

ON THE TOPOLOGY OF THE NONABELIAN TENSOR PRODUCT OF PROFINITE GROUPS

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ABSTRACT. The properties of the nonabelian tensor products are interesting in different contexts of algebraic topology and group theory. We prove two theorems, dealing with the nonabelian tensor products of projective limits of finite groups. The first describes their topology. Then we show a result of embedding in the second homology group of a pro- p -group, via the notion of complete exterior centralizer. We end with some open questions, originating from these two results.

1. Statement of the main theorems

Following [1, 2, 3], two groups G and H act *compatibly* upon each other, if

$$(1.1) \quad x^{(y^z)} = x^{z^{-1}yz}, \quad t^{(z^y)} = t^{y^{-1}zy},$$

for $x, z \in G$ and $y, t \in H$, and if they act upon themselves by conjugation (here $x^y = y^{-1}xy$). The group, presented by

$$(1.2) \quad G \otimes H = \langle x \otimes y \mid xz \otimes y = (z^x \otimes y^x)(x \otimes y), x \otimes yt = (x \otimes y)(x^y \otimes t^y) \rangle,$$

is called *nonabelian tensor product* of G and H . If G and H act trivially upon each other, $G/[G, G] \otimes H/[H, H]$ is the (usual) abelian tensor product (see [3, Proposition 2.4]) of the abelian groups $G/[G, G]$ and $H/[H, H]$. On the other hand, if $G = H$ and all actions are by conjugation, $G \otimes G$ is called *nonabelian tensor square* of G and $\nabla(G) = \langle g \otimes g \mid g \in G \rangle$ is a central subgroup of $G \otimes G$, inducing a short exact sequence $1 \rightarrow \nabla(G) \rightarrow J_2(G) \rightarrow H_2(G, \mathbb{Z}) \rightarrow 1$, where $H_2(G, \mathbb{Z})$ is the second group of integral homology over G (called *Schur multiplier* of G) and $J_2(G) = \ker \kappa$ is the kernel of the surjective homomorphism of groups

$$\kappa : x \otimes y \in G \otimes G \longmapsto [x, y] = x^{-1}y^{-1}xy \in [G, G].$$

Received April 28, 2015.

2010 *Mathematics Subject Classification*. Primary 22C05, 20E18; Secondary 20J05, 20J06.

Key words and phrases. pro- p -groups, nonabelian exterior square, homology, topological groups, profinite groups.

The quotient group $G \wedge G = G \otimes G / \nabla(G)$ is called *nonabelian exterior square* of G and we have a surjective homomorphism of groups

$$\kappa' : x \wedge y \in G \wedge G \mapsto [x, y] = x^{-1}y^{-1}xy \in [G, G]$$

such that $\ker \kappa' = H_2(G, \mathbb{Z})$. The proof of these properties can be found in [2, 3]. In particular, Brown and others [2, 3] show that

$$(1.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \nabla(G) & \longrightarrow & G \otimes G & \longrightarrow & [G, G] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_2(G, \mathbb{Z}) & \longrightarrow & G \wedge G & \longrightarrow & [G, G] \longrightarrow 0, \end{array}$$

is a commutative diagram with central extensions as rows¹.

In order to state our main results, we recall from [6, 12, 13] that a compact (Hausdorff) group G possessing the filter basis

$$\mathcal{P}(G) = \{N = \overline{N} \triangleleft G \mid G/N \text{ is a finite } p\text{-group}\}$$

is said to be a *pro- p -group* (p prime) if $G = \lim_{N \in \mathcal{P}(G)} G/N$, that is, if G is a projective limit of finite p -groups (see [12, Definitions 7.1.12, 7.2.1, 7.2.3, 7.2.4]). Here the open subgroups of G are exactly those subgroups of p -power index [12, Lemma 7.2.2]. Of course, the topology of G is the unique topology induced by $\mathcal{P}(G)$. More generally, we may replace $\mathcal{P}(G)$ with

$$\mathcal{F}(G) = \{N = \overline{N} \triangleleft G \mid G/N \text{ is a finite group}\}$$

and G is said to be a *profinite group* if $G = \lim_{N \in \mathcal{F}(G)} G/N$, that is, if G is a projective limit of finite groups (see again [6, 12, 13] for details). A first question is to understand which topology we get on the nonabelian tensor product of profinite groups. Our first main result answers this question.

