

GORENSTEIN DIMENSIONS OF UNBOUNDED COMPLEXES UNDER BASE CHANGE

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ABSTRACT. Transfer of homological properties under base change is a classical field of study. Let $R \rightarrow S$ be a ring homomorphism. The relations of Gorenstein projective (or Gorenstein injective) dimensions of unbounded complexes between $U \otimes_R^L X$ (or $\mathbf{R}\mathrm{Hom}_R(X, U)$) and X are considered, where X is an R -complex and U is an S -complex. In addition, some sufficient conditions are given under which the equalities $\mathrm{G}\text{-dim}_S(U \otimes_R^L X) = \mathrm{G}\text{-dim}_R X + \mathrm{pd}_S U$ and $\mathrm{Gid}_S(\mathbf{R}\mathrm{Hom}_R(X, U)) = \mathrm{G}\text{-dim}_R X + \mathrm{id}_S U$ hold.

Introduction

It is well known that Gorenstein homological dimensions are refinements of the classical homological dimensions. In a different direction, homological dimensions have been extended to complexes. Avramov and Foxby [2] defined projective, injective and flat dimensions for unbounded complexes of left modules over associative rings. The concepts of Gorenstein projective, injective and flat dimension for homologically bounded complexes were introduced by Christensen [3]. Veliche [10] extended the concept of Gorenstein projective dimension of homologically bounded complexes to the setting of unbounded complexes over associative rings. Dually, Asadollahi and Salarian [1] introduced the concept of Gorenstein injective dimension of unbounded complexes over associative rings.

Transfer of homological properties along ring homomorphisms is a classical field of study (see, for instance, [9] and its references). Let R and S be commutative Noetherian rings. It was shown in [9] that if $\varphi : R \rightarrow S$ is a ring homomorphism such that every S -module of finite flat dimension is of finite projective dimension over R via φ , X is an R -complex and U is an S -complex with finite projective dimension, then one has $\mathrm{Gid}_S(\mathbf{R}\mathrm{Hom}_R(U, X)) \leq \mathrm{Gid}_R X + \mathrm{pd}_S U$. Also a sufficient condition is given under which the equality $\mathrm{Gid}_S(\mathbf{R}\mathrm{Hom}_R(S, X)) = \mathrm{Gid}_R X$ holds. Let R be a commutative Noetherian

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local ring. In the absolute case, $\varphi = 1_R$, some sufficient conditions are given via vanishing of Tate (co)homology for the equalities $\text{G-dim}_R(M \otimes_R^{\mathbf{L}} N) = \text{G-dim}_R M + \text{G-dim}_R N$ and $\text{Gid}_R(\mathbf{RHom}_R(M, N)) = \text{G-dim}_R M + \text{Gid}_R N$ to hold, where M and N are finite (finite generated) modules, see [6, Theorem 3.5] and [12, Theorem 3.2].

In this paper, we study Gorenstein projective (injective) dimensions of unbounded complexes under base change. Let R and S be commutative Noetherian rings. Our main results state as follows.

Theorem A. *Let $\varphi : R \rightarrow S$ be a ring homomorphism such that every S -module of finite flat dimension is of finite projective dimension over R via φ . If X is an R -complex and U is an S -complex with finite projective dimension, then the following inequality holds,*

$$\text{Gpd}_S(U \otimes_R^{\mathbf{L}} X) \leq \text{Gpd}_R X + \text{pd}_S U.$$

Let $\varphi : R \rightarrow S$ be a ring homomorphism. Recall that φ is faithfully flat if S is a faithfully flat R -module and φ is module-finite if S is a finite R -module.

Theorem B. *Let $\varphi : R \rightarrow S$ be a faithfully flat and module-finite ring homomorphism. If $\dim S$ is finite, then for every R -complex X the following equality holds,*

$$\text{Gpd}_S(S \otimes_R^{\mathbf{L}} X) = \text{Gpd}_R X.$$

Let $\varphi : R \rightarrow S$ be a ring homomorphism. Recall that the flat dimension of φ is the flat dimension of S considered as a module over R with the action given by φ .

Theorem C. *Let $\varphi : R \rightarrow S$ be a ring homomorphism of finite flat dimension such that $\dim R$ is finite. If X is an R -complex and U is an S -complex with finite injective dimension, then the following inequality holds,*

$$\text{Gid}_S(\mathbf{RHom}_R(X, U)) \leq \text{Gpd}_R X + \text{id}_S U.$$

Note that the techniques of proofs in the following used in Lemma 2.1, Theorem 2.3 and Theorem 3.2 are modified from [9].

