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A RECURSIVE METHOD FOR DISCRETELY MONITORED GEOMETRIC ASIAN OPTION PRICES

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Abstract. We aim to compute discretely monitored geometric Asian option prices under the Heston model. This method involves explicit formula for multivariate generalized Fourier transform of volatility process and their integrals over different time intervals using a recursive method. As numerical results, we illustrate efficiency and accuracy of our method. In addition, we simulate scenarios which show evidently practical importance of our work.

1. Introduction

Asian options have path dependent payoffs which are functions of average value of the underlying asset over some period. Average values are based on arithmetic average or geometric average and are monitored continuously or at discrete times. From the perspective of risk management, Asian-style options are traded to avoid the risk from large fluctuations of underlying asset near the maturity. Those situations may occur as manipulators intend to disturb the market for their own profits.

Most Asian-style options traded in the markets are based on arithmetic average value of the underlying asset prices monitored discrete times. But it is well-known that there is no analytic formula available for arithmetic Asian option prices even under the Black-Scholes model. On the contrary, continuous-time geometric Asian option prices are more tractable and can be expressed analytically or semi-analytically at least under more relaxed assumptions covering various models such as Black-Scholes, Heston, and Lévy model [2, 3, 4, 7, 11, 12, 13]. In this regard, geometric Asian option prices can be

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utilized as approximate values for the corresponding arithmetic Asian option prices. In addition, geometric Asian option price formulas can be useful as subsidiary tool in computing arithmetic Asian option prices when some numerical methods for arithmetic Asian option are combined with analytic formulas for geometric Asian option.

In practice, most Asian-style options are based on discrete time monitoring, daily or weekly. But for academic perspective, there is more existing literature on continuous-time Asian options mainly due to their mathematical tractability. In general, discrete-time Asian option prices can be approximated by continuous-time counterparts if it is monitored frequently enough. But if monitoring frequency is a day or a week which is not sufficiently small, it is worthwhile to examine substantial difference between continuous-time option prices and discrete-time counterparts.

We stress that computing discrete time option prices is not just routine extension of results for continuous-time counterparts, as it is well-known among the specialists. Mathematically, we derive multivariate generalized Fourier transform of log asset prices at monitoring times recursively. At each step, we need to check if relevant analytic complex-valued functions are well-defined avoiding branch cut.

Since there have been a lot of academic attention and vast existing literature on Asian options, it is hardly possible to enlist them in full generality. Instead, we choose to focus on relatively more recent results for which analytic methods are employed. A more complete review can be found in Boyle and Potapchik [2], Tahani [12] and Kim and Wee [11]. Fusai and Meucci [7] provided analytic closed solutions for discretely monitored geometric Asian option prices and recursive formula for arithmetic Asian option prices under the framework of exponential Lévy model. In Černý and Kyriakou [4], they suggested an improved FFT pricing algorithm in Fourier inversion formula for discretely sampled Asian option prices under exponential Lévy model. In Cai and Kou [3] they presented a closed form solution for the double-Laplace transform of Asian options under the hyper exponential jump diffusion model, and obtained continuous-time arithmetic Asian option by inverting numerically. In Kim and Wee [11], closed form of analytic formulas for geometric Asian options are derived, which involves explicit expression of generalized joint Fourier transform of the square-root process and its three different temporal integrals as multivariable complex-valued functions. Tahani [12] dealt with analogous problem and applied the results when underlying assets are stocks, foreign exchange rate, and interest rates. Here the relevant Fourier transform is computed by numerical algorithm. Umezawa and Yamazaki [13] proposed a semi-analytic pricing method for more general class of discretely monitored path-dependent derivatives under time-changed Lévy process.

The paper is organized as follows. In Section 2, we model the underlying process and present a recursive method for multivariate generalized Fourier transform. In Section 3, we derive geometric Asian option price formula for four different types. In Section 4, we provide numerical results.

2. Preliminaries

We assume that under the risk-neutral measure Q, underlying asset price is given by Heston model [8]:

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t, \tag{1}$$

$$dv_t = \kappa(\theta - v_t)\, dt + \sigma\sqrt{v_t} dZ_t, \tag{2}$$

where r is the interest rate, $\sqrt{v_t}$ is a volatility process, θ is the long-run average of v_t, κ is the rate of mean reversion, and σ is the volatility of volatility. Here W_t and Z_t are two Brownian motions with correlation coefficient $\rho \in [-1, 1]$, i.e., $dW_t dZ_t = \rho dt$. The parameters θ, κ and σ are assumed to be positive constants. It is often convenient to write

$$W_t = \rho Z_t + \sqrt{1-\rho^2}\hat{W}_t, \tag{3}$$

where $\hat{W}_t$ is a standard Brownian motion independent of Z_t. We consider a filtration $\{\mathcal{F}_t : t \geq 0\}$, where $\mathcal{F}_t$ is the smallest σ-algebra generated by $\{Z_u, \hat{W}_u : u \leq t\}$.

The discretely monitored geometric Asian option is based on the geometric mean of the asset prices at predetermined time points $t_1, \ldots, t_n$. Let $I = \{t_1, \ldots, t_n\}$ be a collection of time points such that $0 \leq t_1 < t_2 < \cdots < t_n \leq t_{n+1} = T$, and G_I be the geometric mean of S_t at time points in I, i.e.,

$$G_I = \exp\left(\frac{1}{n}\sum_{k=1}^{n} \ln S_{t_k}\right).$$

The payoffs of the fixed strike geometric Asian call and put options with strike price K and maturity T are given, respectively, by

$$\max\{G_I - K, 0\} \quad \text{and} \quad \max\{K - G_I, 0\}.$$

Floating strike geometric Asian call and put options have payoffs

$$\max\{S_T - G_I, 0\} \quad \text{and} \quad \max\{G_I - S_T, 0\},$$

respectively. First we note that Jensen's inequality implies

$$G_I = e^{\frac{1}{n}\sum_{k=1}^{n} \ln S_{t_k}} \leq e^{\ln \frac{1}{n}\sum_{k=1}^{n} S_{t_k}} = \frac{1}{n}\sum_{k=1}^{n} S_{t_k}.$$

Therefore,

$$\mathbb{E}^{\mathrm{Q}}[G_I] \leq \mathbb{E}^{\mathrm{Q}}\left[\frac{1}{n}\sum_{k=1}^{n} S_{t_k}\right] = \frac{S_0}{n}\sum_{k=1}^{n} e^{rt_k} < \infty, \tag{4}$$

which means that G_I is integrable under the risk-neutral measure Q.

