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REMARKS ON NONTOPOLOGICAL SOLUTIONS IN THE SELF-DUAL CHERN-SIMONS GAUGED $O(3)$ SIGMA MODELS

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Abstract. In this paper, we prove the existence of nontopological solutions to the self-dual equations arising from the Chern-Simons gauged $O(3)$ sigma models. The property of solutions depends on a parameter $\tau \in [-1,1]$ appearing in the nonlinear term. The case $\tau = 1$ lies on the borderline for the existence of solutions in the previous results [4, 5, 7]. We prove the existence of solutions in this case when there are only vortex points. Moreover, if $-1 \le \tau < 1$, we establish solutions which are perturbed from the solutions of singular Liouville equations.

1. Introduction

In this paper we are interested in the following elliptic equation in $\mathbb{R}^2$:

$$\Delta u + \frac{1}{\kappa^2} f(u,\tau) = 4\pi \sum_{j=1}^{d_1} n_j \delta_{p_j} - 4\pi \sum_{j=1}^{d_2} m_j \delta_{q_j} \quad \text{in} \quad \mathbb{R}^2, \tag{1.1}$$

where u is a real-valued function, δ_p stands for the Dirac measure concentrated at p, $\kappa > 0$ is a constant, $n_j, m_j \in \mathbb{N}$, and

$$f(u,\tau) = \frac{e^u\left[(1-\tau) - (1+\tau)e^u\right]}{(1+e^u)^3}, \qquad |\tau| \le 1.$$

Moreover, $\mathcal{P} = \{p_1, p_2, \ldots, p_{d_1}\}$ and $\mathcal{Q} = \{q_1, q_2, \ldots, q_{d_2}\}$ are disjoint sets of distinct points in $\mathbb{R}^2$. The point p_j (q_j, respectively) is called a vortex point of positive (negative, respectively) mass n_j (m_j, respectively). We set

$$N = n_1 + \cdots + n_{d_1}, \quad M = m_1 + \cdots + m_{d_2}.$$

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The equation (1.1) comes from the self-dual Chern-Simons gauged $O(3)$ sigma model in $\mathbb{R}^2$. Classically, the $O(3)$ sigma model describes the planar ferromagnet and its gauged models have been widely studied in recent years. Especially, the $O(3)$ sigma model which is gauged by the Chern-Simons interaction is believed relevant to the planar condensed matter systems where a charge-flux composite obeying fractional statistics plays a major role. The self-dual potential gives a system of equations which produces the minimization of the static energy. This system can be reduced to the single elliptic equation (1.1). One may refer to [7, 13] for the physical model and the derivation of (1.1).

Due to the physical motivation, we have the finite energy condition: $f(u,\tau) \in L^1(\mathbb{R}^2)$. This condition gives us three kinds of boundary conditions near ∞:

$$\begin{aligned} &\text{nontopological BC of type I}: && u(x) \to -\infty, \\ &\text{nontopological BC of type II}: && u(x) \to +\infty, \\ &\text{topological BC}: && u(x) \to \ln \frac{1-\tau}{1+\tau} \quad \text{for} \quad |\tau| < 1, \end{aligned} \tag{1.2}$$

as $|x| \to \infty$. The first two conditions are applied to the case $|\tau| \le 1$, while the third condition is valid only for $|\tau| < 1$. A solution u of (1.1) is called a nontopological solution of type I if it satisfies the nontopological boundary condition of type I. Nontopological solutions of type II and topological solutions are similarly defined. The existence of topological solutions to (1.1) was proved in [9, 17] for arbitrary distribution of vortex points. It was also proved in [9] that the topological solution is unique for sufficiently small $\kappa > 0$.

In this paper, we are interested in nontopological solutions. We note that $f(-u,-\tau) = -f(u,\tau)$. Thus, if u is a nontopological solution of type I to (1.1), then $v = -u$ is a nontopological solution of type II to (1.1) with τ replaced by $-\tau$ and the change of roles of p_j and q_k. In view of this symmetry we may consider only nontopological solutions of type I to (1.1). *From now on, a nontopological solution means a solution of* (1.1) *with nontopological boundary condition of type I.* Finding nontopological solutions have been one of the main issues in the self-dual gauge field theories for the last two decades. It is interesting to compare the equation (1.1) with the Abelian Chern-Simons-Higgs vortex equation ([11, 12])

$$\Delta u = \frac{4}{\kappa^2} e^u (e^u - 1) + 4\pi \sum_{j=1}^{d} n_j \delta_{p_j}, \quad N = n_1 + \cdots + n_d \tag{1.3}$$

which is regarded as the simplest one among the self-dual vortex equations allowing nontopological boundary conditions. It is easy to check by the maximum principle that if u is a solution of (1.3), then $u < 0$. On the other hand, for the equation (1.1) we do not have such pointwise condition by the presence of vortex points of negative mass. Since the pointwise condition $u < 0$ is very powerful in various types of estimates, this gives a big difference of analysis between these two equations. Mathematically, it is a quite interesting to see how the results for (1.3) can be extended to (1.1) without appealing to the pointwise condition $u < 0$. In the following, we briefly review the existence results of nontopological solutions of (1.3) and their extension to (1.1), and then give the main result of this paper.