1. Preliminaries

Throughout this paper, R and S are commutative Noetherian rings and an R -complex is a complex of R -modules. The derived category is written $\mathcal{D}(R)$. A complex

$$X : \cdots \longrightarrow X_{i+1} \xrightarrow{\partial_{i+1}^X} X_i \xrightarrow{\partial_i^X} X_{i-1} \longrightarrow \cdots$$

is called *acyclic* if the homology complex $H(X)$ is the zero-complex. We use the notations $Z_i(X)$ for the kernel of differential ∂_i^X and $C_i(X)$ for the cokernel of the differential ∂_{i+1}^X . The projective, injective and flat dimensions of X are abbreviated as $\text{pd}_R X$, $\text{id}_R X$ and $\text{fd}_R X$. The full subcategories $\mathcal{P}(R)$ and $\mathcal{I}(R)$

of $\mathcal{D}(R)$ consist of complexes of finite projective and injective dimensions. We use the superscript f to denote finite homology.

The notations $\sup X$ and $\inf X$ stand for the supremum and infimum of the set $\{i \in \mathbb{Z} \mid H_i(X) \neq 0\}$, with the conventions $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. A complex X is called *homologically bounded above* if $\sup X$ is finite, it is called *homologically bounded below* if $\inf X$ is finite, and it is called *homologically bounded* if it is homologically bounded above and below. The full subcategories $\mathcal{D}_{\square}(R)$ and $\mathcal{D}_{\square}(R)$ consist of complexes X with, respectively, $\sup X < \infty$ and $\inf X > -\infty$. An R -complex X is said to be *bounded above* if there is an integer u such that $X_i = 0$ for all $i > u$, similarly X is *bounded below* if there is an integer v such that $X_i = 0$ for all $i < v$, and it is called *bounded* if it is bounded above and below.

1.1 (Resolutions). A morphism of R -complexes that induces an isomorphism in homology is called a *quasi-isomorphism* (denoted by “ \simeq ”). An R -complex P is called *semi-projective* if each module P_i is projective, and the functor $\text{Hom}_R(P, -)$ preserves quasi-isomorphisms (equivalently, it preserves acyclicity). Every bounded below complex of projective modules is semi-projective. The definition of semi-injective can be defined dually. Note that in [2] the authors use “DG-projective (DG-injective)” in place of “semi-projective (semi-injective)”. Every R -complex M has a semi-projective resolution. That is, there is a quasi-isomorphism $\pi: P \rightarrow M$, where P is a semi-projective complex with $P_i = 0$ for all $i < \inf M$. Dually, every R -complex M has a semi-injective resolution. We use the standard notations $\mathbf{R}\text{Hom}_R(-, -)$ and $- \otimes_R^{\mathbf{L}} -$ for the derived Hom and derived tensor product of complexes.

1.2 (Complete projective resolutions). An acyclic complex T of projective R -modules is called *totally acyclic*, if the complex $\text{Hom}_R(T, Q)$ is acyclic for every projective R -module Q . An R -module G is called *Gorenstein projective* if there exists such a totally acyclic complex T with $C_0(T) \cong G$. Let X be an R -complex. A *complete projective resolution* of X is a diagram

$$(1.2.1) \quad T \xrightarrow{\tau} P \xrightarrow{\pi} X,$$

where π is a semi-projective resolution, T is a totally acyclic complex of projective R -modules, and τ_i is an isomorphism for $i \gg 0$. The *Gorenstein projective dimension* of X , written $\text{Gpd}_R X$, is the least integer n such that there exists a complete projective resolution (1.2.1) where τ_i is an isomorphism for all $i \geq n$. Note that $\text{Gpd}_R X$ is finite if and only if X has a complete projective resolution. Notice that for homologically bounded below complexes the definition of Gorenstein projective dimension coincides with that of Christensen.

1.3 (Complete injective resolutions). A complex U of injective R -modules is called *totally acyclic* if it is acyclic, and the complex $\text{Hom}_R(J, U)$ is acyclic for every injective R -module J . A *complete injective resolution* of an R -complex

Y is a diagram

$$(1.3.1) \quad Y \xrightarrow{\iota} I \xrightarrow{v} U,$$

where ι is a semi-injective resolution, U is a totally acyclic complex of injective R -modules, and v_i is an isomorphism for $i \ll 0$. An R -module E is called *Gorenstein injective* if there exists a totally acyclic complex U of injective R -modules with $Z_0(U) \cong E$. The *Gorenstein injective dimension* of an R -complex Y , written $\text{Gid}_R Y$, is the least integer n such that there exists a complete injective resolution (1.3.1) where v_i is an isomorphism for all $i \leq -n$. In particular, $\text{Gid}_R Y$ is finite if and only if Y has a complete injective resolution. Notice that for homologically bounded above complexes the definition of Gorenstein injective dimension coincides with that of Christensen.