We introduce k^* as follows. For fixed $t < T$, if $t < t_1$, then $k^* = 0$ and if $t_1 < t < T$, then k^* is the largest integer $j \in \{1, 2, \ldots, n\}$ such that $t_j \le t < t_{j+1}$. Let

$$G_{t,I} \equiv e^{\frac{1}{n}\sum_{k=k^*+1}^{n} \ln S_{t_k}}, \tag{5}$$

$$K_{t,I} \equiv K e^{-\frac{1}{n}\sum_{k=1}^{k^*} \ln S_{t_k}}. \tag{6}$$

Then the price $C_I(t)$ at time t of the fixed strike geometric Asian call option with payoff $\max\{G_I - K, 0\}$ at maturity T, is given by

$$\begin{aligned} C_I(t) &= \mathbb{E}^{\mathrm{Q}}\Big[e^{-r(T-t)} \max\{G_I - K, 0\}\Big| \mathcal{F}_t\Big] \\ &= e^{-r(T-t)+\frac{1}{n}\sum_{k=1}^{k^*} \ln S_{t_k}} \mathbb{E}^{\mathrm{Q}}\left[\max\{G_{t,I} - K_{t,I}, 0\} | \mathcal{F}_t\right]. \end{aligned} \tag{7}$$

The price $\tilde{P}_I(t)$ at time t of the floating strike geometric Asian put option with payoff $\max\{G_I - S_T, 0\}$ at maturity T, is given by

$$\begin{aligned} \tilde{P}_I(t) &= \mathbb{E}^{\mathrm{Q}}\Big[e^{-r(T-t)} \max\{G_I - S_T, 0\}\Big| \mathcal{F}_t\Big] \\ &= e^{-r(T-t)+\frac{1}{n}\sum_{k=1}^{k^*} \ln S_{t_k}} \mathbb{E}^{\mathrm{Q}}\left[\max\{G_{t,I} - S_T e^{-\frac{1}{n}\sum_{k=1}^{k^*} \ln S_{t_k}}, 0\}\Big| \mathcal{F}_t\right]. \end{aligned} \tag{8}$$

The other two geometric Asian options can be treated similarly.

Recall the definition of $G_{t,I}$ from (5) and let, for $(s, w) \in \mathbb{C}^2$, $I \subset [0, T]$ and $0 \le t \le T$, define the generalized Fourier transform of $\ln G_{t,I}$ and $\ln S_T$

$$\Psi_{t,I}(s, w) \equiv \mathbb{E}^{\mathrm{Q}}\left[e^{s \ln G_{t,I} + w \ln S_T} \big| \mathcal{F}_t\right], \tag{9}$$

for complex numbers s and w for which the right-hand side of (9) is well-defined. As we shall see, the discretely sampled geometric Asian option prices are expressed in terms of $\Psi_{t,I}(s, w)$. It is worthwhile to note that $\Psi_{t,I}(s, w)$ can be considered of a multi-period affine transformation which was deal in Duffie, Filipović and Schachermayer [5] and Kang and Kang [9] in general setting of affine diffusion processes. In Kang and Kang [9], multi-period affine transformation provides a key tool to recover finite dimensional distribution of corresponding affine processes. Let

$$\mathcal{D} \equiv \{(s, w) \in \mathbb{C}^2 : \mathrm{Re}(s) \ge 0,\ \mathrm{Re}(w) \ge 0,\ 0 \le \mathrm{Re}(s) + \mathrm{Re}(w) \le 1\}.$$

The following lemma shows that (9) is well-defined for every $(s, w) \in \mathcal{D}$.

Lemma 2.1. *For $(s, w) \in \mathcal{D}$, $I \subset [0, T]$ and $t \in [0, T]$,*

$$\mathbb{E}^{\mathrm{Q}}\left|e^{s \ln G_{t,I} + w \ln S_T}\right| < \infty.$$

Proof. Suppose that $(s, w) \in \mathcal{D}$. If $k^* = n$, then $\mathbb{E}^{\mathrm{Q}}|e^{s \ln G_{t,I} + w \ln S_T}| = \mathbb{E}^{\mathrm{Q}}[(S_T)^{\mathrm{Re}(w)}] \le \mathrm{Re}(w) \le 1$ and $\mathbb{E}^{\mathrm{Q}} S_T = e^{rT} S_0 < \infty$. On the other hand, if $0 \le k^* < n$, then we have

$$\begin{aligned} \left|e^{s \ln G_{t,I} + w \ln S_T}\right| &= e^{\mathrm{Re}(s) \ln G_{t,I} + \mathrm{Re}(w) \ln S_T} \\ &= \left(e^{\frac{1}{n-k^*}\sum_{k=k^*+1}^{n} \ln S_{t_k}}\right)^{\frac{n-k^*}{n}\mathrm{Re}(s)} e^{\mathrm{Re}(w) \ln S_T} \end{aligned}$$

$$\le \Big(\frac{1}{n-k^*}\sum_{k=k^*+1}^{n} S_{t_k}\Big)^{\frac{n-k^*}{n}\mathrm{Re}(s)} (S_T)^{\mathrm{Re}(w)}$$

$$\le \Big(\frac{1}{n-k^*}\sum_{k=k^*+1}^{n} S_{t_k} + S_T\Big)^{\frac{n-k^*}{n}\mathrm{Re}(s)+\mathrm{Re}(w)}. \tag{10}$$

The assertion holds, since $0 \le \frac{n-k^*}{n}\mathrm{Re}(s) + \mathrm{Re}(w) \le 1$ and

$$\mathbb{E}^{\mathrm{Q}}\Big[\frac{1}{n-k^*}\sum_{k=k^*+1}^{n} S_{t_k} + S_T\Big] = \Big(\frac{1}{n-k^*}\sum_{k=k^*+1}^{n} e^{rt_k} + e^{rT}\Big)S_0 \ < \ \infty.$$

□

We adopt the following lemma from [10] to compute $\Psi_{t,I}(s,w)$ recursively in Lemmas 2.3 and 2.4.

Lemma 2.2 (Proposition 1 of Kim et al. [10])**.** *Define*

$$\mathcal{D}_\tau \equiv \Big\{(z_1,z_2)\in\mathbb{C}^2 :\ \mathbb{E}^{\mathrm{Q}}\Big[e^{\mathrm{Re}(z_1)\int_0^\tau v_t dt+\mathrm{Re}(z_2)v_\tau}\Big] < \infty\Big\}, \tag{11}$$

and $F_\tau, \tilde{F}_\tau : \mathbb{C}^2 \to \mathbb{C}$ *as*

$$F_\tau(z_1,z_2) \equiv \begin{cases} \cosh(\frac{\tau}{2}\sqrt{\kappa^2-2z_1\sigma^2})+(\kappa - z_2\sigma^2)\frac{\sinh(\frac{\tau}{2}\sqrt{\kappa^2-2z_1\sigma^2})}{\sqrt{\kappa^2-2z_1\sigma^2}} & \text{if } z_1 \ne \frac{\kappa^2}{2\sigma^2}, \\ 1+\frac{\tau}{2}(\kappa-z_2\sigma^2) & \text{if } z_1 = \frac{\kappa^2}{2\sigma^2}, \end{cases}$$

$$\tilde{F}_\tau(z_1,z_2) \equiv \frac{\sqrt{\kappa^2-2z_1\sigma^2}}{2}\sinh(\tfrac{\tau}{2}\sqrt{\kappa^2-2z_1\sigma^2}) + \frac{\kappa-z_2\sigma^2}{2}\cosh(\tfrac{\tau}{2}\sqrt{\kappa^2-2z_1\sigma^2}).$$