The first approach to obtain nontopological solutions for (1.3) was finding radial solutions for the case that $p_1 = \cdots = p_d = 0$ ([3, 15]). In particular, it was proved in [2] that for each $\beta > 4N+4$ there exist unique radially symmetric nontopological solution $U(x) = U(|x|)$ of (1.3) satisfying the flux relation

$$\beta(u) = \frac{2}{\pi\kappa^2}\int_{\mathbb{R}^2} e^u(1-e^u)dx. \tag{1.4}$$

As a generalization, one may ask whether there exists a nontopological solution of (1.3) satisfying the flux relation (1.4) for any given set of vortex points $\{p_1, \dots, p_d\}$ and $\beta > 4N+4$. The first result in this direction was established in [1] where the authors obtained one parameter family of nontopological solutions $u_\varepsilon(x)$ for small $\varepsilon > 0$ satisfying small flux condition $\beta(u_\varepsilon) = 4N+4+o(1)$ as $\varepsilon \to 0$. This is due to the nature of perturbation argument, a small perturbation of the corresponding Liouville equation by considering the equation of scaled solutions $u_\varepsilon(x) = u(x/\varepsilon)$. Another result was given in [2], where the solution was obtained by patching radial profiles $U(|x-p_j|/\kappa)$ near each vortex point p_j for small $\kappa > 0$. In this case, solutions exist for all large flux $\beta > 8N$ but the vortex points cannot be arbitrarily distributed due to the interaction of each bubble. The most general existence result is [8], where it was shown by the Leray-Schauder degree theory that for any distribution of vortex points and for any $\beta > 4N+4$ satisfying

$$\beta \notin \left\{\frac{4kN}{k-1} : k = 2, 3, \dots, N\right\},$$

(1.3) allows a nontopological solution u with $\beta = \beta(u)$.

Returning to the equation (1.1), it is very interesting to study the structure of nontopological solutions by comparing (1.1) with (1.3). The main difference between these equations is that (1.1) allows singular sources with negative mass which make the nonlinear term change signs. As in the case (1.3), the quantity

$$\beta(u) = \frac{1}{2\pi\kappa^2}\int_{\mathbb{R}^2} f(u,\tau)dx$$

is important in finding solutions. To see this, let $v = u - u_0$, where

$$u_0(x) = \sum_{j=1}^{d_1} n_j \ln|x-p_j|^2 - \sum_{j=1}^{d_2} m_j \ln|x-q_j|^2. \tag{1.5}$$

Then, $\kappa^2\Delta v = -f(u_0+v,\tau) \in L^\infty(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)$. From the standard argument, it follows that

$$\lim_{|x|\to\infty}\frac{v(x)}{\ln|x|} = -\frac{1}{2\pi\kappa^2}\int_{\mathbb{R}^2} f(u_0+v,\tau)dx = -\beta(u).$$

As a consequence, u enjoys the following behavior:

$$u(x) = (2N - 2M - \beta)\ln|x| + o(\ln|x|) \quad \text{as} \quad |x| \to \infty. \tag{1.6}$$

By the integrability condition $f(u,\tau) \in L^1(\mathbb{R}^2)$, a necessary condition is

$$\begin{aligned} &\beta \geq 2N - 2M + 2 \quad \text{for} \quad -1 \leq \tau < 1, \\ &\beta \geq 2N - 2M + 1 \quad \text{for} \quad \tau = 1. \end{aligned} \tag{1.7}$$

The exact range of β, denoted by Λ, can be obtained by the Pohozaev identity which may depends on the values of N, M, and τ. The main problem is to construct a solution u of (1.1) for any prescribed sets $\mathcal{P}$ and $\mathcal{Q}$ and any prescribed numbers $b \in \Lambda$ such that $\beta(u) = b$.

The first rigorous result for nontopological solutions of (1.1) comes from [5, 7] where the authors consider the radial solutions $u(x) = u(|x|)$ for the case $\mathcal{P} \cup \mathcal{Q} = \{0\}$. In this case, we have the following equation

$$\begin{cases} u''(r) + \dfrac{1}{r}u'(r) + \dfrac{1}{\kappa^2}f(u(r),\tau) = 0, \quad 0 < r < \infty \\ u(r) = 2\lambda \ln r + O(1) \quad \text{near} \quad r = 0, \end{cases} \tag{1.8}$$

where λ is an integer. To explain our main results, it is worthwhile to state the results of [5, 7] regarding to the existence of solutions as follows.

Theorem A ([5, 7]). (a) *Suppose that* $-1 \leq \tau < 1$.

(i) *If* $\lambda \geq 0$, *then* $\Lambda = (4\lambda + 4, \infty)$. *Conversely, for each* $\beta > 4\lambda + 4$, (1.8) *has a unique nontopological solution* $U = U(r)$ *which satisfies*

$$U(r) = (2\lambda - \beta)\ln r + I_{\lambda,\beta} + O(r^{2+2\lambda-\beta}) \quad \text{as} \quad r \to \infty, \tag{1.9}$$

and

$$\frac{1}{2\pi\kappa^2}\int_{\mathbb{R}^2} f(U,\tau)dx = \beta, \tag{1.10}$$

where $I_{\lambda,\beta}$ *is a constant.*

(ii) *If* $\lambda = -1$, *then* $\Lambda = (0, \infty)$ *such that for each* $\beta > 0$, (1.8) *has a unique nontopological solution* $u(r)$ *which satisfies* (1.9)-(1.10).