Theorem 1.1. *Let $G = \lim_{N \in \mathcal{F}(G)} G/N$ and $H = \lim_{M \in \mathcal{F}(H)} H/M$ be profinite groups. Then there exists a natural isomorphism of profinite groups such that*

$$\lim_{(N, M) \in \mathcal{F}(G) \times \mathcal{F}(H)} G/N \otimes H/M \simeq \lim_{N \in \mathcal{F}(G)} G/N \otimes \lim_{M \in \mathcal{F}(H)} H/M.$$

In particular, if G and H are pro- p -groups, then

$$\lim_{(N, M) \in \mathcal{P}(G) \times \mathcal{P}(H)} G/N \otimes H/M \simeq \lim_{N \in \mathcal{P}(G)} G/N \otimes \lim_{M \in \mathcal{P}(H)} H/M.$$

¹Having in mind the notion of suspension of an Eilenberg–MacLane space from [19], Brown and Loday [3] show that the third homotopy group of the suspension of an Eilenberg–MacLane space $K(G, 1)$ satisfies $\pi_3(SK(G, 1)) \simeq J_2(G)$. Moreover, if a CW-complex X (see [19] for the definition) is the union $X = A \cup B$ of two path connected CW-subcomplexes A and B whose intersection $C = A \cap B$ is path connected and the canonical homomorphisms $\pi_1(C) \rightarrow \pi_1(A)$ and $\pi_1(C) \rightarrow \pi_1(B)$ are surjective, then $\pi_3(X, A, B) \simeq \pi_2(A, C) \otimes \pi_2(B, C)$, where $\pi_2(A, C)$ and $\pi_2(B, C)$ act upon each other via $\pi_1(C)$. Therefore, restricting the structure of the nonabelian tensor product $\pi_2(A, C) \otimes \pi_2(B, C)$, we get information on $\pi_3(X, A, B)$. This is very interesting in algebraic topology (see [1, 16, 17]).

A consequence of Theorem 1.1 allows us to identify

$$\lim_{(N,M) \in \mathcal{P}(G) \times \mathcal{P}(H)} G/N \otimes H/M$$

with a topological version of $G \otimes H$, properly defined in (2.1) below. This allows us to describe the nonabelian tensor product of projective limits, in terms of suitable generators and relations.

Then we investigate the properties of the operator $\widehat{\wedge}$ and of the set

$$\widehat{C}_G(x) = \{a \in G \mid a \text{ and } x \text{ commute with respect to } \widehat{\wedge}\},$$

when G is a pro- p -group. These notions are relevant in recent contributions [14, 15, 18]. We will see that $\widehat{C}_G(x)$ is a closed subgroup of $G \widehat{\wedge} G$, contained in the (usual) centralizer $C_G(x)$ of x in G . In fact our last main result is of embedding.

Theorem 1.2. *Let G be a pro- p -group. Then $C_G(x)/\widehat{C}_G(x)$ is abelian and isomorphic to a closed subgroup of $H_2(G, \mathbb{Z}_p)$, where \mathbb{Z}_p denotes the additive group of p -adic integers.*

A series of open questions are appended at the end, where we illustrate new problems which originate from an alternative approach via the notion of free products of topological groups with amalgamation. This notion has been studied by Morris and others in [5, 8, 9, 10, 11].

2. Projective limits of nonabelian tensor products

Since (1.2) is defined in terms of generators and relations, we may involve the notion of *profinite presentation* (see [13, pp. 47–50]), in order to define the nonabelian tensor product in the category of profinite groups. From [13, p. 47], the symbol \widehat{F}_d denotes the free profinite group on d generators, that is, \widehat{F}_d is the completion of the discrete free group F_d on d generators x_1, \dots, x_d . Now \widehat{F}_d has the universal property (see [13, p. 47]) and for every element (“word”) w in \widehat{F}_d , every profinite group G and every $g_1, \dots, g_d \in G$, we can evaluate w on g_1, \dots, g_d . Thus we can consider the elements of $w \in \widehat{F}_d$ as “words”, though they are not words in the usual sense. This allow us to speak of presentations of profinite groups by generators and relations, as known in combinatorial group theory.

Definition 2.1 (See [13], p. 47). If $G = \widehat{F}_d/N$ is a profinite group and Y is a subset of N , where N is the minimal closed normal subgroup of \widehat{F}_d containing Y , we say that $G = \langle x_1, \dots, x_d \mid Y \rangle$ is a profinite presentation for G .

Now suppose that G and H are two profinite groups with continuous actions upon each other, and on themselves by conjugation, so that the compatibility relations (1.1) are satisfied. The *profinite nonabelian tensor product* $G \widehat{\otimes} H$ of

G and H is the group *topologically generated* by the symbols $x\widehat{\otimes}y$ such that

$$(2.1) \quad G\widehat{\otimes}H = \langle x\widehat{\otimes}y \mid xz\widehat{\otimes}y = (z^x\widehat{\otimes}y^x)(x\widehat{\otimes}y), x\widehat{\otimes}yt = (x\widehat{\otimes}y)(x^y\widehat{\otimes}t^y) \rangle$$

for all $x, z \in G$ and $y, t \in H$. In view of Definition 2.1, we shall interpret (2.1) as a profinite presentation. The real question is to understand which topology is present on $G\widehat{\otimes}H$ and how it is related to the topologies of G and H .

