\right)\int_0^{\tau_n} \mathrm{e}^{\beta s}|\triangle f(s,Y_s,Z_s)|^2\mathrm{d}s + (\gamma-\beta)\int_0^{\tau_n} \mathrm{e}^{\beta s}|\triangle Y(s)|^2\mathrm{d}s \\ &\quad + 2\left|\int_0^{\tau_n} \mathrm{e}^{\beta s}\triangle Y(s)\triangle Z(s)\mathrm{d}W(s)\right|. \end{aligned}$$

Choosing now υ small enough and γ such that $D_1 = \beta - \gamma > 0$ and $D_2 := 1 - \frac{2L\tilde{\alpha}}{\gamma} > 0$, we obtain

$$\begin{aligned} &\mathbb{E}\Big(\int_0^{\tau_n} \mathrm{e}^{\beta s}|\triangle Z(s)|^2\mathrm{d}s\Big)^{p/2} \\ &\leq C_7\mathbb{E}\left[\mathrm{e}^{p\beta\tau_n/2}|\triangle Y(\tau_n)|^p + \Big(\int_0^{\tau_n} \mathrm{e}^{\beta s}|\triangle f(s,Y_s,Z_s)|^2\mathrm{d}s\Big)^{p/2}\right] \\ &\quad + C_7\mathbb{E}\Big(\sup_{0\leq t\leq T} \mathrm{e}^{\beta t}|\triangle Y(t)|^p\Big) + C_7\mathbb{E}\left|\int_0^{\tau_n} \mathrm{e}^{\beta s}\triangle Y(s)\triangle Z(s)\mathrm{d}W(s)\right|^{p/2}, \end{aligned}$$

where C_7 depends on $p, \beta, \gamma, \tilde{\alpha}, L$ and T. If L is small enough, then C_7 can be chosen as the linear function of $(LT)^{\frac{p}{2}}$. Next thanks to the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned} &\mathbb{E}\left|\int_0^{\tau_n} \mathrm{e}^{\beta s}\triangle Y(s)\triangle Z(s)\mathrm{d}W(s)\right|^{p/2} \\ &\leq \frac{C_8}{2\eta}\mathbb{E}\left(\sup_{0\leq t\leq T} \mathrm{e}^{\beta t}|\triangle Y(t)|^p\right) + \frac{\eta}{2}\mathbb{E}\Big(\int_0^{\tau_n} \mathrm{e}^{\beta s}|\triangle Z(s)|^2\mathrm{d}s\Big)^{p/2}, \end{aligned}$$

where $\eta > 0$ is a constant and $C_8 > 0$ depends only on p. Finally, plugging the last inequality in the previous one, choosing η small enough and using the Fatou Lemma, we obtain that there exists a real constant $\tilde{D}$ depending on $p, \beta, \gamma, \tilde{\alpha}, L$ and T, being the linear function of $(LT)^{\frac{p}{2}}$ such that (3.13) holds.
Step 2. We claim that (3.12) holds.

The estimate of (3.12) can be deduced as follows. Applying the Itô formula to $\mathrm{e}^{\beta t}|\triangle Y(t)|^p$ yields that

$$
\begin{aligned}
(3.14)\qquad & \mathrm{e}^{\beta t}|\triangle Y(t)|^p + \beta \int_t^T \mathrm{e}^{\beta s}|\triangle Y(s)|^p \mathrm{d}s \\
& + c(p) \int_t^T \mathrm{e}^{\beta s}|\triangle Y(s)|^{p-2} 1_{\triangle Y(s)\neq 0}|\triangle Z(s)|^2 \mathrm{d}s \\
\leq\ & \mathrm{e}^{\beta T}|\triangle \xi|^p + p \int_t^T \mathrm{e}^{\beta s}|\triangle Y(s)|^{p-1}\mathrm{sgn}(\triangle Y(s))\triangle f(s, Y_s, Z_s)\mathrm{d}s \\
& + p \int_t^T \mathrm{e}^{\beta s}|\triangle Y(s)|^{p-1}\mathrm{sgn}(\triangle Y(s))[f'(s, Y_s, Z_s) - f'(s, Y'_s, Z'_s)]\mathrm{d}s \\
& + p \int_t^T \mathrm{e}^{\beta s}|\triangle Y(s)|^{p-1}\mathrm{sgn}(\triangle Y(s))\mathrm{d}(\triangle K(s)) \\
& - p \int_t^T \mathrm{e}^{\beta s}|\triangle Y(s)|^{p-1}\mathrm{sgn}(\triangle Y(s))\triangle Z(s)\mathrm{d}W(s),
\end{aligned}
$$