1.4 (Depth). Let (R, \mathfrak{m}, k) be a local ring. The *depth* of an R -complex X is defined as

$$\text{depth}_R X = -\sup \mathbf{R}\text{Hom}_R(k, X).$$

1.5 (Width). Let (R, \mathfrak{m}, k) be a local ring. The *width* of an R -complex X is defined as

$$\text{width}_R X = \inf (k \otimes_R^{\mathbf{L}} X).$$

By [8, 1.5.(1)], for every R -complex X there is an inequality

$$(1.5.1) \quad \text{width}_R X \geq \inf X,$$

and equality holds if X is homologically bounded below with $H(X)$ finite by Nakayama's lemma.

2. Gorenstein projective dimensions

In this section, the Gorenstein projective dimension of unbounded complexes is considered. First we list the following lemmas for later use.

Lemma 2.1. *If T is a homologically trivial R -complex of projective modules and X is a bounded below R -complex of modules with finite projective dimension, then $X \otimes_R T$ is homologically trivial.*

Proof. It is clear that for each projective module P , $P \otimes_R T$ is homologically trivial. Suppose that for $n \geq 1$ and for each module N with $\text{pd}_R N \leq n - 1$, $N \otimes_R T$ is homologically trivial. Let M be a module with $\text{pd}_R M = n$. There exists an exact sequence $0 \rightarrow N \rightarrow P \rightarrow M \rightarrow 0$ with P projective and $\text{pd}_R N = n - 1$. Since every T_l is projective for every $l \in \mathbb{Z}$, one has the following exact sequence of complexes

$$0 \rightarrow N \otimes_R T \rightarrow P \otimes_R T \rightarrow M \otimes_R T \rightarrow 0.$$

Since $N \otimes_R T$ and $P \otimes_R T$ are homologically trivial, so is the complex $M \otimes_R T$. It follows from induction that $M \otimes_R T$ is homologically trivial for any module M of finite projective dimension.

Consequently, if X is a bounded below complex of modules with finite projective dimension, then $X_l \otimes_R T$ is homologically trivial for every $l \in \mathbb{Z}$. Therefore, $X \otimes_R T$ is homologically trivial by [4, Lemma 2.13]. \square

Similarly, one could obtain the next result by using [4, Lemma 2.5].

Lemma 2.2. *If T is a totally acyclic R -complex of projective modules and X is a bounded above R -complex of modules with finite projective dimension, then $\text{Hom}_R(T, X)$ is homologically trivial.*

Theorem 2.3. *Let $\varphi : R \rightarrow S$ be a ring homomorphism such that every S -module of finite flat dimension is of finite projective dimension over R via φ . If $X \in \mathcal{D}(R)$ and $U \in \mathcal{P}(S)$, then the following inequality holds,*

$$\text{Gpd}_S(U \otimes_R^{\mathbf{L}} X) \leq \text{Gpd}_R X + \text{pd}_S U.$$

Proof. We can assume that U and X are homologically non-trivial, otherwise the inequality is trivial. The inequality is also trivial if X is not of finite Gorenstein projective dimension. We set $\text{Gpd}_R X = g \in \mathbb{Z}$. There exists a complete projective resolution $T \xrightarrow{\tau} F \rightarrow X$ where $F \rightarrow X$ is a semi-projective resolution of X , T is a totally acyclic complex of projective R -modules and τ_i is bijective for all $i \geq g$. Since $U \in \mathcal{P}(S)$, there exists a bounded complex P of projective S -modules such that $U \simeq P$ and $P_l = 0$ when $l > u = \text{pd}_S U$ or $l < \inf U$. It is easy to see that U and P are quasi-isomorphic as complexes of R -modules.

Note that for every $l \in \mathbb{Z}$,

$$(P \otimes_R F)_l = \bigoplus_{t \in \mathbb{Z}} P_t \otimes_R F_{l-t}$$

is a projective S -module. For any homologically trivial S -complex E , one has $\text{Hom}_S(P, E)$ is a homologically trivial complex of S -modules as P is semi-projective. Hence $\text{Hom}_S(P, E)$ is homologically trivial as a complex of R -modules. Now adjunction yields

$$\text{Hom}_S(F \otimes_R P, E) \cong \text{Hom}_R(F, \text{Hom}_S(P, E)).$$

Since F is semi-projective, it follows that $\text{Hom}_R(F, \text{Hom}_S(P, E))$ is homologically trivial. Therefore, $P \otimes_R F$ is a semi-projective complex of S -modules. Since F is semi-projective, one has

$$U \otimes_R^{\mathbf{L}} X = U \otimes_R F \simeq P \otimes_R F.$$

Consequently, there is a quasi-isomorphism

$$P \otimes_R F \longrightarrow U \otimes_R^{\mathbf{L}} X$$

by the semi-projectivity of $P \otimes_R F$.