(a) *For every* $(z_1,z_2)\in\mathcal{D}_\tau$*,* $F_\tau(z_1,z_2)\ne 0$*. The argument of* F_τ*,* $\arg F_\tau$*, is uniquely determined on* $\mathcal{D}_\tau$ *with the following properties*:
  (i) *If* z_1 *is real and* $(z_1,z_2)\in\mathcal{D}_\tau$*, then* $-\frac{\pi}{2} < \arg F_\tau(z_1,z_2) < \frac{\pi}{2}$*.*
  (ii) $\arg F_\tau$ *is continuous on* $\mathcal{D}_\tau$*.*

(b) *With the argument of* F_τ *on* $\mathcal{D}_\tau$ *defined as above, we have, for* $(z_1,z_2)\in\mathcal{D}_\tau$*,*

$$\mathbb{E}^{\mathrm{Q}}\Big[e^{z_1\int_0^\tau v_t dt+z_2v_\tau}\Big] = \exp\left(\frac{\kappa v_0+\kappa^2\theta\tau}{\sigma^2} - \frac{2v_0}{\sigma^2}\frac{\tilde{F}_\tau(z_1,z_2)}{F_\tau(z_1,z_2)} - \frac{2\kappa\theta}{\sigma^2}\ln F_\tau(z_1,z_2)\right). \tag{12}$$

For $I\subset[0,T]$, $0\le t\le T$ and $k=k^*,k^*+1,\dots,n+1$, define

$$\tau_k = \tau_k(t) = \begin{cases} t & \text{if } k = k^*, \\ t_k & \text{if } k = k^*+1, k^*+2,\dots,n, \\ T & \text{if } k = n+1. \end{cases} \tag{13}$$

For $(s,w)\in\mathbb{C}^2$, we define $z_k(s,w)$, $k^*+1\le k\le n+1$, and $w_k(s,w)$, $k^*\le k\le n+1$, by

$$z_k(s,w) = \frac{(2\rho\kappa-\sigma)((n-k+1)s+nw)}{2\sigma n} + \frac{(1-\rho^2)((n-k+1)s+nw)^2}{2n^2}, \tag{14}$$

$$w_k(s,w) = \begin{cases} 0 & \text{if } k = k^*, \\ \frac{\rho s}{\sigma n} & \text{if } k^*+1 \le k \le n, \\ \frac{\rho w}{\sigma} & \text{if } k = n+1. \end{cases} \tag{15}$$

Lemma 2.3. *For $I \subset [0,T]$, $0 \le t \le T$, and $(s,w) \in \mathcal{D}$,*

$$\Psi_{t,I}(s,w) = e^a \mathbb{E}^{\mathrm{Q}}\Bigg[\exp\bigg(\sum_{k=k^*+1}^{n+1}\Big(z_k\int_{\tau_{k-1}}^{\tau_k} v_\tau d\tau + w_k v_{\tau_k}\Big)\bigg)\Bigg| v_t\Bigg], \tag{16}$$

where

$$\begin{aligned} a = a(s,w) = &\Big(\frac{s(n-k^*)}{n} + w\Big)\Big(\ln S_t - \frac{\rho v_t}{\sigma} - (r - \frac{\rho\kappa\theta}{\sigma})t\Big) \\ &+ (r - \frac{\rho\kappa\theta}{\sigma})\Big(\frac{s}{n}\sum_{k=k^*+1}^{n} t_k + wT\Big), \end{aligned} \tag{17}$$

$\tau_k = \tau_k(t)$, $z_k = z_k(s,w)$ and $w_k = w_k(s,w)$ are given in (13), (14) *and* (15), *respectively.*

Proof. Combining (1), (2) and (3), we have for $t \le u \le T$,

$$\begin{aligned} \ln S_u = &\ \ln S_t - \frac{\rho}{\sigma}v_t + (r - \frac{\rho\kappa\theta}{\sigma})(u-t) + (\frac{\rho\kappa}{\sigma} - \frac{1}{2})\int_t^u v_\tau d\tau \\ &+ \frac{\rho}{\sigma}v_u + \sqrt{1-\rho^2}\int_t^u \sqrt{v_\tau}d\hat{W}_\tau, \end{aligned} \tag{18}$$

and

$$\begin{aligned} &s\ln G_{t,I} + w\ln S_T \\ &= a + \Big(\frac{\rho\kappa}{\sigma} - \frac{1}{2}\Big)\Big(\frac{s}{n}\sum_{k=k^*+1}^{n}\int_t^{t_k} v_\tau d\tau + w\int_t^T v_\tau d\tau\Big) + \frac{\rho}{\sigma}\Big(\frac{s}{n}\sum_{k=k^*+1}^{n} v_{t_k} + wv_T\Big) \\ &\quad + \sqrt{1-\rho^2}\Big(\frac{s}{n}\sum_{k=k^*+1}^{n}\int_t^{t_k}\sqrt{v_\tau}d\hat{W}_\tau + w\int_t^T\sqrt{v_\tau}d\hat{W}_\tau\Big) \\ &\equiv a + b_1 + b_2, \end{aligned}$$

where

$$\begin{aligned} b_1 \equiv b_1(s,w) = &\Big(\frac{\rho\kappa}{\sigma} - \frac{1}{2}\Big)\sum_{k=k^*+1}^{n+1}\int_{\tau_{k-1}}^{\tau_k}\Big(\frac{(n-k+1)s}{n} + w\Big)v_\tau d\tau \\ &+ \frac{\rho}{\sigma}\Big(\frac{s}{n}\sum_{k=k^*+1}^{n} v_{\tau_k} + wv_{\tau_{n+1}}\Big), \\ b_2 \equiv b_2(s,w) = &\sqrt{1-\rho^2}\sum_{k=k^*+1}^{n+1}\int_{\tau_{k-1}}^{\tau_k}\Big(\frac{(n-k+1)s}{n} + w\Big)\sqrt{v_\tau}d\hat{W}_\tau, \end{aligned}$$