where $\mathrm{sgn}(y) := \frac{y}{|y|}1_{y\neq 0}$. By the Young inequality and since f is Lipschitz, we have

$$
\begin{aligned}
& p \int_t^T \mathrm{e}^{\beta s}|\triangle Y(s)|^{p-1}|f'(s, Y_s, Z_s) - f'(s, Y'_s, Z'_s)|\mathrm{d}s \\
\leq\ & \left((p-1)\kappa^{\frac{p}{p-1}} + \kappa^{-p}(C(T)L\tilde{\alpha}T)^{p/2}\hat{C}_p\right)\left(\sup_{0\leq s\leq T}\mathrm{e}^{\beta s}|\triangle Y(s)|^p\right) \\
& + \kappa^{-p}(C(T)L\tilde{\alpha})^{p/2}\hat{C}_p\left(\int_0^T \mathrm{e}^{\beta s}|\triangle Z(s)|^2\mathrm{d}s\right)^{p/2},
\end{aligned}
$$

for any $\kappa > 0$, where $C(T) = \frac{2e^{\beta T}}{\beta}$.

Note that by the classical discussion (see [17], page13), we have

$$
p \int_t^T \mathrm{e}^{\beta s}|\triangle Y(s)|^{p-1}\mathrm{sgn}(\triangle Y(s))\mathrm{d}(\triangle K(s)) \leq p \int_t^T \mathrm{e}^{\beta s}|\triangle S(s)|^{p-1}\mathrm{d}(\triangle K(s)).
$$

Thus coming back to (3.14), we get

$$
\begin{aligned}
(3.15)\qquad & \mathrm{e}^{\beta t}|\triangle Y(t)|^p + c(p)\int_t^T \mathrm{e}^{\beta s}|\triangle Y(s)|^{p-2}1_{\triangle Y(s)\neq 0}|\triangle Z(s)|^2\mathrm{d}s \\
\leq\ & \mathrm{e}^{\beta T}|\triangle \xi|^p + p\int_t^T \mathrm{e}^{\beta s}|\triangle Y(s)|^{p-1}\mathrm{sgn}(\triangle Y(s))\triangle f(s, Y_s, Z_s)\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big((p-1)\kappa^{\frac{p}{p-1}} + \kappa^{-p}(C(T)L\tilde{\alpha}T)^{p/2}\hat{C}_p\Big)\left(\sup_{0\le s\le T} \mathrm{e}^{\beta s}|\triangle Y(s)|^p\right) \\
&+ \kappa^{-p}(C(T)L\tilde{\alpha})^{p/2}\hat{C}_p\left(\int_0^T \mathrm{e}^{\beta s}|\triangle Z(s)|^2\mathrm{d}s\right)^{p/2} \\
&+ p\int_t^T \mathrm{e}^{\beta s}|\triangle S(s)|^{p-1}\mathrm{d}(\triangle K(s)) \\
&- p\int_t^T \mathrm{e}^{\beta s}|\triangle Y(s)|^{p-1}\mathrm{sgn}(\triangle Y(s))\triangle Z(s)\mathrm{d}W(s).
\end{aligned}
$$

On the other hand the process

$$
\left\{\int_0^t \mathrm{e}^{\beta s}|\triangle Y(s)|^{p-1}\mathrm{sgn}(\triangle Y(s))\triangle Z(s)\mathrm{d}W(s)\right\}_{0\le t\le T}
$$

is a martingale thanks to the Burkholder-Davis-Gundy and Young inequalities. Taking the expectation in (3.15) we have

$$
\begin{aligned}
&c(p)\mathbb{E}\int_t^T \mathrm{e}^{\beta s}|\triangle Y(s)|^{p-2}1_{\triangle Y(s)\neq 0}|\triangle Z(s)|^2\mathrm{d}s \\
\le\ & \mathbb{E}\mathrm{e}^{\beta T}|\triangle\xi|^p + p\mathbb{E}\int_0^T \mathrm{e}^{\beta s}|\triangle Y(s)|^{p-1}|\triangle f(s,Y_s,Z_s)|\mathrm{d}s \\
&+ \Big((p-1)\kappa^{\frac{p}{p-1}} + \kappa^{-p}(C(T)L\tilde{\alpha}T)^{p/2}\hat{C}_p\Big)\mathbb{E}\left(\sup_{0\le s\le T} \mathrm{e}^{\beta s}|\triangle Y(s)|^p\right) \\
&+ \kappa^{-p}(C(T)L\tilde{\alpha})^{p/2}\hat{C}_p\mathbb{E}\left(\int_0^T \mathrm{e}^{\beta s}|\triangle Z(s)|^2\mathrm{d}s\right)^{p/2} \\
&+ p\mathbb{E}\int_0^T \mathrm{e}^{\beta s}|\triangle S(s)|^{p-1}\mathrm{d}(\triangle K(s)).
\end{aligned}
$$