(iii) *If* $\lambda \leq -2$, *then* $(0, \infty) \subset \Lambda$ *such that for each* $\beta > 0$, (1.8) *has a unique nontopological solution* $u(r)$ *which satisfies* (1.9)-(1.10). *Moreover, for each* $\beta \in (2\lambda + 2, 0)$ *there exists* $\tau = \tau_*(\beta)$ *such that* (1.8) *has at least two solutions for each* $\tau \in (\tau_*, 1)$.

(b) *Suppose that* $\tau = 1$. *If* $\lambda \geq 0$, (1.8) *has no nontopological solutions* (*of type I*). *If* $\lambda \leq -1$, *then* $\Lambda = [1 + 2\lambda, 0)$. *For each* $1 + 2\lambda \leq \beta < 0$, (1.8) *has a unique nontopological solution* $U = U(r)$ *which satisfies* (1.10) *and*

$$U(r) = (2\lambda - \beta)\ln r + I_{\lambda,\beta} + O(r^{2+4\lambda-2\beta}) \quad \textit{for} \ \ 1 + 2\lambda < \beta < 0, \tag{1.11}$$

$$U(r) = -\ln r - \ln\ln r + O(1) \quad \textit{for} \ \ \beta = 1 + 2\lambda \tag{1.12}$$

as $r \to \infty$.

Theorem A shows that Λ can be different according to the values of N, M, and τ. For results in the nonradial case, one may find solutions by extending ideas for solving (1.3) to (1.1). The first result in this direction is [6], where the authors established bubbling solutions for small κ when $\mathcal{P}$ and $\mathcal{Q}$ satisfies some compatibility conditions for bubbling. Another result is [4] where the author proved the existence of nontopological solutions of (1.1) following the method of [8]. We state the result of [4] in the following.

Theorem B ([4])**.** (a) *Suppose that* $-1 \leq \tau < 1$ *and* $N \neq M$*. If* $\beta > \max\{0, 4N - 4M + 4\}$ *and*

$$\beta \notin \left\{ \frac{4k(N-M)}{k-1} : N > M,\ k = 2, 3, \ldots, N - M + 1 \right\}, \tag{1.13}$$

(1.1) *possesses a nontopological solution* u *with* $\beta = \beta(u)$.

(b) *Suppose that* $\tau = 1$ *and* $M \geq N + 1$*. If* $1 + 2N - 2M < \beta < 0$*, then there exists a nontopological solution* u *with* $\beta = \beta(u)$.

Comparing Theorem A and Theorem B or comparing (1.3) and (1.1), we are led to the following questions.

(1) Can we remove the condition (1.13)?
(2) Are there any nonradial nontopological solutions for the case $\tau = 1$ and $\beta = 1 + 2N - 2M$ as in the radial case in Theorem A? Do those solutions satisfy the asymptotic behavior (1.12)?
(3) Are there any nontopological solutions of (1.1) which are perturbed from the Liouville equation as in the case (1.3)?

In this paper, we provide some answers to the second and the third questions. First, we establish nonradial nontopological solutions for the case $\tau = 1$ and $\beta = 1 + 2N - 2M$ when $N = 0$. Moreover, our solutions enjoy the asymptotic behavior (1.12). So, our result gives a partial answer to the second question. Next, for $-1 \leq \tau < 1$ and $M = 0$, we construct solutions which are perturbed from the singular Liouville equations. This also gives a partial answer to the third question. We state our two main results as follows.

Theorem 1.1. *Suppose that* $\tau = 1$*. If* $M \geq 1$ *and* $N = 0$*, then* (1.1) *admits a nontopological solution* (*of type I*) *such that*

$$u(x) = -\ln|x| - \ln\ln|x| + O(1) \quad \text{as} \quad |x| \to \infty. \tag{1.14}$$

Theorem 1.2. *Suppose that* $M = 0$ *and* $-1 \leq \tau < 1$*. Then, there exists* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$, (1.1) *admits a nontopological solution* u_ε *satisfying that*

$$u_\varepsilon(x) = -\left\{2N + 4 + \varepsilon^2(c_0 + o(1))\right\}\ln(1 + |x|) + \ln\varepsilon^2 + O(1) \tag{1.15}$$

as $|x| \to \infty$*. Here,* c_0 *is a positive constant which is independent of* ε*. Furthermore,*

$$\frac{1}{2\pi\kappa^2}\int_{\mathbb{R}^2} f(u_\varepsilon, \tau)dx = 4N + 4 + \varepsilon^2(c_0 + o(1)). \tag{1.16}$$

In the next section, we prove Theorems 1.1 and 1.2.