$a = a(s,w)$ is defined in (17). Let $\mathcal{G}$ be the σ-field generated by $\mathcal{F}_t$ and $\{Z_u: t < u \le T\}$. For $(s,w) \in \mathcal{D}$,

$$\begin{aligned}
&\Psi_{t,I}(s,w)\\
&= e^a \mathbb{E}^{\mathrm{Q}}\big[e^{b_1}\mathbb{E}^{\mathrm{Q}}\big[e^{b_2}\big|\mathcal{G}\big]\big|\mathcal{F}_t\big]\\
&= e^a \mathbb{E}^{\mathrm{Q}}\Big[\exp\Big(b_1 + \frac{1-\rho^2}{2}\sum_{k=k^*+1}^{n+1}\int_{\tau_{k-1}}^{\tau_k}\Big(\frac{(n-k+1)s}{n}+w\Big)^2 v_\tau d\tau\Big)\Big|\mathcal{F}_t\Big]\\
&= e^a \mathbb{E}^{\mathrm{Q}}\Big[\exp\Big\{\sum_{k=k^*+1}^{n+1}\Big(\Big(\frac{\rho\kappa}{\sigma}-\frac{1}{2}\Big)\frac{(n-k+1)s+w}{n}\\
&\quad + \frac{1-\rho^2}{2}\Big(\frac{(n-k+1)s}{n}+w\Big)^2\Big)\int_{\tau_k-1}^{\tau_k} v_\tau d\tau\\
&\quad + \sum_{k=k^*+1}^{n}\frac{\rho s}{\sigma n}v_{\tau_k} + \frac{\rho}{\sigma}w v_{\tau_{n+1}}\Big\}\Big|\mathcal{F}_t\Big],
\end{aligned}$$

since the conditional distribution of b_2, given $\mathcal{G}$, is the normal distribution with mean 0 and variance $(1-\rho^2)\sum_{k=k^*+1}^{n+1}\int_{\tau_{k-1}}^{\tau_k}\Big(\frac{(n-k+1)s}{n}+w\Big)^2 v_\tau\, d\tau$. The assertion follows by the definition of z_k and τ_k and the Markov property of $\{v_u : 0 \le u \le T\}$. □

For $t \ge 0$ and $y > 0$, let $\mathrm{Q}_{t,y}$ be a probability measure under which $\{v_u : u \ge t\}$ is a volatility process starting from y at time t and satisfies

$$dv_u = \kappa(\theta - v_u)d_u + \sigma\sqrt{v_u}dZ_u,\ u \ge t,$$

where $\{Z_u : u \ge 0\}$ is a Brownian motion. For $l = k^*, k^*+1, \ldots, n$ and $y > 0$, let

$$\begin{aligned}
\mathcal{D}_l' \equiv \Big\{&(z_{l+1}', \ldots, z_{n+1}'; w_{l+1}', \ldots, w_{n+1}') \in \mathbb{C}^{2(n-l+1)} :\\
&\mathbb{E}^{\mathrm{Q}_{\tau_l,y}}\Big[\exp\Big(\sum_{k=l+1}^{n+1}\Big(\mathrm{Re}(z_k')\int_{\tau_{k-1}}^{\tau_k} v_\tau d\tau + \mathrm{Re}(w_k')v_{\tau_k}\Big)\Big)\Big] < \infty\Big\}.
\end{aligned}$$

It is easy to show that $\mathcal{D}_l'$ does not depend on y. For $(z_{l+1}', \ldots, z_{n+1}'; w_{l+1}', \ldots, w_{n+1}') \in \mathcal{D}_l'$, we define $\tilde{w}_k'$, $l+1 \le k \le n+1$, by the backward recursion:

$$\tilde{w}_{n+1}' = w_{n+1}',$$

$$\tilde{w}_{k-1}' = w_{k-1}' + \frac{\kappa}{\sigma^2} - \frac{2}{\sigma^2}\frac{\tilde{F}_{\tau_k-\tau_{k-1}}(z_k', \tilde{w}_k')}{F_{\tau_k-\tau_{k-1}}(z_k', \tilde{w}_k')},\ k = n+1, n, \ldots, l+2. \tag{19}$$

Lemma 2.4. *For $l = k^*, k^*+1, \ldots, n$ and $(z_{l+1}', \ldots, z_{n+1}'; w_{l+1}', \ldots, w_{n+1}') \in \mathcal{D}_l'$, the following hold.*

(a) $F_{\tau_k-\tau_{k-1}}(z_k', \tilde{w}_k') \ne 0$, $l+1 \le k \le n+1$.

(b) *The argument of $F_{\tau_k-\tau_{k-1}}(z_k', \tilde{w}_k')$, $\arg F_{\tau_k-\tau_{k-1}}(z_k', \tilde{w}_k')$, is uniquely determined with the following properties*:

(i) $\arg F_{\tau_k-\tau_{k-1}}(z'_k,\tilde{w}'_k)=0$ *if* $(z'_{l+1},\ldots,z'_{n+1};w'_{l+1},\ldots,w'_{n+1})\in\mathcal{D}'_l\cap\mathbb{R}^{2(n-l+1)}$.

(ii) $\arg F_{\tau_k-\tau_{k-1}}(z'_k,\tilde{w}'_k)$ *is continuous in* $(z'_{l+1},\ldots,z'_{n+1};w'_{l+1},\ldots,w'_{n+1})$ *on* $\mathcal{D}'_l$.

(c) *For* $(z'_{l+1},\ldots,z'_{n+1};w'_{l+1},\ldots,w'_{n+1})\in\mathcal{D}'_l$ *and* $y>0$,

$$\begin{aligned}&\mathbb{E}^{\mathrm{Q}_{\tau_l,y}}\Big[\exp\Big(\sum_{k=l+1}^{n+1}\Big(z'_k\int_{\tau_{k-1}}^{\tau_k}v_\tau d\tau+w'_k v_{\tau_k}\Big)\Big)\Big]\\&=\exp\left(x_l y+\frac{\kappa^2\theta(\tau_{n+1}-\tau_l)}{\sigma^2}-\frac{2\kappa\theta}{\sigma^2}\sum_{k=l+1}^{n+1}\ln F_{\tau_k-\tau_{k-1}}(z'_k,\tilde{w}'_k)\right),\end{aligned}$$

where

$$x_l=\frac{\kappa}{\sigma^2}-\frac{2}{\sigma^2}\frac{\tilde{F}_{\tau_{l+1}-\tau_l}(z'_{l+1},\tilde{w}'_{l+1})}{F_{\tau_{l+1}-\tau_l}(z'_{l+1},\tilde{w}'_{l+1})}.$$