Coming back to (3.15), taking the supremum and then expectation we get

(3.16)

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le T} \mathrm{e}^{\beta t}|\triangle Y(t)|^p\right) + c(p)\mathbb{E}\int_t^T \mathrm{e}^{\beta s}|\triangle Y(s)|^{p-2}1_{\triangle Y(s)\neq 0}|\triangle Z(s)|^2\mathrm{d}s \\
\le\ & 2\mathbb{E}\mathrm{e}^{\beta T}|\triangle\xi|^p + 2p\mathbb{E}\int_0^T \mathrm{e}^{\beta s}|\triangle Y(s)|^{p-1}|\triangle f(s,Y_s,Z_s)|\mathrm{d}s \\
&+ 2\Big((p-1)\kappa^{\frac{p}{p-1}} + \kappa^{-p}(C(T)L\tilde{\alpha}T)^{p/2}\hat{C}_p\Big)\mathbb{E}\left(\sup_{0\le s\le T} \mathrm{e}^{\beta s}|\triangle Y(s)|^p\right) \\
&+ 2\kappa^{-p}(C(T)L\tilde{\alpha})^{p/2}\hat{C}_p\mathbb{E}\left(\int_0^T \mathrm{e}^{\beta s}|\triangle Z(s)|^2\mathrm{d}s\right)^{p/2}
\end{aligned}
$$

$$
\begin{aligned}
&+2p\mathbb{E}\int_t^T \mathrm{e}^{\beta s}|\triangle S(s)|^{p-1}\mathrm{d}(\triangle K(s)) \\
&+p\mathbb{E}\sup_{0\le t\le T}\left|\int_t^T \mathrm{e}^{\beta s}|\triangle Y(s)|^{p-1}\mathrm{sgn}(\triangle Y(s))\triangle Z(s)\mathrm{d}W(s)\right|.
\end{aligned}
$$

Now with the Hölder inequality

$$
\begin{aligned}
&\mathbb{E}\int_0^T \mathrm{e}^{\beta s}|\triangle S(s)|^{p-1}\mathrm{d}(\triangle K(s)) \\
&\le \left[\mathbb{E}\left(\sup_{0\le t\le T}\mathrm{e}^{\beta t}|\triangle S(t)|^p\right)\right]^{\frac{p-1}{p}}\left[\mathbb{E}\left(\mathrm{e}^{\beta T}|\triangle K(T)|^p\right)\right]^{\frac{1}{p}}
\end{aligned}
$$

and since $\mathbb{E}|\triangle K(T)|^p \le C_p(\mathbb{E}|K(T)|^p + \mathbb{E}|K'(T)|^p)$, using inequality (3.8) and Proposition 3.1, we deduce that

$$
\mathbb{E}\mathrm{e}^{\beta T}|\triangle K(T)|^p \le \tilde{C}(p)\Psi_T,
$$

where $\tilde{C}(p) > 0$ depends on $p, \beta, \gamma, \tilde{\alpha}, L$ and T.

Next using the Burkholder-Davis-Gundy inequality we have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le T}\left|\int_t^T \mathrm{e}^{\beta s}|\triangle Y(s)|^{p-1}\mathrm{sgn}(\triangle Y(s))\triangle Z(s)\mathrm{d}W(s)\right|\right) \\
&\le \iota p\left(\mathbb{E}\sup_{0\le t\le T}\mathrm{e}^{\beta t}|\triangle Y(t)|^p\right) + \frac{p}{\iota}\mathbb{E}\left(\int_0^T \mathrm{e}^{\beta s}|\triangle Y(s)|^{p-2}1_{\triangle Y(s)\neq 0}|\triangle Z(s)|^2\mathrm{d}s\right)
\end{aligned}
$$

for any $\iota > 0$. Furthermore,

$$
\begin{aligned}
&2p\int_0^T \mathrm{e}^{\beta s}|\triangle Y(s)|^{p-1}|\triangle f(s, Y_s, Z_s)|\mathrm{d}s \\
&\le 2p\rho^{\frac{p}{p-1}}\sup_{0\le t\le T}\mathrm{e}^{\beta t}|\triangle Y(t)|^p + \frac{2p\left(\tilde{C}(T)\right)^{\frac{p}{2}}}{\rho^p}\left(\int_0^T |\triangle f(s, Y_s, Z_s)|^2\mathrm{d}s\right)^{p/2},
\end{aligned}
$$

where $\tilde{C}(T) = \frac{\mathrm{e}^{2\beta T}}{\beta}$. Taking into account of Step 1, we now plug the above three inequalities in (3.16) and we obtain