Next we will show that $P \otimes_R T$ is a totally acyclic complex of projective S -modules. Firstly from the assumption that every S -module of finite flat dimension is of finite projective dimension over R via φ , it follows that $\text{pd}_R(P_l)$

is finite for all $l \in \mathbb{Z}$. By Lemma 2.1, one has $P \otimes_R T$ is homologically trivial. Secondly, for every $l \in \mathbb{Z}$, one has

$$(P \otimes_R T)_l = \bigoplus_{t \in \mathbb{Z}} P_t \otimes_R T_{l-t}$$

is a projective S -modules. Finally, assume that Q is any projective S -module. Since P is a bounded S -complex, the S -complex $\text{Hom}_S(P, Q)$ is in particular a bounded R -complex. For any $l \in \mathbb{Z}$, one has $(\text{Hom}_S(P, Q))_l = \text{Hom}_S(P_{-l}, Q)$ is a flat S -module. It follows that $\text{Hom}_R(T, \text{Hom}_S(P, Q))$ is homologically trivial by Lemma 2.2. By adjointness, one has

$$\text{Hom}_S(T \otimes_R P, Q) \cong \text{Hom}_R(T, \text{Hom}_S(P, Q)).$$

Therefore, $P \otimes_R T$ is a totally acyclic complex of projective S -modules.

For any $l \in \mathbb{Z}$ with $l \geq u + g$, one has

$$\begin{aligned} (P \otimes_R T)_l &= \bigoplus_{k \in \mathbb{Z}} P_k \otimes_R T_{l-k} \\ &= \bigoplus_{u \geq k \in \mathbb{Z}} P_k \otimes_R T_{l-k} \\ &\cong \bigoplus_{u \geq k \in \mathbb{Z}} P_k \otimes_R F_{l-k} \\ &= \bigoplus_{k \in \mathbb{Z}} P_k \otimes_R F_{l-k} \\ &= (P \otimes_R F)_l. \end{aligned}$$

Now from the complete projective resolution

$$P \otimes_R T \xrightarrow{P \otimes_R \tau} P \otimes_R F \longrightarrow U \otimes_R^{\mathbf{L}} X$$

it follows that

$$\text{Gpd}_S(U \otimes_R^{\mathbf{L}} X) \leq g + u = \text{Gpd}_R X + \text{pd}_S U. \quad \square$$

Let R be a local ring. The following equality is well-known as the Auslander-Buchsbaum formula; it holds for any $X \in \mathcal{P}^f(R)$,

$$(2.1.1) \quad \text{pd}_R X = \text{depth } R - \text{depth}_R X.$$

For homologically bounded below complex with finite homology, for finite modules in particular, the Gorenstein projective dimension coincides with Auslander and Bridger’s notion of G-dimension; see [4, Proposition 3.8]. The next equality is known as the Auslander-Bridger formula; it holds for every homologically bounded below complex X with $H(M)$ finite and $\text{G-dim}_R X$ finite,

$$(2.1.2) \quad \text{G-dim}_R X = \text{depth } R - \text{depth}_R X.$$

Let (R, \mathfrak{m}, k) and (S, \mathfrak{n}, l) be local rings. Recall that a homomorphism of rings $\varphi : R \rightarrow S$ is said to be local if $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$.

Corollary 2.4. *Let $\varphi : R \rightarrow S$ be a local ring homomorphism of finite flat dimension. If $X \in \mathcal{D}_{\square}^f(R)$ with $\text{G-dim}_R X$ finite and $U \in \mathcal{P}^f(S)$, then the following equality holds,*

$$\text{G-dim}_S(U \otimes_R^{\mathbf{L}} X) = \text{G-dim}_R X + \text{pd}_S U.$$