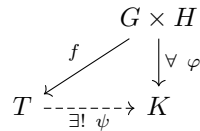
In order to answer this question, we note that the original properties of [2, 3] have been formulated in terms of functors and categories by Inassaridze [7] (see also [1]). The universal property of the (usual) abelian tensor product (see [6, Theorem A3.28]) may be subject of generalizations, when we deal with the nonabelian tensor product (see [1]). The role of the bilinear maps is replaced by new maps, which preserve the compatibility of the actions (1.1). This motivates the notion of crossed pairings, which we recall below.

Definition 2.2 (See [2, 3]). Let G, H and K be groups. A map $\varphi : G \times H \rightarrow K$ is said to be a *crossed pairing*, if for all $g, a \in G$ and $h, b \in H$

$$\begin{aligned} \varphi(ag, h) &= \varphi(g^a, h^g) \varphi(g, h), & \varphi(g, h) \varphi(a, b) &= \varphi(a^{[g,h]}, b^{[g,h]}) \varphi(g, h), \\ \varphi(g, bh) &= \varphi(g, h) \varphi(g^h, h^b), & \varphi(a, b) \varphi(g, h) &= \varphi(g, h) \varphi(a^{[h,g]}, b^{[h,g]}). \end{aligned}$$

The following result describes the universal property for crossed pairings.

Proposition 2.3 (See [3, 7]). *Let G, H and T be groups and $f : (g, h) \in G \times H \mapsto f(g, h) \in T$ be a crossed pairing. Then $T \simeq G \otimes H$ if and only if for every group K and every crossed pairing $\varphi : G \times H \rightarrow K$ there exists a unique homomorphism $\psi : T \rightarrow K$ which makes commutative the following diagram:*



i.e., $\psi(f(g, h)) = \varphi(g, h)$ for all $g \in G$ and $h \in H$.

Without difficulties, one can check that (1.2) satisfies Proposition 2.3. Of course, a crossed pairing must be continuous in the profinite case.

Definition 2.4. If G and H are profinite groups and the map φ of Definition 2.2 is continuous, we say that φ is a *profinite crossed pairing*.

The proof of the following lemma is standard.

Lemma 2.5. *Let G, H and T be profinite groups and $f : (g, h) \in G \times H \mapsto f(g, h) \in T$ be a profinite crossed pairing. Then $T \simeq G\widehat{\otimes}H$ if and only if for every group K and every profinite crossed pairing $\varphi : G \times H \rightarrow K$ there*

exists a unique homomorphism of profinite groups $\psi : T \rightarrow K$ which makes commutative the following diagram:

$$\begin{array}{ccc}
 & G \times H & \\
 f \swarrow & & \downarrow \varphi \\
 T & \dashrightarrow & K \\
 & \exists! \psi &
 \end{array}$$

i.e., $\psi(f(g, h)) = \varphi(g, h)$ for all $g \in G$ and $h \in H$.

Proof. Assume that we have $G \widehat{\otimes} H$ and the profinite crossed pairing $f : (g, h) \in G \times H \mapsto f(g, h) = g \widehat{\otimes} h \in G \widehat{\otimes} H$. Define $\psi : g \widehat{\otimes} h \in G \widehat{\otimes} H \mapsto \psi(g \widehat{\otimes} h) = f(g, h) \in K$. Clearly, ψ makes commutative the above diagram. Moreover, if $\lambda : G \widehat{\otimes} H \rightarrow K$ is another map which satisfies the condition $\lambda(f(g, h)) = \varphi(g, h)$ for all $g \in G$ and $h \in H$, then $\lambda(f(g, h)) = \psi(f(g, h))$ coincides with ψ over $f(g, h)$ and this means over all $G \otimes H$, since f is clearly surjective.

Conversely, suppose that a unique homomorphism of profinite groups $\psi : T \rightarrow K$ always exists. We will have the following commutative diagram

$$\begin{array}{ccc}
 G \widehat{\otimes} H & \xrightarrow{\alpha} & T \\
 f \uparrow & & \downarrow \psi \\
 G \times H & \xrightarrow{\varphi} & K
 \end{array}$$

where $\alpha \circ \psi = \psi$ by the unicity of ψ . This implies that α is the identity homomorphism of profinite groups and the result follows. □

Again one can check that $G \widehat{\otimes} H$, presented by (2.1), satisfies Lemma 2.5. Here is appropriate to note that Inassaridze [7, Theorem 2] shows that the nonabelian tensor product has a good behaviour with respect to projective limits.