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le T}\mathrm{e}^{\beta t}|\triangle Y(t)|^p\right) \\
&\le (2 + \frac{p}{c(p)\iota})(1 + \kappa^{-p}(C(T)L\tilde{\alpha})^{p/2}\tilde{D}\hat{C}_p)\mathbb{E}\mathrm{e}^{\beta T}|\triangle\xi|^p \\
&\quad + (2 + \frac{p}{c(p)\iota})\Big(\frac{p\left(\tilde{C}(T)\right)^{\frac{p}{2}}}{\rho^p} + \kappa^{-p}(C(T)L\tilde{\alpha})^{p/2}\tilde{D}\hat{C}_p\Big) \\
&\quad \times\mathbb{E}\left(\int_0^T |\triangle f(s, Y_s, Z_s)|^2\mathrm{d}s\right)^{p/2}
\end{aligned}
$$

$$+\left\{(2+\frac{p}{c(p)\iota})\Big((p-1)\kappa^{\frac{p}{p-1}}+\kappa^{-p}(C(T)L\tilde{\alpha}T)^{p/2}\hat{C}_p\right.$$
$$\left.+\,2\kappa^{-p}(C(T)L\tilde{\alpha})^{p/2}\tilde{D}\hat{C}_p+2p\rho^{\frac{p}{p-1}}\Big)+\iota p\right\}\mathbb{E}\sup_{0\le t\le T}\mathrm{e}^{\beta t}|\triangle Y(t)|^p$$
$$+\,(1+\frac{p^2}{c(p)\iota})\left[\mathbb{E}\left(\sup_{0\le t\le T}\mathrm{e}^{\beta t}|\triangle S(t)|^p\right)\right]^{\frac{p-1}{p}}\left[\tilde{C}(p)\Psi_T\right]^{\frac{1}{p}}.$$

Finally, it is enough to choose $\iota=\frac{1}{2p}$, $\kappa=T^{\frac{1}{2}}$, T and ρ small enough in the above inequality to obtain (3.12). The desired result of the proposition follows. □

We deduce immediately the following uniqueness result from Proposition 3.2 with $\xi'=\xi$, $f'=f$ and $S'=S$.

Corollary 3.1. *Under the assumptions* (i)–(v)*, if the Lipschitz constant L of the generator f and the time horizon T are small enough and for two positive constants β and γ the conditions of Proposition 3.2 are satisfied, then there exists at most one solution of the RBSDE with time delayed generators* (1)–(4).

Proof. Using the previous Proposition 3.2, we obtain immediately $Y=Y'$ and $Z=Z'$. Therefore we have $K=K'$, whence uniqueness of the solution of the reflected BSDE with time delayed generator associated with (ξ,f,S). □

3.2. Existence and uniqueness of the solution

To begin with, let us first assume that f does not depend on (y,z), that is, it is a given $\mathcal{F}_t$-progressively measurable process satisfying that

(ii′) $\mathbb{E}\left(\int_0^T|f(t)|^2\mathrm{d}t\right)^{\frac{p}{2}}<\infty.$

A solution to the backward reflection problem (BRP, in short) is a triple (Y,Z,K) which satisfies (1), (3), (4) and

(2′) $Y(t)=\xi+\int_t^T f(s)\mathrm{d}s+K(T)-K(t)-\int_t^T Z(s)\mathrm{d}W(s),\quad 0\le t\le T.$

The following proposition is from Hamadène et al. ([17], Theorem 2).

Proposition 3.3. *Under the assumptions* (i), (ii′) *and* (v)*, the BRP* (1), (2′), (3), (4) *has a unique L^p- solution* $\{(Y(t),Z(t),K(t));0\le t\le T\}$.

We now deal with the general case of generator, i.e., f depends on (y,z).

Theorem 3.1. *Assume the assumptions* (i)–(v) *hold. If the Lipschitz constant L of the generator f and the time horizon T are small enough and for two positive constants β and γ the conditions of Proposition 3.2 are satisfied, then the RBSDE with time delayed generator* (1)–(4) *has a unique solution* $\{(Y(t),Z(t),K(t));0\le t\le T\}$.

Proof. Denote by $\mathcal{L}$ the space of progressively measurable $\{(Y(t), Z(t)); 0 \leq t \leq T\}$ with values in $\mathbb{R} \times \mathbb{R}$ which satisfy (1) and (3).