2. Proof of Theorem 1.1

Let $\tau = 1$ and rewrite (1.1) as

$$\Delta u = \frac{2e^{2u}}{\kappa^2(1+e^u)^3} - 4\pi \sum_{j=1}^{d_2} m_j \delta_{q_j}, \qquad (2.1)$$
$$u \to -\infty \text{ as } |x| \to \infty.$$

We will use super- and sub-solution method following the argument of [10]. If $d_2 = 1$, we are done by Theorem A(b). Suppose that $d_2 \geq 2$ and let $M = m_1 + \cdots + m_{d_2} \geq 2$. By induction, we may assume that there exists a nontopological solution u_1 of

$$\Delta u_1 = \frac{2e^{2u_1}}{\kappa^2(1+e^{u_1})^3} - 8\pi \sum_{j=1}^{d_2-1} m_j \delta_{q_j},$$

which satisfies (1.14). We also define u_2 to be a solution of

$$\Delta u_2 = \frac{2e^{2u_2}}{\kappa^2(1+e^{u_2})^3} - 8\pi m_{d_2} \delta_{q_{d_2}},$$

which satisfies (1.14).

For a solution u of (2.1) and a number $\varepsilon \in (0,1)$, let $v_\varepsilon(x) = u(x/\varepsilon)$. Then, (2.1) becomes

$$\Delta v_\varepsilon = \frac{2e^{2v_\varepsilon}}{\kappa^2\varepsilon^2(1+e^{v_\varepsilon})^3} - 4\pi \sum_{j=1}^{d_2} m_j \delta_{q_j^\varepsilon},$$

where $q_j^\varepsilon = \varepsilon q_j$. Let

$$\eta_\varepsilon(x) = \sum_{j=1}^{d_2} m_j \ln\left(1 + |x - q_j^\varepsilon|^{-2}\right) \quad \text{and} \quad g_{1,\varepsilon}(x) = \sum_{j=1}^{d_2} \frac{4m_j}{\left(1+|x-q_j^\varepsilon|^2\right)^2}.$$

If we set $V_\varepsilon = v_\varepsilon - \eta_\varepsilon$, then

$$\Delta V_\varepsilon = \frac{2e^{2\eta_\varepsilon + 2V_\varepsilon}}{\kappa^2\varepsilon^2(1+e^{V_\varepsilon+\eta_\varepsilon})^3} - g_{1,\varepsilon}.$$

Let V_0 be a smooth function on $\mathbb{R}^2$ with $V_0(x) = -\ln|x| - \ln\ln|x|$ for $|x| \geq 2$ and $V_0 = 0$ for $|x| \leq 1$. If we put $V_\varepsilon = V_0 + w$, $K = e^{V_0}$ and $g_\varepsilon = g_{1,\varepsilon} + \Delta V_0$, we obtain

$$\Delta w = \frac{2K^2 e^{2\eta_\varepsilon + 2w}}{\kappa^2\varepsilon^2(1+Ke^{w+\eta_\varepsilon})^3} - g_\varepsilon. \qquad (2.2)$$

For small ε, we may assume that $\sup|q_j^\varepsilon|^2 < 1/36$. Let

$$\phi_\varepsilon = \sum_{j=1}^{d_2} \frac{2m_j}{1+|x-q_j^\varepsilon|^2}.$$

We claim that ϕ_ε is a bounded supersolution of (2.2) for all small ε. We note that

$$\Delta\phi_\varepsilon + g_\varepsilon = \sum_{j=1}^{d_2} \frac{4m_j(3|x-{q_j}^\varepsilon|^2-1)}{(1+|x-{q_j}^\varepsilon|^2)^3} + \Delta V_0.$$

First, suppose that $|x|^2 \le 1/12$. Since $3|x-{q_j}^\varepsilon|^2 \le 3(2|x|^2+2|{q_j}^\varepsilon|^2) \le 2/3$ and $V_0 = 0$. Then,

$$\Delta\phi_\varepsilon + g_\varepsilon < 0 \le \frac{2K^2 e^{2\eta_\varepsilon+2\phi_\varepsilon}}{\kappa^2\varepsilon^2(1+Ke^{\eta_\varepsilon+\phi_\varepsilon})^3}.$$

Next, suppose that $|x|^2 \ge 1/12$. Since $|x-{q_j}^\varepsilon|^2 \ge \frac{1}{2}|x|^2 - |{q_j}^\varepsilon|^2 \ge 1/72$, η_ε and ϕ_ε are uniformly bounded with respect to ε. We note that $K = (|x|\ln|x|)^{-1}$ and $\Delta V_0 = (|x|\ln|x|)^{-2}$ for $|x| \ge 2$. Hence, for all sufficiently small ε

$$\Delta\phi_\varepsilon + g_\varepsilon < \frac{2K^2 e^{2\eta_\varepsilon+2\phi_\varepsilon}}{\kappa^2\varepsilon^2(1+Ke^{\eta_\varepsilon+\phi_\varepsilon})^3}, \qquad \forall\ |x|^2 \ge 1/12.$$

and the claim follows.