Lemma 2.6 (See [7], Theorem 2). *Assume that $\{(G_i, \phi_i)\}_{i \in I}$ is a projective system of groups, A a group and for every $i \in I$ the groups A, G_i act upon each other (and on themselves) by conjugation in a compatible way. Assume further that the homomorphisms ϕ_i preserve the actions for all $i \in I$. Then there is a natural isomorphism*

$$\left(\lim_{i \in I} G_i \right) \otimes A \simeq \lim_{i \in I} (G_i \otimes A).$$

Finally, we consider the filter basis of a compact group G

$$\mathcal{N}(G) = \{N = \overline{N} \triangleleft G \mid G/N \text{ is a Lie group}\}$$

and recall that G satisfies the Approximation Theorem by Compact Lie Groups [6, Corollary 2.43], that is, $G = \lim_{N \in \mathcal{N}(G)} G/N$. We summarize some useful information in the following lemma.

Lemma 2.7 (See [6], Theorem 1.34, Exercise E1.12 (iii)). *Let G be a compact group. With the above notations, $\mathcal{P}(G) \subseteq \mathcal{F}(G) \subseteq \mathcal{N}(G)$. Moreover, if the identity component G_0 of G is trivial, then $\mathcal{N}(G) = \mathcal{F}(G)$.*

3. Proof of Theorem 1.1 and consequences

We are ready to describe the topology of nonabelian tensor products of profinite groups, generalizing a result of Moravec [14] for pro- p -groups. Lemma 2.6 would be a crucial point in this perspective.

Proof of Theorem 1.1. We begin to prove that

$$\left(\lim_{N \in \mathcal{F}(G)} G/N \right) \otimes H \simeq \lim_{N \in \mathcal{F}(G)} (G/N \otimes H).$$

We have the projective system $\{(G/N, \varphi_N)\}_{N \in \mathcal{F}(G)} = \{(L_N, \varphi_N)\}_{N \in \mathcal{F}(G)}$ such that $G = \lim_{N \in \mathcal{F}(G)} L_N$. Now for each $N \in \mathcal{F}(G)$ the actions by conjugation $(x, l_N) \in H \times L_N \mapsto x^{l_N} \in L_N$ and $(l_N, x) \in L_N \times H \mapsto l_N^x \in H$ are compatible among themselves and with the action by conjugation of H on itself and of L_N on itself. Furthermore, φ_N preserves these actions for all $N \in \mathcal{F}(G)$, therefore we can form the nonabelian tensor product $L_N \otimes H$ for all $N \in \mathcal{F}(G)$ and consider the projective limit of the system $\{(L_N \otimes H, \varphi_N \otimes 1_H)\}_{N \in \mathcal{F}(G)}$, where $\varphi_N \otimes 1_H : l_N \otimes x \in L_N \otimes H \mapsto l_K \otimes x \in L_K \otimes H$ with $N \subseteq K$ and $K \in \mathcal{F}(G)$. In other words, $\varphi_N \otimes 1_H$ fix the second component and is defined as the homomorphism φ_N , which we have already, on the first component. Therefore we may consider the projective limit $\lim_{N \in \mathcal{F}(G)} (L_N \otimes H)$. Now define for all $N \in \mathcal{F}(G)$ the action of $\left(\lim_{N \in \mathcal{F}(G)} L_N \right)$ over H by the rule $(l_N, x) \in \left(\lim_{N \in \mathcal{F}(G)} L_N \right) \times H \mapsto l_N^x \in H$ and the action of H over $\left(\lim_{N \in \mathcal{F}(G)} L_N \right)$ by the rule $(x, l_N) \in H \times \left(\lim_{N \in \mathcal{F}(G)} L_N \right) \mapsto x^{l_N} \in \left(\lim_{N \in \mathcal{F}(G)} L_N \right)$. These are compatible among themselves and with respect to the conjugation of H on itself and of $\left(\lim_{N \in \mathcal{F}(G)} L_N \right)$ on itself. This means that it is well defined the nonabelian tensor product $\left(\lim_{N \in \mathcal{F}(G)} L_N \right) \otimes H$. It remains to define an isomorphism among $\left(\lim_{N \in \mathcal{F}(G)} G/N \right) \otimes H$ and $\lim_{N \in \mathcal{F}(G)} (G/N \otimes H)$ and the following is the natural way to do it:

$$f_N : (l_N) \otimes x \in \left(\lim_{N \in \mathcal{F}(G)} L_N \right) \otimes H \mapsto f_N((l_N) \otimes x) = (l_N \otimes x) \in \lim_{N \in \mathcal{F}(G)} (L_N \otimes H)$$

and it is easy to check that f_N is a homomorphism of (abstract, until now) groups both injective and surjective. Now we may repeat the argument, fixing the first component, instead of the second component as we have just done. We will find

$$G \otimes \left(\lim_{M \in \mathcal{F}(H)} H/M \right) \simeq \lim_{M \in \mathcal{F}(H)} (G \otimes H/M)$$

via the isomorphism

$$h_M : x \otimes (l_M) \in G \otimes \left(\lim_{M \in \mathcal{F}(H)} L_M \right) \mapsto$$

$$h_M(x \otimes (l_M)) = (x \otimes l_M) \in \lim_{M \in \mathcal{F}(G)} \left(G \otimes L_M \right).$$