We define a mapping Φ from $\mathcal{L}$ into itself as follows. Given $(U, V) \in \mathcal{L}$, we define $(Y, Z) = \Phi(U, V)$ where (Y, Z, K) is the solution of the RBSDE associated with $(\xi, f(t, U_t, V_t), S)$, i.e.,

$$
\begin{aligned}
&(Y, Z) \in \mathcal{L},\ K \in \mathbb{S}^p_\beta(\mathbb{R});\\
&Y(t) = \xi + \int_t^T f(s, U_s, V_s)\mathrm{d}s + K(T) - K(t) - \int_t^T Z(s)\mathrm{d}W(s), \quad 0 \leq t \leq T;\\
&Y(t) \geq S(t) \text{ and } \int_0^T \mathrm{e}^{\beta t}(Y(t) - S(t))\mathrm{d}K(t) = 0,\ \text{for some } \beta > 0.
\end{aligned}
$$

The solution of this equation exists and is unique thanks to Proposition 3.3. Now for (U', V') in $\mathcal{L}$, we define in the same way $(Y', Z') = \Phi(U', V')$ and

$$
\bar{U} = U - U', \quad \bar{V} = V - V', \quad \bar{Y} = Y - Y', \quad \bar{Z} = Z - Z'.
$$

We are now going to prove that the mapping Φ is a strict contraction on $\mathcal{L}$ equipped with the norm

$$
\|(Y, Z)\|_\beta = \left(\mathbb{E}\left[\sup_{0\leq t\leq T} \mathrm{e}^{\beta t}|Y(t)|^p\right]\right)^{1/p} + \left(\mathbb{E}\left[\int_0^T \mathrm{e}^{\beta t}|Z(t)|^2\mathrm{d}t\right]^{\frac{p}{2}}\right)^{1/p}.
$$

In fact, it follows from the arguments similar to those in the proofs of Propositions 3.1 and 3.2 that

(3.17)

$$
\begin{aligned}
&\mathrm{e}^{\beta t}|\bar{Y}(t)|^p + \beta\int_t^T \mathrm{e}^{\beta s}|\bar{Y}(s)|^p\mathrm{d}s + c(p)\int_t^T \mathrm{e}^{\beta s}|\bar{Y}(s)|^{p-2}1_{\bar{Y}(s)\neq 0}|\bar{Z}(s)|^2\mathrm{d}s\\
&\leq p\int_t^T \mathrm{e}^{\beta s}|\bar{Y}(s)|^{p-1}\mathrm{sgn}(\bar{Y}(s))[f(s, U_s, V_s) - f(s, U'_s, V'_s)]\mathrm{d}s\\
&\quad + p\int_t^T \mathrm{e}^{\beta s}|\bar{Y}(s)|^{p-1}\mathrm{sgn}(\bar{Y}(s))\mathrm{d}\bar{K}(s)\\
&\quad - p\int_t^T \mathrm{e}^{\beta s}|\bar{Y}(s)|^{p-1}\mathrm{sgn}(\bar{Y}(s))\bar{Z}(s)\mathrm{d}W(s).
\end{aligned}
$$

Now for $\tau > 0$, using the Young inequality, we have

$$
\begin{aligned}
&p\int_t^T \mathrm{e}^{\beta s}|\bar{Y}(s)|^{p-1}\mathrm{sgn}(\bar{Y}(s))[f(s, U_s, V_s) - f(s, U'_s, V'_s)]\mathrm{d}s\\
&\leq (p-1)\tau^{\frac{p}{p-1}}\left(\sup_{0\leq t\leq T} \mathrm{e}^{\beta t}|\bar{Y}(t)|^p\right)\\
&\quad + \tau^{-p}(C(T)L\tilde{\alpha}T)^{p/2}\hat{C}(p)\left(\sup_{0\leq t\leq T} \mathrm{e}^{\beta t}|\bar{U}(t)|^p\right)
\end{aligned}
$$

$$+\tau^{-p}(C(T)L\tilde{\alpha})^{p/2}\hat{C}(p)\left(\int_0^T e^{\beta s}|\bar{V}(s)|^2 ds\right)^{p/2},$$

where $C(T)=\frac{2e^{\beta T}}{\beta}$. Moreover, $\int_t^T e^{\beta s}|\bar{Y}(s)|^{p-1}\mathrm{sgn}(\bar{Y}(s))d\bar{K}(s)\leq 0$. Coming back to (3.17), we obtain