Proof. These can be proved by backward induction on l. If $l=n$, then the assertions hold by Lemma 2.2. Suppose that the assertions hold for some $l\in\{k^*+1,\ldots,n\}$. We have to show that the assertions also hold when $l-1$ is substituted for l. Suppose that $(z'_l,\ldots,z'_{n+1};w'_l,\ldots,w'_{n+1})\in\mathcal{D}'_{l-1}$. Then $(z'_{l+1},\ldots,z'_{n+1};w'_{l+1},\ldots,w'_{n+1})\in\mathcal{D}'_l$, which implies that $F_{\tau_k-\tau_{k-1}}(z'_k,\tilde{w}'_k)\neq 0$, $l+1\le k\le n+1$, by the induction hypothesis. Let $\mathcal{H}_{\tau_l}$ be the σ-field generated by $\{Z_u:\tau_{l-1}\le u\le\tau_l\}$. For $y>0$,

$$\begin{aligned}&\mathbb{E}^{\mathrm{Q}_{\tau_{l-1},y}}\Big[\exp\Big(\sum_{k=l}^{n+1}\Big(z'_k\int_{\tau_{k-1}}^{\tau_k}v_\tau d\tau+w'_k v_{\tau_k}\Big)\Big)\Big]\\&=\mathbb{E}^{\mathrm{Q}_{\tau_{l-1},y}}\Big[\exp\Big(z'_l\int_{\tau_{l-1}}^{\tau_l}v_\tau d\tau+w'_l v_{\tau_l}\Big)\\&\quad\times\mathbb{E}^{\mathrm{Q}_{\tau_{l-1},y}}\Big[\exp\Big(\sum_{k=l+1}^{n+1}\Big(z'_k\int_{\tau_{k-1}}^{\tau_k}v_\tau d\tau+w'_k v_{\tau_k}\Big)\Big)\Big|\mathcal{H}_{\tau_l}\Big]\Big]\\&=\mathbb{E}^{\mathrm{Q}_{\tau_{l-1},y}}\Big[\exp\Big(z'_l\int_{\tau_{l-1}}^{\tau_l}v_\tau d\tau+w'_l v_{\tau_l}\Big)\\&\quad\times\mathbb{E}^{\mathrm{Q}_{\tau_l,v_{\tau_l}}}\Big[\exp\Big(\sum_{k=l+1}^{n+1}\Big(z'_k\int_{\tau_{k-1}}^{\tau_k}v_\tau d\tau+w'_k v_{\tau_k}\Big)\Big)\Big]\Big]\\&=\mathbb{E}^{\mathrm{Q}_{\tau_{l-1},y}}\Big[\exp\Big(z'_l\int_{\tau_{l-1}}^{\tau_l}v_\tau d\tau+w'_l v_{\tau_l}+x_l v_{\tau_l}+\frac{\kappa^2\theta(\tau_{n+1}-\tau_l)}{\sigma^2}\\&\quad-\frac{2\kappa\theta}{\sigma^2}\sum_{k=l+1}^{n+1}\ln F_{\tau_k-\tau_{k-1}}(z'_k,\tilde{w}'_k)\Big)\Big]\end{aligned}$$

$$= \exp\Big(\frac{\kappa^2\theta(\tau_{n+1}-\tau_l)}{\sigma^2} - \frac{2\kappa\theta}{\sigma^2}\sum_{k=l+1}^{n+1}\ln F_{\tau_k-\tau_{k-1}}(z_k', \tilde{w}_k')\Big)$$

$$\times\, \mathbb{E}^{\mathrm{Q}_{\tau_{l-1},y}}\Big[\exp\Big(z_l'\int_{\tau_{l-1}}^{\tau_l} v_\tau d\tau + \tilde{w}_l' v_{\tau_l}\Big)\Big], \tag{20}$$

where the first equality follows from the induction hypothesis and the second equality follows from the definition of $\tilde{w}_l'$. Since $(z_l', \dots, z_{n+1}'; w_l', \dots, w_{n+1}') \in \mathcal{D}_{l-1}'$, (20) implies that $(z_l', \tilde{w}_l') \in \mathcal{D}_{\tau_l-\tau_{l-1}}$, which completes the proof of (b) by Lemma 2.2(a). By making use of Lemma 2.2(b) we complete the proof of (c). □

Theorem 2.5. *Let $I \subset [0,T]$ and $0 \le t \le T$.*

(a) *If $(s,w) \in \mathcal{D}$, then for $k = k^*+1, \dots, n+1$, $F_{\tau_k-\tau_{k-1}}(z_k, \tilde{w}_k) \neq 0$.*

(b) *If $(s,w) \in \mathcal{D}\cap\mathbb{R}^2$, then for $k = k^*+1, \dots, n+1$, $F_{\tau_k-\tau_{k-1}}(z_k, \tilde{w}_k) > 0$.*

(c) *For $t \in [0,T]$ and $(s,w) \in \mathcal{D}$, $\Psi_{t,I}(s,w)$ is given by*

$$\Psi_{t,I}(s,w) = \exp\left(a + \tilde{w}_{k^*}v_t + \frac{\kappa^2\theta(T-t)}{\sigma^2} - \frac{2\kappa\theta}{\sigma^2}\sum_{k=k^*+1}^{n+1}\ln F_{\tau_k-\tau_{k-1}}(z_k, \tilde{w}_k)\right), \tag{21}$$

where $a = a(s,w)$, $\tilde{w}_k = \tilde{w}_k(s,w)$ and $z_k = z_k(s,w)$ are defined in (17)*,* (19) *and* (14)*, respectively.*

Proof. By Lemmas 2.1 and 2.3,

$$(z_{k^*+1}, z_{k^*+2}, \dots, z_{n+1}; w_{k^*+1}, w_{k^*+2}, \dots, w_{n+1}) \in \mathcal{D}_{k^*}'.$$

Parts (a), (b) and (c) follow from Lemma 2.4(a), Lemma 2.4(b) and Lemma 2.4(c), respectively, combined with Lemma 2.3. □

3. Main results

To derive semi-analytic formula for geometric Asian option prices, we basically use the classical Fourier inversion formula in general setting as follows:

Let X be a random variable with characteristic function ϕ_X under P,

$$\phi_X(\xi) = \mathbb{E}^{\mathrm{P}}[e^{i\xi x}].$$

Then

$$\mathrm{P}(X \ge x) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big(\phi_X(\xi)\frac{e^{-i\xi x}}{i\xi}\Big)d\xi \tag{22}$$

for any continuous point x of distribution function of X.

This formula is basically analogous to approach used in Heston [8] for option pricing to extend Black-Scholes model. It is very well-known that Heston model is exemplified not only as a stochastic volatility model but also as a multidimensional affine model. Although there has been huge abundant literature on affine models, Duffie, Filipović and Schachermayer [5] and Duffie, Pan and Singleton [6] are attributed as pioneering breakthrough for affine models and Fourier transform method adopted on them. This is commonly named as transformation analysis in asset pricing which provides most useful and tractable formulas for wide range of option pricing especially when underlying assets follow affine models. Following the same spirit, we present our main results in explicit and computable form under Heston model in Theorem 3.1 and Theorem 3.2. Recall definitions of $K_{t,I}$ and $\Psi_{t,T}$ from (6) and (21).