Now, let us fix $\varepsilon \in (0,1)$ such that (2.2) allows a bounded supersolution w^+. For an appropriate subsolution of (2.2), we use u_1 and u_2. We change the equations of u_1 and u_2 into the regularized forms by letting

$$W_1(x) = u_1(x/\varepsilon) - \eta_{1,\varepsilon}(x) - V_0(x) \text{ and } W_2(x) = u_2(x/\varepsilon) - \eta_{2,\varepsilon}(x) - V_0(x),$$

where

$$\eta_{1,\varepsilon}(x) = \sum_{j=1}^{d_2-1} 2m_j \ln(1+|x-{q_j}^\varepsilon|^{-2}) \text{ and } \eta_{2,\varepsilon}(x) = 2m_{d_2}\ln(1+|x-{q_{d_2}}^\varepsilon|^{-2}).$$

Then, we have

$$\Delta W_1 = \zeta\big(x, \eta_{1,\varepsilon}(x)+W_1(x)\big) - h_{1,\varepsilon} \text{ and } \Delta W_2 = \zeta\big(x,\eta_{2,\varepsilon}(x)+W_2(x)\big) - h_{2,\varepsilon},$$

where $\zeta(x,t) = [2K^2(x)e^{2t}]/[\kappa^2\varepsilon^2(1+K(x)e^t)^3]$ for $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$ and

$$h_{1,\varepsilon} = \Delta V_0 + \sum_{j=1}^{d_2-1} \frac{8m_j}{(1+|x-{q_j}^\varepsilon|^2)^2} \quad \text{and} \quad h_{2,\varepsilon} = \Delta V_0 + \frac{8m_{d_2}}{(1+|x-{q_j}^\varepsilon|^2)^2}.$$

We observe that $\eta_\varepsilon = (\eta_{1,\varepsilon}+\eta_{2,\varepsilon})/2$ and $g_\varepsilon = (h_{1,\varepsilon}+h_{2,\varepsilon})/2$. Since K is bounded, there exists $\alpha \in \mathbb{R}$ such that for all $t < \alpha$

$$\frac{\partial\zeta}{\partial t} = \frac{2K^2e^{2t}(2-Ke^t)}{\kappa^2\varepsilon^2(1+Ke^t)^4} > 0 \quad \text{and} \quad \frac{\partial^2\zeta}{\partial t^2} = \frac{2K^2e^{2t}(K^2e^{2t}-7Ke^t+4)}{\kappa^2\varepsilon^2(1+Ke^t)^5} > 0.$$

Thus, for each $x \in \mathbb{R}^2$, $\zeta(x,\cdot)$ is increasing and convex with respect to t for $t < \alpha$. On the other hand, since u_k satisfies (1.14) by induction assumption, there exists $c_k < 0$ such that $w_k = W_k + c_k < \alpha$ for $k = 1,2$, which implies that

$$\Delta w_1 > \zeta\big(x,\eta_{1,\varepsilon}(x)+w_1(x)\big) - h_{1,\varepsilon} \text{ and } \Delta w_2 > \zeta\big(x,\eta_{2,\varepsilon}(x)+w_2(x)\big) - h_{2,\varepsilon}.$$

Let $w_0 = (w_1 + w_2)/2$. Then,

$$\Delta w_0 > \frac{1}{2}\Big\{\zeta\big(x, \eta_{1,\varepsilon} + w_1\big) + \zeta\big(x, \eta_{2,\varepsilon} + w_2\big)\Big\} - g_\varepsilon \geq \zeta(x, \eta_\varepsilon + w_0) - g_\varepsilon.$$

Since w_0 is bounded above, we can choose a constant $c_0 < 0$ such that $w^- = w_0 + c_0 < w^+$ on $\mathbb{R}^2$. Then w^- satisfies

$$\Delta w^- > \zeta\big(x, \eta_\varepsilon(x) + w^-(x)\big) - g_\varepsilon,$$

which implies that w^- is a subsolution of (2.2) with $w^- < w^+$. Now applying the method of super and subsolutions (e.g., see Theorem 2.10 of [14]), we get a bounded solution w of (2.2). Obviously, $u(x) = w(\varepsilon x) + \eta_\varepsilon(\varepsilon x) + V_0(\varepsilon x)$ becomes a solution of (2.1) and the proof is complete.

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We will construct solutions by reducing (1.1) as a perturbation of singular Liouville equations. This method was initiated from [1].

Let $M = 0$, $-1 \leq \tau < 1$ and rewrite (1.1) as

$$\Delta u + \frac{e^u\big[(1-\tau) - (1+\tau)e^u\big]}{\kappa^2(1+e^u)^3} = 4\pi \sum_{j=1}^{d_1} n_j \delta_{p_j}. \tag{3.1}$$

Set $u_\varepsilon(z) = u(z/\varepsilon) - \ln \varepsilon^2$. Then, we have

$$\Delta u_\varepsilon + \frac{e^{u_\varepsilon}\big[(1-\tau) - \varepsilon^2(1+\tau)e^{u_\varepsilon}\big]}{\kappa^2(1+\varepsilon^2 e^{u_\varepsilon})^3} = 4\pi \sum_{j=1}^{d_1} n_j \delta_{\varepsilon p_j}. \tag{3.2}$$

If ε is small enough, (3.2) can be regarded as a perturbation of the following Liouville problem:

$$\begin{aligned} &\Delta u_\varepsilon + \lambda e^{u_\varepsilon} = 4\pi \sum_{j=1}^{d_1} n_j \delta_{\varepsilon p_j}, \\ &\int_{\mathbb{R}^2} e^{u_\varepsilon} dx < \infty, \end{aligned} \tag{3.3}$$

where $\lambda = (1-\tau)/\kappa^2$.