(3.18)

$$\begin{aligned}&e^{\beta t}|\bar{Y}(t)|^p+c(p)\int_t^T e^{\beta s}|\bar{Y}(s)|^{p-2}1_{\bar{Y}(s)\neq 0}|\bar{Z}(s)|^2 ds\\ &\leq (p-1)\tau^{\frac{p}{p-1}}\left(\sup_{0\leq t\leq T} e^{\beta t}|\bar{Y}(t)|^p\right)+\tau^{-p}(C(T)L\tilde{\alpha}T)^{p/2}\hat{C}(p)\left(\sup_{0\leq t\leq T} e^{\beta t}|\bar{U}(t)|^p\right)\\ &\quad+\tau^{-p}(C(T)L\tilde{\alpha})^{p/2}\hat{C}(p)\left(\int_0^T e^{\beta s}|\bar{V}(s)|^2 ds\right)^{p/2}\\ &\quad-p\int_t^T e^{\beta s}|\bar{Y}(s)|^{p-1}\mathrm{sgn}(\bar{Y}(s))\bar{Z}(s)dW(s).\end{aligned}$$

But as in the proof of uniqueness, the process

$$\left\{M_t=\int_0^t e^{\beta s}|\bar{Y}(s)|^{p-1}\mathrm{sgn}(\bar{Y}(s))\bar{Z}(s)dW(s)\right\}_{0\leq t\leq T}$$

is a uniformly integrable martingale. Therefore with (3.18), we obtain

(3.19)

$$\begin{aligned}&c(p)\mathbb{E}\int_t^T e^{\beta s}|\bar{Y}(s)|^{p-2}1_{\bar{Y}(s)\neq 0}|\bar{Z}(s)|^2 ds\\ &\leq (p-1)\tau^{\frac{p}{p-1}}\mathbb{E}\left(\sup_{0\leq t\leq T} e^{\beta t}|\bar{Y}(t)|^p\right)+\tau^{-p}(C(T)L\tilde{\alpha}T)^{p/2}\hat{C}(p)\mathbb{E}\sup_{0\leq t\leq T} e^{\beta t}|\bar{U}(t)|^p\\ &\quad+\tau^{-p}(C(T)L\tilde{\alpha})^{p/2}\hat{C}(p)\mathbb{E}\left(\int_0^T e^{\beta s}|\bar{V}(s)|^2 ds\right)^{p/2}\end{aligned}$$

and

$$\begin{aligned}(3.20)\qquad &\left(1-(p-1)\tau^{\frac{p}{p-1}}\right)\mathbb{E}\left(\sup_{0\leq t\leq T} e^{\beta t}|\bar{Y}(t)|^p\right)\\ &\leq \tau^{-p}(C(T)L\tilde{\alpha}T)^{p/2}\hat{C}(p)\mathbb{E}\left(\sup_{0\leq t\leq T} e^{\beta t}|\bar{U}(t)|^p\right)\\ &\quad+\tau^{-p}(C(T)L\tilde{\alpha})^{p/2}\hat{C}(p)\mathbb{E}\left(\int_0^T e^{\beta s}|\bar{V}(s)|^2 ds\right)^{p/2}+p\mathbb{E}\langle M,M\rangle_T^{1/2}.\end{aligned}$$

For the last inequality we have made use of the Burkholder-Davis-Gundy inequality. Note that

$$\begin{aligned}
&E\langle M, M\rangle_T^{1/2} \\
&\leq \mathbb{E}\left[\left(\sup_{0\leq t\leq T} \mathrm{e}^{\beta t/2}|\triangle Y(t)|^{p/2}\right)\left(\int_0^T \mathrm{e}^{\beta s}|\bar{Y}(s)|^{p-2}1_{\bar{Y}(s)\neq 0}|\bar{Z}(s)|^2\mathrm{d}s\right)^{1/2}\right] \\
&\leq \frac{1}{2p}\mathbb{E}\left(\sup_{0\leq t\leq T} \mathrm{e}^{\beta t}|\triangle Y(t)|^{p}\right) + \frac{p}{2}\mathbb{E}\int_0^T \mathrm{e}^{\beta s}|\bar{Y}(s)|^{p-2}1_{\bar{Y}(s)\neq 0}|\bar{Z}(s)|^2\mathrm{d}s.
\end{aligned}$$