Theorem 3.1. (a) *The price $C_I(t)$ of a fixed strike geometric Asian call option at time $t \in [0,T]$ is given by*

$$C_I(t) = e^{-r(T-t)+\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}} \Bigg[\frac{\Psi_{t,I}(1,0)-K_{t,I}}{2}$$

$$+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\Big(\Psi_{t,I}(1+i\xi,0)-K_{t,I}\Psi_{t,I}(i\xi,0)\Big)\frac{e^{-i\xi\ln K_{t,I}}}{i\xi}\right)d\xi\Bigg]. \tag{23}$$

(b) *The price $P_I(t)$ of a fixed strike geometric Asian put option at time $t \in [0,T]$ is given by*

$$P_I(t) = e^{-r(T-t)+\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}} \Bigg[\frac{K_{t,I}-\Psi_{t,I}(1,0)}{2}$$

$$+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\Big(\Psi_{t,I}(1+i\xi,0)-K_{t,I}\Psi_{t,I}(i\xi,0)\Big)\frac{e^{-i\xi\ln K_{t,I}}}{i\xi}\right)d\xi\Bigg]. \tag{24}$$

Proof. We introduce a probability measure Q^* by

$$d\mathrm{Q}^* = \frac{G_I}{\mathbb{E}^{\mathrm{Q}}[G_I]}d\mathrm{Q} \tag{25}$$

and rewrite (7)

$$\begin{aligned} C_I(t) &= e^{-r(T-t)+\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}\Big(\mathbb{E}^{\mathrm{Q}}[G_{t,I}\mathbb{1}_{\{G_{t,I}\ge K_{t,I}\}}|\mathcal{F}_t] \\ &\quad - K_{t,I}\mathrm{Q}(G_{t,I}\ge K_{t,I}|\mathcal{F}_t)\Big) \\ &= e^{-r(T-t)+\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}\Big\{\Psi_{t,I}(1,0)\mathrm{Q}^*(G_{t,I}\ge K_{t,I}|\mathcal{F}_t) \\ &\quad - K_{t,I}\mathrm{Q}(G_{t,I}\ge K_{t,I}|\mathcal{F}_t)\Big\}. \end{aligned} \tag{26}$$

By Fourier inversion formula (22), we have

$$\begin{aligned} \mathrm{Q}(G_{t,I}\ge K_{t,I}|\mathcal{F}_t) &= \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big(\mathbb{E}^{\mathrm{Q}}[e^{i\xi\ln G_{t,I}}|\mathcal{F}_t]\frac{e^{-i\xi\ln K_{t,I}}}{i\xi}\Big)d\xi \\ &= \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big(\Psi_{t,I}(i\xi,0)\frac{e^{-i\xi\ln K_{t,I}}}{i\xi}\Big)d\xi, \end{aligned} \tag{27}$$

and

$$Q^*(G_{t,I}\ge K_{t,I}|\mathcal{F}_t) = \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big(\mathbb{E}^{\mathrm{Q}^*}[e^{i\xi\ln G_{t,I}}|\mathcal{F}_t]\frac{e^{-i\xi\ln K_{t,I}}}{i\xi}\Big)d\xi$$

$$= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big(\frac{\Psi_{t,I}(1+i\xi,0)}{\Psi_{t,I}(1,0)}\frac{e^{-i\xi \ln K_{t,I}}}{i\xi}\Big)d\xi, \tag{28}$$

since the conditional characteristic function under Q^* is given as follows:

$$\mathbb{E}^{\mathrm{Q}^*}[e^{i\xi \ln G_{t,I}}|\mathcal{F}_t] = \frac{\mathbb{E}^{\mathrm{Q}}[G_{t,I}e^{i\xi \ln G_{t,I}}|\mathcal{F}_t]}{\mathbb{E}^{\mathrm{Q}}[G_{t,I}|\mathcal{F}_t]} = \frac{\Psi_{t,I}(1+i\xi,0)}{\Psi_{t,I}(1,0)}.$$

Combining (26), (27) and (28) yields (23). Equation (24) follows from the put-call parity;

$$\begin{aligned} C_I(t) - P_I(t) &= e^{-r(T-t)+\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}\mathbb{E}^{\mathrm{Q}}[G_{t,I} - K_{t,I}|\mathcal{F}_t] \\ &= e^{-r(T-t)+\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}(\Psi_{t,I}(1,0) - K_{t,I}). \end{aligned}$$

□

Theorem 3.2. (a) *The price $\tilde{C}_I(t)$ of a floating strike geometric Asian call option at time $t \in [0,T]$ is given by*

$$\begin{aligned} \tilde{C}_I(t) = e^{-r(T-t)}\Big[&\frac{1}{2}\Big(e^{r(T-t)}S_t - e^{\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}\Psi_{t,I}(1,0)\Big) \\ &+ \frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big\{\Big(e^{\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}\Psi_{t,I}(1+i\xi,-i\xi) \\ &- \Psi_{t,I}(i\xi,1-i\xi)\Big)\frac{e^{\frac{i\xi}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}}{i\xi}\Big\}d\xi\Big]. \end{aligned} \tag{29}$$

(b) *The price $\tilde{P}_I(t)$ of a floating strike geometric Asian put option at time $t \in [0,T]$ is given by*

$$\begin{aligned} \tilde{P}_I(t) = e^{-r(T-t)}\Big[&\frac{1}{2}\Big(e^{\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}\Psi_{t,I}(1,0) - e^{r(T-t)}S_t\Big) \\ &+ \frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big\{\Big(e^{\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}\Psi_{t,I}(1+i\xi,-i\xi) \\ &- \Psi_{t,I}(i\xi,1-i\xi)\Big)\frac{e^{\frac{i\xi}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}}{i\xi}\Big\}d\xi\Big]. \end{aligned} \tag{30}$$

Proof. We introduce a probability measure Q^{**} by

$$d\mathrm{Q}^{**} = \frac{S_T}{e^{rT}S_0}d\mathrm{Q}.$$

Starting from (8), we have

$$\begin{aligned} \tilde{P}_I(t) &= e^{-r(T-t)+\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}\mathbb{E}^{\mathrm{Q}}\Big[G_{t,I}\mathbb{1}_{\{G_{t,I}\geq S_T e^{-\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}\}}\Big|\mathcal{F}_t\Big] \\ &\quad - e^{-r(T-t)}\mathbb{E}^{\mathrm{Q}}\Big[S_T\mathbb{1}_{\{G_{t,I}\geq S_T e^{-\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}\}}\Big|\mathcal{F}_t\Big] \\ &= e^{-r(T-t)+\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}\Psi_{t,I}(1,0)\mathrm{Q}^*\Big(\frac{G_{t,I}}{S_T} \geq e^{-\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}\Big|\mathcal{F}_t\Big) \\ &\quad - S_t\mathrm{Q}^{**}\Big(\frac{G_{t,I}}{S_T} \geq e^{-\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}\Big|\mathcal{F}_t\Big) \end{aligned}$$