Hereafter, we identify $\mathbb{R}^2$ and $\mathbb{C}$ by the relation $x = (x_1, x_2) \leftrightarrow z = x_1 + ix_2$. Let

$$g_\varepsilon(z) = (N+1)\prod_{j=1}^{d_1}(z - \varepsilon p_j)^{n_j} \quad \text{and} \quad G_\varepsilon(z) = \int_0^z g_\varepsilon(s)ds.$$

Then, it is well known that $\ln \rho_{\varepsilon,a}$ is a solution of (3.3), where $a \in \mathbb{C}$ and

$$\rho_{\varepsilon,a}(z) = \frac{8|g_\varepsilon(z)|^2}{\lambda(1 + |G_\varepsilon(z) + a|^2)^2}.$$

We set

$$u_\varepsilon(z) = \ln \rho_{\varepsilon,a}(z) + \varepsilon^2 w_\varepsilon(z). \tag{3.4}$$

Then w satisfies

$$\Delta w_\varepsilon = -\frac{(1-\tau)\rho_{\varepsilon,a}}{\kappa^2(1+\varepsilon^2\rho_{\varepsilon,a}e^{\varepsilon^2 w_\varepsilon})^3}\Big(\frac{e^{\varepsilon^2 w_\varepsilon}-1}{\varepsilon^2}\Big) + H_{\varepsilon,a}(w_\varepsilon), \tag{3.5}$$

where

$$\begin{aligned} H_{\varepsilon,a}(w_\varepsilon) =& \frac{(1+\tau)\rho_{\varepsilon,a}^2 e^{2\varepsilon^2 w_\varepsilon}}{\kappa^2(1+\varepsilon^2\rho_{\varepsilon,a}e^{\varepsilon^2 w_\varepsilon})^3} + \frac{3(1-\tau)\rho_{\varepsilon,a}^2 e^{\varepsilon^2 w_\varepsilon}}{\kappa^2(1+\varepsilon^2\rho_{\varepsilon,a}e^{\varepsilon^2 w_\varepsilon})^3} \\ &+ \frac{(1-\tau)\big(3\varepsilon^2\rho_{\varepsilon,a}^3 e^{2\varepsilon^2 w_\varepsilon} + \varepsilon^4\rho_{\varepsilon,a}^4 e^{3\varepsilon^2 w_\varepsilon}\big)}{\kappa^2(1+\varepsilon^2\rho_{\varepsilon,a}e^{\varepsilon^2 w_\varepsilon})^3}. \end{aligned}$$

Since the right hand side of (3.5) is regular at $(\varepsilon, a) = (0,0)$, for sufficiently small ε, a we may regard (3.5) as perturbation of the case $(\varepsilon, a) = (0,0)$:

$$\mathcal{L}w_0 := \Delta w_0 + \rho w_0 = \frac{4-2\tau}{\kappa^2\lambda^2}\rho^2, \tag{3.6}$$

where for $r = |z|$,

$$\rho(r) = \lambda\rho_{0,0}(r) = \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2}.$$

It is known that (3.6) has a unique radial solution $w_0(r)$. Moreover, as $r = |z| \to \infty$, we have

$$\begin{cases} w_0(r) = -c_0\ln(1+r) + O(1), \\ w_0'(r) = -\dfrac{c_0}{r} + o\Big(\dfrac{1}{r}\Big), \end{cases} \tag{3.7}$$

for some positive c_0. See [1] or Corollary 3.4.21 of [16] . Finally, we set

$$w_\varepsilon = w_0 + v. \tag{3.8}$$

Then, w_ε is a solution of (3.5) if and only if $P(\varepsilon, a, v) = 0$ where

$$\begin{aligned} P(\varepsilon,a,v) = \Delta v &+ \frac{\lambda\rho_{\varepsilon,a}}{(1+\varepsilon^2\rho_{\varepsilon,a}e^{\varepsilon^2(w_0+v)})^3}\Big(\frac{e^{\varepsilon^2(w_0+v)}-1}{\varepsilon^2}\Big) + H_{\varepsilon,a}(w_0+v) \\ &- \rho w_0 + \frac{4-2\tau}{\kappa^2\lambda^2}\rho^2. \end{aligned}$$

We observe that $P(0,0,0) = 0$. We will apply the standard Implicit Function Theorem to the operator P to find solutions of $P(\varepsilon, a, v) = 0$.

First, let us introduce two Hilbert spaces:

$$\begin{aligned} X &= \Big\{ u \in L^2_{loc}(\mathbb{R}^2) : \int_{\mathbb{R}^2} (1+|z|^{2+\frac14})|u|^2\, dz < \infty \Big\}, \\ Y &= \Big\{ u \in W^{2,2}_{loc}(\mathbb{R}^2) : \Delta u \in X \ \text{ and } \ \int_{\mathbb{R}^2} \frac{|u|^2}{1+|z|^{2+\frac14}}\, dz < \infty \Big\}. \end{aligned}$$

The inner products are defined by

$$(u,v)_X = \int_{\mathbb{R}^2} (1+|z|^{2+\frac{1}{4}})uv\,dz,$$

$$(u,v)_Y = (\Delta u, \Delta v)_X + \int_{\mathbb{R}^2} \frac{uv}{1+|z|^{2+\frac{1}{4}}}\,dz.$$