Plugging now that inequality in (3.20) and (3.19) to obtain

$$\begin{aligned}
&\left(\frac{1}{2} - \Big(1+\frac{p^2}{2c(p)}\Big)(p-1)\tau^{\frac{p}{p-1}}\right)\mathbb{E}\left(\sup_{0\leq t\leq T} \mathrm{e}^{\beta t}|\bar{Y}(t)|^{p}\right) \\
&\leq \tau^{-p}(C(T)L\tilde{\alpha})^{p/2}\hat{C}(p)\Big(1+\frac{p^2}{2c(p)}\Big)\max(T^{\frac{p}{2}},1)\left(\mathbb{E}\left(\sup_{0\leq t\leq T} \mathrm{e}^{\beta t}|\bar{U}(t)|^{p}\right)\right. \\
&\quad\left. +\mathbb{E}\left(\int_0^T \mathrm{e}^{\beta s}|\bar{V}(s)|^2\mathrm{d}s\right)^{p/2}\right).
\end{aligned}$$

It is enough to choose τ such that $\frac{1}{2} - \Big(1+\frac{p^2}{2c(p)}\Big)(p-1)\tau^{\frac{p}{p-1}} > 0$. Thus

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq t\leq T} \mathrm{e}^{\beta t}|\bar{Y}(t)|^{p}\right) \\
&\leq \hat{C}\left(\mathbb{E}\left(\sup_{0\leq t\leq T} \mathrm{e}^{\beta t}|\bar{U}(t)|^{p}\right) + \mathbb{E}\left(\int_0^T \mathrm{e}^{\beta s}|\bar{V}(s)|^2\mathrm{d}s\right)^{p/2}\right),
\end{aligned} \tag{3.21}$$

where $\hat{C}$ depends on $p, \beta, \gamma, \tilde{\alpha}, \tau, L$ and T. Since L and T are small enough, we will have $\hat{C}$ small enough.

We next focus on the same estimate for $\bar{Z}$. For each integer $n \geq 1$ let us set:

$$\tau_n = \inf\left\{t\in[0,T], \int_0^t \mathrm{e}^{\beta s}|\bar{Z}(s)|^2\mathrm{d}s\right\}\wedge T.$$

Therefore using the Itô formula leads to

$$\begin{aligned}
&|\bar{Y}(0)|^2 + \int_0^{\tau_n} \mathrm{e}^{\beta s}|\bar{Z}(s)|^2\mathrm{d}s \\
&= \mathrm{e}^{\beta\tau_n}|\bar{Y}_{\tau_n}|^2 - \beta\int_0^{\tau_n} \mathrm{e}^{\beta s}|\bar{Y}(s)|^2\mathrm{d}s + 2\int_0^{\tau_n} \mathrm{e}^{\beta s}\bar{Y}(s)[f(s,U_s,V_s) - f(s,U'_s,V'_s)]\mathrm{d}s \\
&\quad + 2\int_0^{\tau_n} \mathrm{e}^{\beta s}\bar{Y}(s)\mathrm{d}\bar{K}(s) - 2\int_0^{\tau_n} \mathrm{e}^{\beta s}\bar{Y}(s)\bar{Z}(s)\mathrm{d}W(s)
\end{aligned} \tag{3.22}$$

$$\leq e^{\beta\tau_n}|\bar{Y}_{\tau_n}|^2 + (\gamma-\beta)\int_0^{\tau_n} e^{\beta s}|\bar{Y}(s)|^2 ds + \frac{L\tilde{\alpha}}{\gamma}\left[T\sup_{0\leq t\leq T} e^{\beta t}|\bar{U}(t)|^2 + \int_0^{\tau_n} e^{\beta s}|\bar{V}(s)|^2 ds\right] - 2\int_0^{\tau_n} e^{\beta s}\bar{Y}(s)\bar{Z}(s)dW(s),$$

for any $\gamma > 0$, where we have used the inequality $\int_0^{\tau_n} e^{\beta s}\bar{Y}(s)d\bar{K}(s) \leq 0$. Now, we obtain

$$\int_0^{\tau_n} e^{\beta s}|\bar{Z}(s)|^2 ds \leq (\gamma-\beta)\int_0^{\tau_n} e^{\beta s}|\bar{Y}(s)|^2 ds + \frac{L\tilde{\alpha}}{\gamma}\left[T\sup_{0\leq t\leq T} e^{\beta t}|\bar{U}(t)|^2 + \int_0^{\tau_n} e^{\beta s}|\bar{V}(s)|^2 ds\right] + e^{\beta\tau_n}|\bar{Y}_{\tau_n}|^2 + 2\left|\int_0^{\tau_n} e^{\beta s}\bar{Y}(s)\bar{Z}(s)dW(s)\right|.$$