Proof. By Theorem 2.3, one has $\text{G-dim}_S(U \otimes_R^{\mathbf{L}} X)$ is finite. By hypothesis, it is not hard to see that $U \otimes_R^{\mathbf{L}} X \in \mathcal{D}_{\square}^f(S)$ and $U \in \mathcal{P}(R)$. Since $\text{G-dim}_R X$ is finite, the complex X is homologically bounded above. Now the first equality in the computation below follows from the Auslander-Bridger formula (2.1.2), the second follows by [5, Theorem 6.2(i)] and the last follows from the Auslander-Bridger formula (2.1.2) and the Auslander-Buchsbaum formula (2.1.1).

$$\begin{aligned} \text{G-dim}_S(U \otimes_R^{\mathbf{L}} X) &= \text{depth } S - \text{depth}_S(U \otimes_R^{\mathbf{L}} X) \\ &= \text{depth } S - \text{depth}_S U - \text{depth}_R X + \text{depth } R \\ &= \text{G-dim}_R X + \text{pd}_S U. \end{aligned} \quad \square$$

Corollary 2.5. *Let $\varphi : R \rightarrow S$ be a ring homomorphism of finite flat dimension and let $X \in \mathcal{D}(R)$ be a complex. If $\dim R$ is finite, then the following equality holds,*

$$\text{Gpd}_S(S \otimes_R^{\mathbf{L}} X) \leq \text{Gpd}_R X.$$

Proof. Note that under the condition that $\varphi : R \rightarrow S$ is a ring homomorphism of finite flat dimension and $\dim R$ is finite, one has every S -module of finite flat dimension is of finite projective dimension over R via φ . Now the result follows from Theorem 2.3. □

Next we consider when the equality in Corollary 2.5 holds. To this end we need the next two lemmas.

Lemma 2.6. ([11, Lemma 2.2]) *Let $\varphi : R \rightarrow S$ be a module-finite faithfully flat ring homomorphism. If P is a projective R -module, then it is a direct summand (as an R -module) of the projective S -module $S \otimes_R P$.*

Lemma 2.7. ([11, Lemma 2.3]) *Let $\varphi : R \rightarrow S$ be a faithfully flat ring homomorphism. If $\dim R$ is finite, then an R -module M is Gorenstein projective if and only if $S \otimes_R M$ is a Gorenstein projective S -module and $\text{Ext}_R^i(M, P) = 0$ for all projective R -module P and all $i \geq 1$.*

Theorem 2.8. *Let $\varphi : R \rightarrow S$ be a module-finite faithfully flat ring homomorphism. If $\dim S$ is finite, then for every R -complex X there is an equality,*

$$\text{Gpd}_R X = \text{Gpd}_S(S \otimes_R^{\mathbf{L}} X).$$

Proof. Since $\varphi : R \rightarrow S$ is a faithfully flat ring homomorphism and $\dim S$ is finite, one has $\dim R$ is finite. Hence by Theorem 2.3 it is enough to show that $\text{Gpd}_R X \leq \text{Gpd}_S(S \otimes_R^{\mathbf{L}} X)$.

Assume that $\text{Gpd}_S(S \otimes_R^{\mathbf{L}} X) = g \in \mathbb{Z}$. By [10, Theorem 3.4], it follows that $\text{sup}(S \otimes_R^{\mathbf{L}} X) \leq g$ and for every semi-projective resolution $F \rightarrow S \otimes_R^{\mathbf{L}} X$ the

module $C_g(F)$ is a Gorenstein projective S -module. Next we show that the complex X is homologically bounded above.

Consider a semi-projective resolution $P \rightarrow X$ over R . One has

$$S \otimes_R^{\mathbf{L}} X = S \otimes_R P.$$

Obviously, $S \otimes_R P$ is a complex of projective S -modules. For every homologically trivial S -complex E , by adjunction, the complex

$$\mathrm{Hom}_S(P \otimes_R S, E) \cong \mathrm{Hom}_R(P, E)$$

is homologically trivial since P is a semi-projective R -complex and E is homologically trivial as an R -complex. Therefore, $P \otimes_R S$ is a semi-projective resolution of $S \otimes_R^{\mathbf{L}} X$ and so $\mathrm{sup}(P \otimes_R S) \leq g$. Hence the sequence

$$\cdots \rightarrow P_{g+2} \otimes_R S \rightarrow P_{g+1} \otimes_R S \rightarrow P_g \otimes_R S$$

is exact. Clearly, it is exact as a sequence of R -modules. Since S is a faithfully flat R -module, the sequence $\cdots \rightarrow P_{g+2} \rightarrow P_{g+1} \rightarrow P_g$ is exact. Consequently, one has $\mathrm{sup} P \leq g$ and so $\mathrm{sup} X \leq g$.