Therefore we conclude that the following is an isomorphism of abstract groups

$$(3.1) \quad \lim_{N \in \mathcal{F}(G)} G/N \otimes \lim_{M \in \mathcal{F}(H)} H/M \simeq \lim_{(N,M) \in \mathcal{F}(G) \times \mathcal{F}(H)} G/N \otimes H/M.$$

It remains to prove that in the both sides of this relation we are dealing with profinite groups and that the isomorphism among them is continuous, that is, an isomorphism of profinite groups.

In order to check that the group on the right side of (3.1) is profinite, we note that this is a projective limit of the system $\{(G/N \otimes H/M, \varphi_N \otimes \varphi_M)\}_{(N,M) \in \mathcal{F}(G) \times \mathcal{F}(H)}$. Assuming that $G/N \otimes H/M$ is a finite group, we have $\varphi_N \otimes \varphi_M : gN \otimes hM \in G/N \otimes H/M \mapsto (\varphi_N \otimes \varphi_M)(gN \otimes hM) = \varphi_N(gN) \otimes \varphi_M(hM) = gA \otimes hB \in G/A \otimes H/B$ for some $A, B \in \mathcal{F}(G)$ such that $A \supseteq N$ and $B \supset M$. Then the group would be profinite by definition. In other words, it is enough to prove that $G/N \otimes G/M$ is a finite group, in order to conclude our claim. But this is clear applying Lemma 2.5, because G/N and H/M are finite and the crossed pairing $f : G/N \times H/M \rightarrow G/N \otimes H/M$ is a surjective continuous homomorphism. Then $G/N \otimes H/M$ is a homomorphic image under f of the finite group $G/N \times H/M$.

On the other hand, the group on the left side of (3.1) is exactly $G \widehat{\otimes} H$ (see Lemma 2.5), then (3.1) is saying that $G \widehat{\otimes} H$ is a profinite group which is isomorphic to the projective limit of finite groups $G/N \otimes H/M$ via $F_N^M = f_N \otimes h_M$, defined by f_N and h_M , which are continuous. Therefore F_N^M is continuous and the result follows. □

In other words, the nonabelian tensor product of the profinite groups G and H has the topology which is projective limit of the topologies of each factor with respect to the operator of nonabelian tensor product. This passage to the limit is induced naturally by the isomorphism in Theorem 1.1. We note that Theorem 1.1 was known by [14, Theorem 2.1] in case of pro- p -groups via a different argument.

Remark 3.1. Looking at Lemma 2.7 and at the proof of Theorem 1.1, we may replace $\mathcal{F}(G)$ with $\mathcal{N}(G)$, and $\mathcal{F}(H)$ with $\mathcal{N}(H)$, respectively. This means that Theorem 1.1 is true even for compact groups, not only for profinite groups.

By Theorem 1.1, we get an important description of $G \widehat{\otimes} G$ when G is profinite. Of course, the case of pro- p -group will be also true, mutatis mutandis.

Corollary 3.2. *Given a profinite group $G = \lim_{N \in \mathcal{F}(G)} G/N$, there exists an isomorphism of profinite groups such that $G \widehat{\otimes} G \simeq \lim_{(N,M) \in \mathcal{F}(G) \times \mathcal{F}(G)} G/N \otimes G/M$.*

Proof. It is enough to apply Lemma 2.5, where $H = K = G$, φ is the natural projection of $G \times G = \lim_{N \in \mathcal{F}(G)} G/N \times \lim_{M \in \mathcal{F}(G)} G/M$ onto G , $T = \lim_{(N,M) \in \mathcal{F}(G) \times \mathcal{F}(G)} G/N \otimes G/M \simeq \lim_{N \in \mathcal{F}(G)} G/N \otimes \lim_{M \in \mathcal{F}(G)} G/M$ is profinite by Theorem 1.1 and

$$f : (a, b) \in \lim_{N \in \mathcal{F}(G)} G/N \times \lim_{M \in \mathcal{F}(G)} G/M \mapsto a \otimes b \in \lim_{N \in \mathcal{F}(G)} G/N \otimes \lim_{M \in \mathcal{F}(G)} G/M.$$

□

The following consequence is interesting for compact groups.