It is known from [1] that there exists a constant $C>0$ such that

$$|v(z)| \le C\|v\|_Y \ln(2+|z|), \quad \forall \in Y. \tag{3.9}$$

Given $\delta > 0$, we define

$$\Omega_\delta = \left\{(\varepsilon, a, v) \in \mathbb{R}\times\mathbb{C}\times Y \;:\; |\varepsilon| + |a| + \|v\|_Y < \delta\right\}.$$

Lemma 3.1. *If $\delta > 0$ is small enough, then P maps Ω_δ into X. Moreover, P is continuously differentiable on Ω_δ.*

Proof. Let $(\varepsilon, a, v) \in \Omega_\delta$. We deduce from (3.7) and (3.9) that

$$\rho_{\varepsilon,a}(z)e^{\varepsilon^2(w_0(z)+v(z))} \le C(2+|z|)^{-2N-4+\delta^2(c_0+C\delta)}. \tag{3.10}$$

Hence, if $\delta > 0$ is sufficiently small, then $H_{\varepsilon,a}(w_0+v) \in X$. Similarly, other terms of $P(\varepsilon,a,v)$ belong to X. Moreover, since $P(\varepsilon,a,v)$ is regular with respect to ε, one can easily check that P is C^1 in Ω_δ. We omit the details. □

Now, let

$$\Gamma = P'_{(a,v)}(0,0,0) : \mathbb{C}\times Y \longrightarrow X.$$

By direct calculation we obtain

$$\Gamma(b,\varphi) = \mathcal{L}\varphi - 4\phi_+ b_1\eta - 4\phi_- b_2\eta,$$

where

$$\eta = \rho w_0 - \frac{2(4-2\tau)}{\kappa^2\lambda^2}\rho^2, \qquad \psi_0(r) = \frac{r^{N+1}}{1+r^{2N+2}},$$

$$\phi_+(r) = \psi_0(r)\cos\big((N+1)\theta\big), \qquad \phi_-(r) = \psi_0(r)\sin\big((N+1)\theta\big).$$

Lemma 3.2. *We have*

$$\int_{\mathbb{R}^2} \eta\phi_\pm^2\,dx < 0. \tag{3.11}$$

Proof. Let $\sigma = (1+r^{2N+2})^{-2}$. By direct computation, we obtain

$$\mathcal{L}\sigma = \frac{16(N+1)^2 r^{4N+2}}{(1+r^{2N+2})^4} = 2\rho\psi_0^2. \tag{3.12}$$

Then,

$$\begin{aligned}\int_{\mathbb{R}^2} \eta\phi_+^2\,dx &= \pi\int_0^\infty \Big(\rho w_0 - \frac{2(4-2\tau)}{\kappa^2\lambda^2}\rho^2\Big)\psi_0^2 r\,dr \\ &= \pi\int_0^\infty \Big(\frac{1}{2}(\mathcal{L}\sigma)w_0 - \frac{2(4-2\tau)}{\kappa^2\lambda^2}\rho^2\psi_0^2\Big) r\,dr\end{aligned}$$

$$= \pi \int_0^\infty \Big(\frac{1}{2}\sigma\mathcal{L}w_0 - \frac{2(4-2\tau)}{\kappa^2\lambda^2}\rho^2\psi_0^2\Big) rdr$$
$$= \frac{\pi(4-2\tau)}{\kappa^2\lambda^2}\int_0^\infty \rho^2\Big(\frac{1}{2}\sigma - 2\psi_0^2\Big) rdr.$$

Letting $t = r^2$, we see that

$$\int_0^\infty \rho^2\Big(\frac{1}{2}\sigma - 2\psi_0^2\Big) rdr$$
$$= 16\pi(N+1)^4\Big[\int_0^\infty \frac{t^{2N}}{(1+t^{N+1})^5}dt - \int_0^\infty \frac{5t^{(3N+1)}}{(1+t^{N+1})^6}dt\Big]$$
$$= 16\pi(N+1)^4\Big(1 - \frac{2N+1}{N+1}\Big)\int_0^\infty \frac{t^{2N}}{(1+t^{(N+1)})^5}dt < 0.$$

By a similar argument, we can prove the result for ϕ_-. □

Lemma 3.3. Γ *is surjective and* $\mathrm{Ker}\Gamma = \{0\} \times \mathrm{span}\{\phi_0, \phi_+, \phi_-\} \subset \mathbb{C} \times Y$, *where*

$$\phi_0(r) = \frac{1 - r^{2N+2}}{1 + r^{2N+2}}.$$

Proof. First, we recall from that [1] that

$$\text{(3.13)} \qquad \mathrm{Im}\,\mathcal{L} = \{h \in X \ : \ \int_{\mathbb{R}^2} h\phi_\pm = 0\}, \quad \mathrm{Ker}\mathcal{L} = \mathrm{span}\{\phi_0, \phi_+, \phi_-\}.$$

Given $f \in X$, we define

$$b_1 = -\frac{\int f\phi_+}{4\int \eta\phi_+^2}, \quad b_2 = -\frac{\int f\phi_-}{4\int \eta\phi_-^2}.$$

By Lemma 3.2, b_1 and b_2 are well defined. Then,

$$\int_{\mathbb{R}^2} (f + 4\phi_+ b_1\eta + 4\phi_- b_2\eta)\phi_\pm dx = 0.$$

Hence, by (3.13) there exists $w \in Y$ such that $\mathcal{L}w = f + 4\phi_+ b_1\eta + 4\phi_- b_2\eta$. In other words, $\Gamma(b_1 + ib_2, w) = f$. Thus, Γ is surjective.