Next we use Lemma 2.7 to prove that $C_g(P)$ is a Gorenstein projective R -module. For $i > g$, one has $H_i(S \otimes_R P) = 0$. Right-exactness of the functor $S \otimes_R -$ yields an isomorphism $\mathrm{Coker} \partial_n^{S \otimes_R P} \cong S \otimes_R \mathrm{Coker} \partial_n^P$ for each n . Set $K = C_g(P)$. By [10, Theorem 3.4], one has the S -module $C_g(S \otimes_R P) \cong S \otimes_R K$ is Gorenstein projective. For every projective R -module P , one has P is a direct summand of a projective S -module Q by Lemma 2.6. Let P be a projective resolution of R -module K . For all $i \geq 1$, one has

$$\begin{aligned} \mathrm{Ext}_R^i(K, Q) &= H_{-i}(\mathrm{Hom}_R(P, Q)) \\ &= H_{-i}(\mathrm{Hom}_R(P, \mathrm{Hom}_S(S, Q))) \\ &= H_{-i}(\mathrm{Hom}_S(S \otimes_R P, Q)) \\ &= \mathrm{Ext}_S^i(S \otimes_R K, Q) \\ &= 0. \end{aligned}$$

Therefore, one has $\mathrm{Ext}_R^i(K, P) = 0$ and so K is a Gorenstein projective R -module. It follows from [10, Theorem 3.4] that $\mathrm{Gpd}_R X$ is finite.

To prove the equality of Gorenstein projective dimensions, choose a projective R -module Q such that $\mathrm{Gpd}_R X = -\inf \mathbf{RHom}_R(X, Q)$; see [10, Theorem 3.8]. By Lemma 2.6, one has Q is a direct summand of a projective S -module \tilde{Q} . Therefore, one has

$$\begin{aligned} \mathrm{Gpd}_S(S \otimes_R^{\mathbf{L}} X) &\geq -\inf \mathbf{RHom}_S(S \otimes_R^{\mathbf{L}} X, \tilde{Q}) \\ &= -\inf \mathbf{RHom}_R(X, \mathbf{RHom}_S(S, \tilde{Q})) \\ &= -\inf \mathbf{RHom}_R(X, \tilde{Q}) \\ &\geq -\inf \mathbf{RHom}_R(X, Q) \\ &= \mathrm{Gpd}_R X. \end{aligned}$$

The first step in the computation above is by [10, Theorem 3.8], the second one is from Hom-tensor adjointness, the fourth one follows from Q is a direct summand of a projective S -module \tilde{Q} and the last one comes from the choice of Q . This completes the proof. \square

Lemma 2.9 ([11, Theorem 2.6]). *Let $\varphi : R \rightarrow S$ be a ring homomorphism of finite flat dimension. Assume that $\dim R$ is finite. For every homologically bounded R -complex X there is an inequality $\text{Gpd}_R X \geq \text{Gpd}_S(S \otimes_R^{\mathbf{L}} X)$. If φ is faithfully flat and module-finite, then equality holds.*

For every homologically bounded R -complex X , we also have the next result.

Proposition 2.10. *Let $\varphi : R \rightarrow S$ be a ring homomorphism with $\dim R$ finite. For every homologically bounded R -complex X and each projective S -module N with $\text{fd}_R N$ finite, there is an inequality*

$$\text{Gpd}_R X \geq \text{Gpd}_S(N \otimes_R^{\mathbf{L}} X).$$

If φ is a module-finite faithfully flat local ring homomorphism and $X \in \mathcal{D}^f(R)$, then equality holds; in particular, the dimensions are simultaneously finite in this case.

Proof. Assume that X has finite Gorenstein projective dimension. There is an isomorphism $X \simeq P$ in $\mathcal{D}(R)$, where P is a bounded complex of Gorenstein projective R -modules. Since $\text{pd}_R N$ is finite, one has $N \otimes_R^{\mathbf{L}} X \simeq N \otimes_R P$ by [4, Corollary 2.16]. By [5, Ascent table II(b)], $N \otimes_R P$ is a bounded complex of Gorenstein projective S -modules. Therefore, there is an inequality $\text{Gpd}_R X \geq \text{Gpd}_S(N \otimes_R^{\mathbf{L}} X)$.