Corollary 3.3. *A compact group admits always a profinite quotient satisfying the thesis of Theorem 1.1 and Corollary 3.2.*

Proof. From [6, Exercise E1.13], G/G_0 is profinite. Lemma 2.7 implies $\mathcal{N}(G/G_0) = \mathcal{F}(G/G_0)$ and we may apply Theorem 1.1 and Corollary 3.2 to G/G_0 . □

4. Proof of Theorem 1.2 and consequences

Moravec [14] introduces the *complete nonabelian tensor square* $G \widehat{\otimes} G$ of a pro- p -group G as the group topologically generated by the symbols $x \widehat{\otimes} y$, subject to the relations $xy \widehat{\otimes} z = (x^y \widehat{\otimes} z^y)(y \widehat{\otimes} z)$ and $y \widehat{\otimes} tz = (y \widehat{\otimes} z)(y^z \widehat{\otimes} t^z)$ for all $x, y, z, t \in G$. This is a special situation of (2.1). The subgroup $\widehat{\nabla}(G) = \overline{\langle x \otimes x \mid x \in G \rangle}$ is a closed central subgroup of $G \widehat{\otimes} G$ and the quotient group $G \widehat{\otimes} G / \widehat{\nabla}(G) = G \widehat{\wedge} G$ is called *complete nonabelian exterior square* of the pro- p -group G . The diagram (1.3) can be modified in the case of pro- p -groups, where $H_2(G, \mathbb{Z}_p)$ is also called Schur multiplier of the pro- p -group G (see [4, 12]):

Lemma 4.1 (See [14]). *There exist a pro- p -group $G \widehat{\otimes} G$ and a closed normal subgroup $\widehat{\nabla}(G)$ such that the following diagram is commutative:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widehat{\nabla}(G) & \longrightarrow & G \widehat{\otimes} G & \longrightarrow & \overline{[G, G]} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_2(G, \mathbb{Z}_p) & \longrightarrow & G \widehat{\wedge} G & \longrightarrow & \overline{[G, G]} \longrightarrow 0, \end{array}$$

where $G \widehat{\wedge} G = G \widehat{\otimes} G / \widehat{\nabla}(G)$. Moreover the sequences above are central extensions.

In fact Moravec [14, §2] noted that the maps

$$\widehat{\kappa} : x \widehat{\otimes} y \in G \widehat{\otimes} G \mapsto [x, y] \in \overline{[G, G]} \text{ and } \widehat{\kappa}' : x \widehat{\wedge} y \in G \widehat{\wedge} G \mapsto [x, y] \in \overline{[G, G]},$$

are epimorphisms of pro- p -groups with $\ker \widehat{\kappa} \simeq \widehat{\nabla}(G)$ and $\ker \widehat{\kappa}' \simeq H_2(G, \mathbb{Z}_p)$. Moravec [14] also noted that $H_2(G, \mathbb{Z}_p)$ is a closed central subgroup of $G \widehat{\wedge} G$ and

$$\widehat{\varepsilon} : x \widehat{\otimes} y \in G \widehat{\otimes} G \mapsto x \widehat{\wedge} y \in G \widehat{\wedge} G \text{ and } \widehat{\varphi} : x \widehat{\otimes} x \in \widehat{\nabla}(G) \mapsto x \widehat{\wedge} x \in H_2(G, \mathbb{Z}_p)$$

are epimorphisms of pro- p -groups.

In [15, 16], it was studied the notion of *exterior centralizer*

$$C_E^\wedge(x) = \{a \in E \mid a \wedge x = 1_{E \wedge E}\}$$

of an element x of a finite group E and this was a powerful tool, in order to detect whether a finite group is *capable*² or not. The *complete exterior centralizer* $\widehat{C}_G(x)$ has been introduced in the unpublished note [18]: it may be rewritten more properly

$$\widehat{C}_G(x) = \{a \in G \mid a \widehat{\wedge} x = 1_{G \widehat{\wedge} G}\}.$$

We will list some of its properties.

Lemma 4.2. *Let G be a pro- p -group. Then $\widehat{C}_G(x)$ is a closed normal subgroup of $C_G(x)$.*

Proof. If $a, b \in \widehat{C}_G(x)$, then the defining relations of complete nonabelian exterior square imply

$$ab^{-1} \widehat{\wedge} x = (a^{b^{-1}} \widehat{\wedge} x^{b^{-1}}) (b^{-1} \widehat{\wedge} x) = (a \widehat{\wedge} x)^{b^{-1}} (b^{-1} \widehat{\wedge} x) = 1_{G \widehat{\wedge} G}$$

so $ab^{-1} \in \widehat{C}_G(x)$. Then $\widehat{C}_G(x)$ is not only a set, but a subgroup. If $g \in \widehat{C}_G(x)$ and $y \in \widehat{C}_G(x)$, then $g^y \widehat{\wedge} x = (g \widehat{\wedge} x)^y = 1_{G \widehat{\wedge} G}$ thus $\widehat{C}_G(x)$ is normal in $C_G(x)$. The fact that $\widehat{C}_G(x)$ is the stabilizer of a point, under the action of the operator $\widehat{\wedge}$, together with Theorem 1.1, ensure that it is closed. \square

Now, we may prove the structural result which is related to $H_2(G, \mathbb{Z}_p)$.