Next, suppose that $\Gamma(b_1 + ib_2, w) = 0$. Then, by (3.13),

$$0 = (\mathcal{L}w, \phi_+)_{L^2} = (4\phi_+ b_1\eta + 4\phi_- b_2\eta, \phi_+)_{L^2} = -4b_1\int_{\mathbb{R}^2} \eta\phi_+^2,$$

which implies by Lemma 3.2 that $b_1 = 0$. Similarly, $b_2 = 0$. Moreover, $w \in \mathrm{Ker}\mathcal{L}$. □

Proof of Theorem 1.2. By Lemma 3.1, P is a C^1 map on Ω_δ. Moreover, $\Gamma = P'_{(a,v)}(0,0,0)$ is surjective. Since Γ is not injective, we decompose $Y = \mathrm{Ker}\mathcal{L} \oplus Z$ with $Z = (\mathrm{Ker}\mathcal{L})^\perp$ and denote by Q the restriction of P on $\mathbb{R} \times \mathbb{C} \times Z$. Then, by Lemma 3.3, $Q'_{(a,v)}(0,0,0) : \mathbb{R} \times \mathbb{C} \times Z \to X$ is a bijective. Applying the standard Implicit Function Theorem to the equation $Q(\varepsilon, a, v) = 0$, we conclude that there exist $\varepsilon_0 > 0$ and a C^1-map $\varepsilon \mapsto (a_\varepsilon, v_\varepsilon) \in \mathbb{C} \times Y$ for $|\varepsilon| < \varepsilon_0$ such that $Q(\varepsilon, a_\varepsilon, v_\varepsilon) = 0$. Then, by (3.4) and (3.8)

$$u_\varepsilon(z) := \ln \varepsilon^2 + \ln \rho_{\varepsilon,a}(\varepsilon z) + \varepsilon^2 w_0(\varepsilon z) + \varepsilon^2 v_\varepsilon(\varepsilon z). \tag{3.14}$$

is a solution of (3.2) for $0 < \varepsilon < \varepsilon_0$.

It remains to show the estimates (1.15)–(1.16). Since $\varepsilon \mapsto (a_\varepsilon, v_\varepsilon)$ is C^1, $\|v_\varepsilon\|_Y \to 0$ as $\varepsilon \to 0$. Thus, it follows from (3.9) that

$$|v_\varepsilon(z)| \le o(1) \ln(2 + |z|) \quad \text{as} \quad |z| \to \infty, \tag{3.15}$$

which yields the estimate (1.15) by (3.7).

Next, we prove (1.16). For small $\delta > 0$, $U_\delta := \mathbb{R}^2 \setminus \cup_{j=1}^{d_1} B_\delta(p_j)$. Then,

$$\frac{1}{\kappa^2} \int_{\mathbb{R}^2} f\big(u_\varepsilon, \tau\big) dx = -\lim_{\delta \to 0} \int_{U_\delta} \Delta u_\varepsilon dx. \tag{3.16}$$

It follows from (3.14) that for $z \in U_\delta$,

$$\Delta u_\varepsilon = \Delta \left\{ -2 \ln\big(1 + |G_\varepsilon(\varepsilon z) + a_\varepsilon|^2\big) + \varepsilon^2 w_0(\varepsilon z) + \varepsilon^2 v_\varepsilon(\varepsilon z) \right\}.$$

We note that

$$\left| \int_{\mathbb{R}^2} \Delta\big(v_\varepsilon(\varepsilon z)\big) dz \right| \le \big\|(1 + |z|)^{-1-\frac{1}{8}}\big\|_{L^2} \cdot \|\Delta v_\varepsilon\|_X \le C \|v_\varepsilon\|_Y = o(1).$$

We also deduce from (3.7) that

$$\int_{\mathbb{R}^2} \Delta\big(w_0(\varepsilon z)\big) dz = \lim_{R \to \infty} \int_{|z|=R} \frac{\partial}{\partial r} w_0(r) dS = -2\pi c_0.$$

Moreover,

$$\begin{aligned} &\int_{\mathbb{R}^2} \Delta \ln\big(1 + |G_\varepsilon(\varepsilon z) + a_\varepsilon|^2\big) dz \\ &= \lim_{R \to \infty} \int_{|z|=R} \frac{\partial_r |G_\varepsilon(\varepsilon z) + a_\varepsilon|^2}{1 + |G_\varepsilon(\varepsilon z) + a_\varepsilon|^2} dS \\ &= \lim_{R \to \infty} \int_0^{2\pi} \left\{ \frac{(2N+2)\varepsilon^{2N+2} R^{2N+1}}{1 + \varepsilon^{2N+2} R^{2N+2}} + O\Big(\frac{1}{R^2}\Big) \right\} R d\theta \\ &= 4\pi(N+1). \end{aligned}$$

Inserting these estimates into (3.16), we obtain (1.16). This completes the proof of Theorem 1.2. □

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