Now let φ be a module-finite faithfully flat local ring homomorphism. Assume that $M \in \mathcal{D}^f(R)$ and $\text{Gpd}_S(N \otimes_R^{\mathbf{L}} X)$ is finite. In particular, $\text{Gpd}_R X$ is finite by Lemma 2.9. Let Q be a projective S -module. One has

$$\begin{aligned} \text{Gpd}_S(N \otimes_R^{\mathbf{L}} X) &\geq -\inf \mathbf{R}\text{Hom}_S(N \otimes_R^{\mathbf{L}} X, Q) \\ &= -\inf \mathbf{R}\text{Hom}_R(X, N \otimes_S^{\mathbf{L}} Q) \\ &\geq -\text{width}_S \mathbf{R}\text{Hom}_R(X, N \otimes_S^{\mathbf{L}} Q) \\ &= \text{depth } R - \text{depth}_R X - \text{width}_S(N \otimes_S^{\mathbf{L}} Q) \\ &= \text{depth } R - \text{depth}_R X \\ &= \text{Gpd}_R X. \end{aligned}$$

The first inequality in the computation above follows from [4, Theorem 3.1] and the second is by (1.5.1). The first equality is by Hom-tensor adjointness, the second by [5, Theorem 6.3(ii)] as $N \otimes_S^{\mathbf{L}} Q = N \otimes_S Q$ is a projective S -module and the last follows from the Auslander-Bridger formula (2.1.2). \square

3. Gorenstein injective dimensions

In this section, we consider the Gorenstein injective dimension. By analogy with the proof of Lemma 2.1, one has the next result by using [4, Lemma 2.5].

Lemma 3.1. *If T is a homologically trivial R -complex of projective modules and X is a bounded above R -complex of modules with finite injective dimension, then $\text{Hom}_R(T, X)$ is homologically trivial.*

The next result could be compared with [9, Theorem 7].

Theorem 3.2. *Let $\varphi : R \rightarrow S$ be a ring homomorphism of finite flat dimension with $\dim R$ finite. If $X \in \mathcal{D}(R)$ and $U \in \mathcal{I}(S)$, then the following inequality holds,*

$$\text{Gid}_S(\mathbf{R}\text{Hom}_R(X, U)) \leq \text{Gpd}_R X + \text{id}_S U.$$

Proof. We can assume that U and X are homologically non-trivial, otherwise the inequality is trivial. The inequality is also trivial if X is not of finite Gorenstein projective dimension. We set $\text{Gpd}_R X = g \in \mathbb{Z}$. There exists a complete resolution $T \xrightarrow{\tau} P \rightarrow X$ where $P \rightarrow X$ is a semi-projective resolution of X , T is a totally acyclic complex of projective R -modules and τ_i is bijective for all $i \geq g$. Since $U \in \mathcal{I}(S)$, there exists a bounded complex I of injective S -modules such that $U \simeq I$ and $I_l = 0$ when $l > \sup U$ or $l < -v$, where $v = \text{id}_S U$. It is easy to see that U and I are quasi-isomorphic as complexes of R -modules.

Note that for every $l \in \mathbb{Z}$,

$$(\text{Hom}_R(P, I))_l = \prod_{t \in \mathbb{Z}} \text{Hom}_R(P_t, I_{l+t})$$

is an injective S -module. For any homologically trivial S -complex E , one has $\text{Hom}_S(E, I)$ is a homologically trivial complex of S -modules as I is semi-injective. Hence $\text{Hom}_S(E, I)$ is homologically trivial as a complex of R -modules. Now swap yields

$$\text{Hom}_S(E, \text{Hom}_R(P, I)) \cong \text{Hom}_R(P, \text{Hom}_S(E, I)).$$

Since P is semi-projective, it follows that $\text{Hom}_S(E, \text{Hom}_R(P, I))$ is homologically trivial. Therefore, $\text{Hom}_R(P, I)$ is a semi-injective complex of S -modules. Since P is semi-projective, one has

$$\mathbf{R}\text{Hom}_R(X, U) = \text{Hom}_R(P, U) \simeq \text{Hom}_R(P, I).$$

Consequently, there is a quasi-isomorphism

$$\mathbf{R}\text{Hom}_R(X, U) \longrightarrow \text{Hom}_R(P, I)$$

by the semi-injectivity of $\text{Hom}_R(P, I)$.