Since $\beta - \gamma > 0$, it follows that

$$\left(\int_0^{\tau_n} e^{\beta s}|\bar{Z}(s)|^2 ds\right)^{p/2} \leq 2^{(p-1)}\left(\frac{L\tilde{\alpha}}{\gamma}\right)^{p/2}\left[T^{p/2}\sup_{0\leq t\leq T} e^{\beta pt/2}|\bar{U}(t)|^p + \left(\int_0^{\tau_n} e^{\beta s}|\bar{V}(s)|^2 ds\right)^{p/2}\right] + 2^{(p-1)}\left(e^{\beta\tau_n}|\bar{Y}_{\tau_n}|^2\right)^{p/2} + 2^{(p-1)}\cdot 2^{p/2}\left|\int_0^{\tau_n} e^{\beta s}\bar{Y}(s)\bar{Z}(s)dW(s)\right|^{p/2}.$$

By the Burkholder-Davis-Gundy inequality we have

$$\mathbb{E}\left|\int_0^{\tau_n} e^{\beta s}\bar{Y}(s)\bar{Z}(s)dW(s)\right|^{p/2} \leq \bar{c}_p^2 2^{3p/2}\mathbb{E}\left(\sup_{0\leq t\leq T} e^{\beta pt/2}|\bar{Y}(t)|^p\right) + 2^{-3p/2}\mathbb{E}\left(\int_0^{\tau_n} e^{\beta s}|\bar{Z}(s)|^2 ds\right)^{p/2},$$

where $\bar{c}_p > 0$ is a constant depending on p. Therefore plugging this inequality in the previous one, we obtain

$$\frac{1}{2}\mathbb{E}\left(\int_0^{\tau_n} e^{\beta s}|\bar{Z}(s)|^2 ds\right)^{p/2} \leq 2^{(p-1)}\left(\frac{L\tilde{\alpha}}{\gamma}\right)^{p/2}\mathbb{E}\left[T^{p/2}\sup_{0\leq t\leq T} e^{\beta t}|\bar{U}(t)|^p + \left(\int_0^{\tau_n} e^{\beta s}|\bar{V}(s)|^2 ds\right)^{p/2}\right] + 2^{(p-1)}\mathbb{E}e^{\beta\tau_n}|\bar{Y}_{\tau_n}|^p + \bar{c}_p^2 2^{(3p-1)}\mathbb{E}\left(\sup_{0\leq t\leq T} e^{\beta t}|\bar{Y}(t)|^p\right).$$

Next using the Fatou Lemma yields that

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left(\int_0^T e^{\beta s}|\bar{Z}(s)|^2 ds\right)^{p/2}\\
&\le 2^{(p-1)}\Big(\frac{L\tilde{\alpha}}{\gamma}\Big)^{p/2}\mathbb{E}\left[T^{p/2}\sup_{0\le t\le T} e^{\beta t}|\bar{U}(t)|^p + \left(\int_0^T e^{\beta s}|\bar{V}(s)|^2 ds\right)^{p/2}\right]\\
&\quad + \bar{c}_p^2 2^{(3p-1)}\mathbb{E}\left(\sup_{0\le t\le T} e^{\beta t}|\bar{Y}(t)|^p\right).
\end{aligned}$$

Using (3.21) and choosing β great enough, we obtain

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T e^{\beta s}|\bar{Z}(s)|^2 ds\right)^{p/2}\\
&\le 2^{(p-1)}\Big(\frac{L\tilde{\alpha}}{\gamma}\Big)^{p/2}\mathbb{E}\left[T^{p/2}\sup_{0\le t\le T} e^{\beta t}|\bar{U}(t)|^p + \left(\int_0^T e^{\beta s}|\bar{V}(s)|^2 ds\right)^{p/2}\right]\\
&\quad + \bar{c}_p^2 2^{(3p-1)}\hat{C}\left(\mathbb{E}\sup_{0\le t\le T} e^{\beta t}|\bar{U}(t)|^p + \mathbb{E}\left(\int_0^T e^{\beta s}|\bar{V}(s)|^2 ds\right)^{p/2}\right)\\
&\le \check{C}\left(\mathbb{E}\left(\sup_{0\le t\le T} e^{\beta t}|\bar{U}(t)|^p\right) + \mathbb{E}\left(\int_0^T e^{\beta s}|\bar{V}(s)|^2 ds\right)^{p/2}\right),
\end{aligned}$$

where $\check{C} = 2^{(p-1)}\left(\frac{L\tilde{\alpha}}{\gamma}\right)^{p/2}\max(T^{p/2},1) + \bar{c}_p^2 2^{(3p-1)}\hat{C}$. Since L and T are small enough, we will have $\check{C} < 1/2$.

Hence, the mapping Φ is a strict contraction on $\mathcal{L}$ and therefore it has a unique fixed point, which in combination with the associated K is the unique solution of the RBSDE with time delayed generator. □

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