$$= e^{-r(T-t)+\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}\Psi_{t,I}(1,0)$$

$$\times\Big\{\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big(\mathbb{E}^{\mathrm{Q}^*}[e^{i\xi\ln(G_{t,I}/S_T)}|\mathcal{F}_t]\frac{e^{\frac{i\xi}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}}{i\xi}\Big)d\xi\Big\}$$

$$-S_t\Big\{\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big(\mathbb{E}^{\mathrm{Q}^{**}}[e^{i\xi\ln(G_{t,I}/S_T)}|\mathcal{F}_t]\frac{e^{\frac{i\xi}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}}{i\xi}\Big)d\xi\Big\}, \tag{31}$$

where in the last equality Fourier inversion formula is employed. Note that the conditional characteristic functions are given by

$$\mathbb{E}^{\mathrm{Q}^*}[e^{i\xi\ln(G_{t,I}/S_T)}|\mathcal{F}_t] = \frac{\mathbb{E}^{\mathrm{Q}}[G_{t,I}e^{i\xi\ln(G_{t,I}/S_T)}|\mathcal{F}_t]}{\mathbb{E}^{\mathrm{Q}}[G_{t,I}|\mathcal{F}_t]} = \frac{\Psi_{t,I}(1+i\xi,-i\xi)}{\Psi_{t,I}(1,0)},$$

$$\mathbb{E}^{\mathrm{Q}^{**}}[e^{i\xi\ln(G_{t,I}/S_T)}|\mathcal{F}_t] = \frac{\mathbb{E}^{\mathrm{Q}}[S_Te^{i\xi\ln(G_{t,I}/S_T)}|\mathcal{F}_t]}{\mathbb{E}^{\mathrm{Q}}[S_T|\mathcal{F}_t]} = \frac{\Psi_{t,I}(i\xi,1-i\xi)}{e^{r(T-t)}S_t}.$$

Plugging these into (31) yields (30). Equation (29) is obtained from the put-call parity

$$\begin{aligned}\tilde{C}_I(t)-\tilde{P}_I(t) &= e^{-r(T-t)+\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}\mathbb{E}^{\mathrm{Q}}[e^{-\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}S_T-G_{t,I}|\mathcal{F}_t]\\ &= e^{-r(T-t)}\mathbb{E}^{\mathrm{Q}}[S_T|\mathcal{F}_t]-e^{-r(T-t)+\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}\mathbb{E}^{\mathrm{Q}}[G_{t,I}|\mathcal{F}_t]\\ &= S_t-e^{-r(T-t)+\frac{1}{n}\sum_{k=1}^{k^*}\ln S_{t_k}}\Psi_{t,I}(1,0).\end{aligned}$$ □

4. Numerical results

TABLE 1. Fixed strike geometric Asian call option prices of different maturities, varying the number of time points; K=90.

$K = 90$		Analytic formula		Simulation		
T	Δt	Price	CPU	Price	Confidence interval	CPU
1 mo.	weekly	10.2732	0.19	10.2930	(10.2582, 10.3278)	26.77
	daily	10.1714	0.67	10.1524	(10.1215, 10.1833)	27.16
	continuously	10.1513	0.12	10.1462	(10.1162, 10.1762)	28.69
3 mo.	weekly	10.9554	0.45	10.9607	(10.9122, 11.0091)	27.42
	daily	10.8454	2.07	10.8610	(10.8143, 10.9078)	29.13
	continuously	10.8175	0.11	10.8108	(10.7645, 10.8571)	27.60
6 mo.	weekly	11.9916	0.82	12.0153	(11.9550, 12.0757)	27.96
	daily	11.8924	3.65	11.8554	(11.7963, 11.9145)	28.66
	continuously	11.8664	0.12	11.9129	(11.8539, 11.9719)	27.89
1 yr.	weekly	13.6950	1.98	13.7005	(13.6259, 13.7751)	28.28
	daily	13.6144	9.14	13.6490	(13.5749, 13.7232)	28.25
	continuously	13.5931	0.12	13.6247	(13.5510, 13.6984)	27.79
2 yr.	weekly	16.1773	3.56	16.1172	(16.0255, 16.2089)	28.01
	daily	16.1153	17.03	16.1548	(16.0633, 16.2463)	34.10
	continuously	16.0988	0.12	16.1171	(16.0260, 16.2081)	27.98
3 yr.	weekly	18.0146	5.53	17.9957	(17.8920, 18.0993)	27.92
	daily	17.9617	24.72	17.9147	(17.8114, 18.0179)	34.26
	continuously	17.9475	0.14	18.0015	(17.8981, 18.1049)	27.23

TABLE 2. Fixed strike geometric Asian call option prices of different maturities, varying the number of time points; K=100.

$K = 100$		Analytic formula		Simulation		
T	Δt	Price	CPU	Price	Confidence interval	CPU
1 mo.	weekly	2.4389	0.03	2.4246	(2.4035, 2.4457)	24.28
	daily	2.1222	0.14	2.1410	(2.1224, 2.1596)	25.69
	continuously	2.0472	0.06	2.0470	(2.0292, 2.0649)	27.62
3 mo.	weekly	3.7881	0.12	3.7836	(3.7517, 3.8156)	28.06
	daily	3.6179	0.62	3.5922	(3.5616, 3.6228)	29.79
	continuously	3.5735	0.03	3.5849	(3.5546, 3.6151)	25.16
6 mo.	weekly	5.2132	0.32	5.1836	(5.1408, 5.2264)	30.99
	daily	5.0910	1.51	5.0861	(5.0443, 5.1279)	32.10
	continuously	5.0588	0.06	5.0860	(5.0445, 5.1276)	25.19
1 yr.	weekly	7.2243	0.67	7.2180	(7.1609, 7.2751)	25.77
	daily	7.1364	2.37	7.1526	(7.0962, 7.2089)	25.77
	continuously	7.1132	0.03	7.0957	(7.0394, 7.1520)	24.86
2 yr.	weekly	9.9948	1.26	10.0694	(9.9943, 10.1445)	26.09
	daily	9.9309	6.09	9.8934	(9.8190, 9.9679)	30.37
	continuously	9.9139	0.09	9.8978	(9.8236, 9.9721)	30.34
3 yr.	weekly	12.0639	1.77	12.0221	(11.9347, 12.1095)	27.01
	daily	12.0102	10.54	12.0336	(11.9461, 12.1210)	31.20
	continuously	11.9959	0.06	12.0301	(11.9423, 12.1179)	27.95

TABLE 3. Fixed strike geometric Asian call option prices of different maturities, varying the number of time points; K=110.