Next we will show that $\text{Hom}_R(T, I)$ is a totally acyclic complex of injective S -modules. Firstly, by hypothesis, one has every S -module of finite injective dimension is of finite injective dimension over R via φ . It follows that $\text{id}_R(I_l)$ is finite for all $l \in \mathbb{Z}$. By Lemma 3.1, one has $\text{Hom}_R(T, I)$ is homologically trivial. Secondly, for every $l \in \mathbb{Z}$, one has

$$(\text{Hom}_R(T, I))_l = \prod_{t \in \mathbb{Z}} \text{Hom}_R(T_t, I_{l+t})$$

is an injective S -modules. Finally, assume that E is any injective S -module. Since I is a bounded S -complex, the S -complex $\text{Hom}_S(E, I)$ is in particular a bounded R -complex. For any $l \in \mathbb{Z}$, one has $(\text{Hom}_S(E, I))_l = \text{Hom}_S(E, I_l)$ is a flat S -module. Since φ is of finite flat dimension and $\dim R$ is finite, one has $\text{pd}_R \text{Hom}_S(E, I_l) < \infty$. It follows that $\text{Hom}_R(T, \text{Hom}_S(E, I))$ is homologically trivial by Lemma 2.2. By swap, one has

$$\text{Hom}_S(E, \text{Hom}_R(T, I)) \cong \text{Hom}_R(T, \text{Hom}_S(E, I)).$$

Therefore, $\text{Hom}_R(T, I)$ is a totally acyclic complex of injective S -modules.

For any $l \in \mathbb{Z}$ with $l \leq -v - g$, one has

$$\begin{aligned} (\text{Hom}_R(T, I))_l &= \prod_{k \in \mathbb{Z}} \text{Hom}_R(T_k, I_{l+k}) \\ &= \prod_{-v-l \leq k \in \mathbb{Z}} \text{Hom}_R(T_k, I_{l+k}) \\ &\cong \prod_{-v-l \leq k \in \mathbb{Z}} \text{Hom}_R(P_k, I_{l+k}) \\ &= \prod_{k \in \mathbb{Z}} \text{Hom}_R(P_k, I_{l+k}) \\ &= (\text{Hom}_R(P, I))_l. \end{aligned}$$

Now from the complete injective resolution

$$\mathbf{R}\text{Hom}_R(X, U) \longrightarrow \text{Hom}_R(P, I) \xrightarrow{\text{Hom}_R(\tau, I)} \text{Hom}_R(T, I)$$

it follows that

$$\text{Gid}_S(\mathbf{R}\text{Hom}_R(X, U)) \leq v + g = \text{Gpd}_R X + \text{id}_S U. \quad \square$$

Let R be a local ring. The following equality is well-known as the Bass formula; it holds for any $X \in \mathcal{I}^f(R)$,

$$(3.1.1) \quad \text{id}_R X = \text{depth } R - \text{width}_R X.$$

The next equality is also known as the Bass formula for Gorenstein injective dimension; see [7, Corollary 2.3]. It holds for every homologically bounded above complex X with $H(X)$ finite and $\text{Gid}_R X$ finite,

$$(3.1.2) \quad \text{Gid}_R X = \text{depth } R - \text{width}_R X.$$

Corollary 3.3. *Let $\varphi : R \rightarrow S$ be a local ring homomorphism of finite flat dimension. If $X \in \mathcal{D}_{\square}^f(R)$ with $\text{G-dim}_R X$ finite and $U \in \mathcal{I}^f(S)$, then the following equality holds,*

$$\text{Gid}_S \mathbf{R}\text{Hom}_R(X, U) = \text{G-dim}_R X + \text{id}_S U.$$

Proof. By Theorem 3.2, one has $\text{Gid}_S \mathbf{R}\text{Hom}_R(X, U)$ is finite. By hypothesis, it is not hard to see that $\mathbf{R}\text{Hom}_R(X, U) \in \mathcal{D}_{\square}^f(S)$ and $U \in \mathcal{I}(R)$. Since $\text{G-dim}_R X$ is finite, the complex X is homologically bounded above. Now the first equality

in the computation below follows from the Bass formula (3.1.2), the second follows by [5, Theorem 6.2(ii)] and the last follows from the Auslander-Bridger formula (2.1.2) and the Bass formula (3.1.1).

$$\begin{aligned} \text{Gid}_S \mathbf{R}\text{Hom}_R(X, U) &= \text{depth } S - \text{width}_S \mathbf{R}\text{Hom}_R(X, U) \\ &= \text{depth } S - \text{depth}_R X + \text{depth } R - \text{width}_S U \\ &= \text{G-dim}_R X + \text{id}_S U. \end{aligned} \quad \square$$

Similarly one has the next result by using [9, Theorem 7].

Corollary 3.4. *Let $\varphi : R \rightarrow S$ be a local ring homomorphism of finite flat dimension. If $X \in \mathcal{D}_{\square}^f(R)$ with $\text{Gid}_R X$ finite and $U \in \mathcal{P}^f(S)$, then the following equality holds,*

$$\text{Gid}_S \mathbf{R}\text{Hom}_R(U, X) = \text{Gid}_R X + \text{pd}_S U.$$

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