Proof of Theorem 1.2. Applying Corollary 3.2 to $G = \lim_{N \in \mathcal{P}(G)} G/N$ and noting that $\widehat{\nabla}(G) = \lim_{x \in G} \langle x \otimes x \rangle$, we find

$$G \widehat{\wedge} G = \frac{G \widehat{\otimes} G}{\widehat{\nabla}(G)} \simeq \frac{\lim_{N \in \mathcal{P}(G)} G/N \otimes \lim_{M \in \mathcal{P}(G)} G/M}{\lim_{x \in G} \langle x \otimes x \rangle}$$

and so $G \widehat{\wedge} G$ has the quotient topology induced on $G \widehat{\otimes} G$ by $\widehat{\nabla}(G)$. On the other hand, Lemma 4.1 implies that $H_2(G, \mathbb{Z}_p)$ has the topology induced by $G \widehat{\wedge} G$. Now we consider the map

$$\widehat{\psi} : y \in C_G(x) \mapsto \widehat{\psi}(y) = y \widehat{\wedge} x \in H_2(G, \mathbb{Z}_p)$$

and note that it is a homomorphism, since for all $a, b \in C_G(x)$

$$\widehat{\psi}(ab) = ab \widehat{\wedge} x = (a^b \widehat{\wedge} x^b) (b \widehat{\wedge} x) = (a \widehat{\wedge} x)^b (b \widehat{\wedge} x)$$

²A group is called capable, when it is isomorphic to the group of inner automorphisms of another group. See [15] for details.

$$= a^b \widehat{\wedge} x^b (b \widehat{\wedge} x) = (a \widehat{\wedge} x) (b \widehat{\wedge} x) = \widehat{\psi}(a) \widehat{\psi}(b).$$

Because of the initial considerations, if $y \widehat{\wedge} x \in H_2(G, \mathbb{Z}_p)$, then

$$\widehat{\psi}^{-1}(y \widehat{\wedge} x) = \widehat{\psi}^{-1} \left(\lim_{\substack{x \in G \\ y \in C_G(x)}} y \wedge x \right) = \lim_{\substack{x \in G \\ y \in C_G(x)}} \widehat{\psi}^{-1}(y \wedge x) = \lim_{x \in G} y = y,$$

that is, the counterimage of a point is a point and so $\widehat{\psi}$ is continuous. Since it is clear that $\ker \widehat{\psi} = \widehat{C}_G(x)$, the result follows from Lemma 4.2 and from the First Homomorphism Theorem in the form $\widehat{\psi}(C_G(x)) \simeq C_G(x)/\widehat{C}_G(x)$. \square

An interesting consequence is listed below.

Corollary 4.3. *If a pro- p -group G has an element x such that $C_G(x) \neq \widehat{C}_G(x)$, then $H_2(G, \mathbb{Z}_p)$ is nontrivial.*

We end with an example, which has inspired most of the above arguments.

Example 4.4. The notion of coclass of a pro- p -group can be found in [12, Definition 7.4.1] and has been largely studied by Leedham-Green and Newman: a finite p -group of order p^n ($n \geq 1$) and nilpotency class c has coclass r if $n - c = r$; a pro- p -group G has coclass r if there exists some $u \geq 2$ such that $G/\gamma_i(G)$ has coclass r for all $i \geq u$, where $\gamma_i(G)$ denotes the i -th term of the lower central series of G . From [12], the *rank* of a pro- p -group G is defined by $\text{rk}(G) = \sup_{H=\overline{H} \leq G} d(H)$, where $d(H)$ denotes the minimal number of elements

which are necessary to generate topologically H . If G is a torsion-free pro- p -group, then $l = \text{rk}(G) = \text{tf}(G)$ is called torsion-free rank. From [4, p. 148] (see also [12, §9]), we may describe the low dimensional homology of an *infinite pro- p -group of finite coclass and central exponent $t \geq 1$* , that is, of the group

$$K_t = \mathbb{Z}_p^{d_t} \rtimes C_{p^t} = \langle t_1, \dots, t_{d_t} \rangle \rtimes \langle g \rangle = \langle g, t_1, \dots, t_{d_t} \mid g^{p^t} = 1,$$

$$g^{-1}t_1g = t_{d_t}^{-1}, g^{-1}t_i g = t_{i-1}t_{d_t}^{-e_i} \ (1 < i \leq d_t), [t_i, t_j] = 1 \ (1 \leq j < i \leq d_t)\rangle,$$

where p is a prime, $e_i = 1$ if p^{t-1} divides $i-1$, $e_i = 0$ if p^{t-1} does not divide $i-1$, $\mathbb{Z}_p^{d_t}$ is the direct product of $d_t = p^{t-1}(p-1)$ copies of \mathbb{Z}_p , C_{p^t} is a cyclic group of order p^t acting uniserially on $\mathbb{Z}_p^{d_t}$. In fact $H_2(K_t, \mathbb{Z}_p) = T(K_t) \times F(K_t)$, where $T(K_t)$ is a finite p -group and $F(K_t) \simeq \mathbb{Z}_p^l$. An interesting bound was proved few years ago in [4, Theorem A]: $\text{tf}(H_2(K_t, \mathbb{Z}_p)) = \frac{1}{2}d_t$ for all $p > 2$ and $t > 1$, but the case $p = 2$ needs to be treated separately (see [4, p. 148 and §7]). Now Theorem 1.1, specialized to K_t , illustrates that $K_t \widehat{\wedge} K_t \simeq \mathbb{Z}_p^v$ (for a suitable $v \leq l$) may be written as projective limit of finite p -groups. Similarly, this happens for $K_t \widehat{\otimes} K_t \simeq \mathbb{Z}_p^u$ for some $v \leq u \leq l$.

5. Open problems

Following [17, §2], if $\phi : H \rightarrow H^\phi$ is an isomorphism (and G and H two groups acting compatibly upon each other by conjugation), we may define

$$\eta(G, H) = \langle G, H^\phi \mid [g, h^\phi]^{g'} = [g^{g'}, (h^{g'})^\phi], [g, h^\phi]^{h'\phi} = [g^{h'}, (h^{h'})^\phi], \forall g, g' \in G, h, h' \in H \rangle,$$

where G and H are canonically embedded via $\iota_G : G \rightarrow \eta(G, H)$ and $\iota_H : H \rightarrow \eta(G, H)$. If one looks at G and H as normal subgroups of a bigger group M , then we denote with X_G a generating set for G and with X_H one for H and let $X = X_G \cup X_H$. Now denote by \tilde{X}_G a generating set of G , closed under conjugation by elements of X , and by \tilde{X}_H the corresponding for H . One gets

$$(5.1) \quad \eta(G, H) \simeq (G * H)/K,$$

where

$$K = \langle x[g, h]x^{-1}[g, h]^x \mid x \in X, g \in \tilde{X}_G, h \in \tilde{X}_H \rangle^{G*H}.$$

Further results (see [1, 17]) show that $\eta(G, H)$ is isomorphic to the free product with amalgamation on a suitable subgroup of G and H . The following results hold.

Proposition 5.1 (See [17], Lemma 2.1). *Let G and H be two normal subgroups (acting compatibly and by conjugation upon each other) of a group M . Then*

- (i) $((G \otimes H) \rtimes H) \rtimes G \simeq \eta(G, H)$;
- (ii) $G \otimes H \simeq G^{\eta(G, H)} \cap H^{\eta(G, H)}$;

Proposition 5.1 shows that we may connect the notion of nonabelian tensor product with that of free product, because of (5.1). Then it is interesting to understand the role of the topology in the free products of topological groups with amalgamation in this perspective. More specifically:

Question 5.2. Can we get a topological version of Proposition 5.1(i) and (ii)?

Looking at the fundamental paper of Graev [5], free products of topological groups have not perfect analogies with the abstract case. The presence of topology gives complications. The works of Morris and others [8, 9, 10, 11] illustrate how we should proceed. In fact another problem arises and it is strictly related to the previous question:

Question 5.3. Can we define properly $\eta(G, H)$, when G and H are two topological groups? If so, which are the topological properties of $\eta(G, H)$?

The suspect is that we could find a further argument for the proof of Theorem 1.1, after an appropriate answer to the above two questions.

A final question originates from possible generalizations of Theorem 1.2. For instance, we might replace the role of $H_2(G, \mathbb{Z}_p)$ by $H^2(G, \mathbb{T})$ when G is not a totally disconnected compact group (here \mathbb{T} denotes the torus group).

In fact $H_2(G, \mathbb{Z}_p)$ is related to the extension theory of G , when G is a pro- p -group and, when G would be compact, the second dimensional cohomology group with coefficients in \mathbb{T} seems to be more appropriate, instead of the two dimensional homology group with coefficients in \mathbb{Z}_p . Then:

Question 5.4. How can we extend Theorem 1.2 to compact groups?

Acknowledgements. I am very grateful to the editor and the referee for their excellent job. Finally, the present researches are supported in part by NRF (South Africa) for the Grant No. 93652 and in part from the Launching Grant No. 459235 of the University of Cape Town (South Africa).

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