$K = 110$		Analytic formula		Simulation		
T	Δt	Price	CPU	Price	Confidence interval	CPU
1 mo.	weekly	0.1012	0.14	0.1012	(0.0973, 0.1051)	28.49
	daily	0.0447	0.54	0.0425	(0.0402, 0.0447)	27.82
	continuously	0.0350	0.10	0.0337	(0.0318, 0.0357)	27.68
3 mo.	weekly	0.5949	0.36	0.5899	(0.5778, 0.6021)	27.43
	daily	0.5086	1.60	0.5005	(0.4897, 0.5114)	28.80
	continuously	0.4869	0.10	0.4894	(0.4786, 0.5002)	27.51
6 mo.	weekly	1.4444	0.81	1.4343	(1.4121, 1.4566)	28.28
	daily	1.3597	3.40	1.3471	(1.3260, 1.3683)	29.07
	continuously	1.3376	0.09	1.3269	(1.3058, 1.3479)	27.68
1 yr.	weekly	2.9479	1.56	2.9443	(2.9077, 2.9809)	28.70
	daily	2.8759	7.92	2.8865	(2.8507, 2.9223)	29.28
	continuously	2.8569	0.11	2.8440	(2.8084, 2.8797)	28.38
2 yr.	weekly	5.3531	3.26	5.3520	(5.2962, 5.4077)	28.66
	daily	5.2957	16.78	5.2729	(5.2177, 5.3282)	35.63
	continuously	5.2804	0.11	5.3213	(5.2659, 5.3767)	27.95
3 yr.	weekly	7.3315	5.27	7.3267	(7.2566, 7.3968)	28.31
	daily	7.2815	24.68	7.2639	(7.1944, 7.3334)	34.60
	continuously	7.2682	0.12	7.3103	(7.2405, 7.3801)	27.70

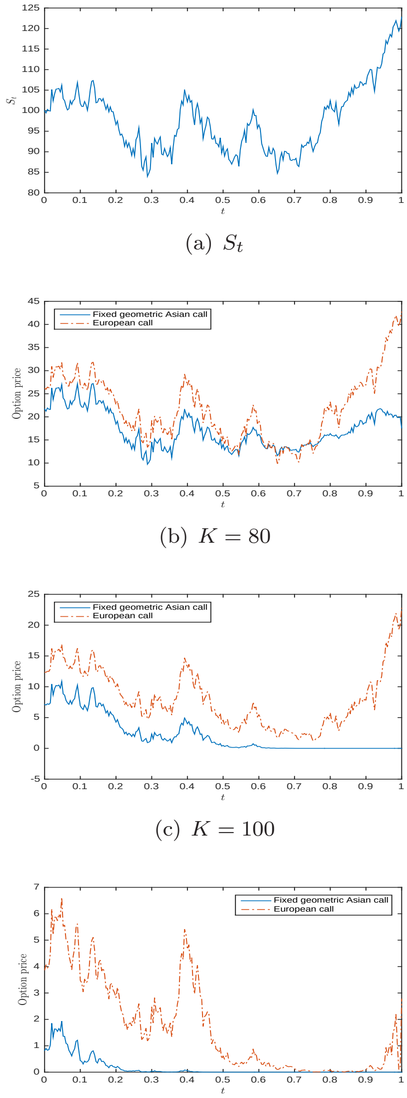
Throughout this section, we present numerical results under Heston model with parameters $r = 0.05$, $v_0 = 0.09$ and $S_0 = 100$, along with the parameters reported in Bakshi et al. [1], $\kappa = 1.15$, $\theta = 0.0348$, $\sigma = 0.39$, and $\rho = -0.64$.

TABLE 4. Floating strike geometric Asian call option prices of different maturities, varying the number of time points.

Floating strike		Analytic formula		Simulation		
T	Δt	Price	CPU	Price	Confidence interval	CPU
1 mo.	weekly	1.6624	0.04	1.6681	(1.6536, 1.6826)	29.92
	daily	1.9852	0.17	1.9869	(1.9697, 2.0041)	27.84
	continuously	2.0601	0.20	2.0644	(2.0466, 2.0821)	26.84
3 mo.	weekly	3.3068	0.12	3.3125	(3.2848, 3.3402)	25.30
	daily	3.4763	0.49	3.4570	(3.4281, 3.4859)	26.47
	continuously	3.5203	0.21	3.5086	(3.4793, 3.5380)	25.33
6 mo.	weekly	4.6791	0.26	4.7144	(4.6764, 4.7525)	25.70
	daily	4.7995	1.07	4.8115	(4.7727, 4.8502)	25.89
	continuously	4.8311	0.24	4.8053	(4.7666, 4.8441)	25.86
1 yr.	weekly	6.4613	0.54	6.4755	(6.4262, 6.5248)	28.28
	daily	6.5470	2.40	6.5958	(6.5452, 6.6464)	25.80
	continuously	6.5694	0.29	6.5859	(6.5351, 6.6366)	25.72
2 yr.	weekly	9.2207	1.09	9.2653	(9.1997, 9.3310)	26.20
	daily	9.2827	4.82	9.2883	(9.2218, 9.3548)	31.04
	continuously	9.2989	0.39	9.3319	(9.2654, 9.3983)	29.84
3 yr.	weekly	11.8065	1.59	11.8244	(11.7422, 11.9067)	28.47
	daily	11.8588	8.22	11.8688	(11.7863, 11.9513)	31.80
	continuously	11.8724	0.32	11.9253	(11.8427, 12.0080)	31.94

In Tables 1-4 we compare our analytic pricing method with the plain Monte Carlo method to illustrate efficiency and accuracy of our method. We present numerical results for the prices of the discretely sampled geometric Asian call options with fixed and floating strikes at time 0. In Tables 1-4, we denote the strike price by K, maturity by T and monitoring period by Δt. We provide prices for discrete-time geometric Asian option as well as continuous-time counterparts obtained by Kim and Wee [11] for purpose of comparison. Although daily monitored geometric Asian option prices are close to continuous time counterparts, we observe substantial difference between weekly monitored geometric Asian and continuous-time counterparts especially for short maturity options. We present fixed strike geometric Asian call option prices, $C_I(0)$, for $K = 90$, 100, 110 in Tables 1-3, respectively and floating strike geometric Asian call option prices, $\tilde{C}_I(0)$, in Table 4. In each table, for Monte Carlo simulation results with confidence intervals of 95% are provided. CPU time spent for computation is included in the tables to illustrate the efficiency of our analytic method. The results show that Monte Carlo results have accuracy comparable to our method, although efficiency is far below our method.

In Figures 1 and 2, we simulate sample paths of underlying assets under two extremely different scenarios when there are large fluctuations near the maturity. Under each scenario, we compute and graph fixed strike geometric Asian call option prices and European call option prices when $K = 80, 100, 120$ and T is one year. In Figure 1, we simulate a sample path of underlying asset when there is a sharp rise and in Figure 2, when there is a sharp decline near the maturity. As we understand and expect intuitively, geometric Asian



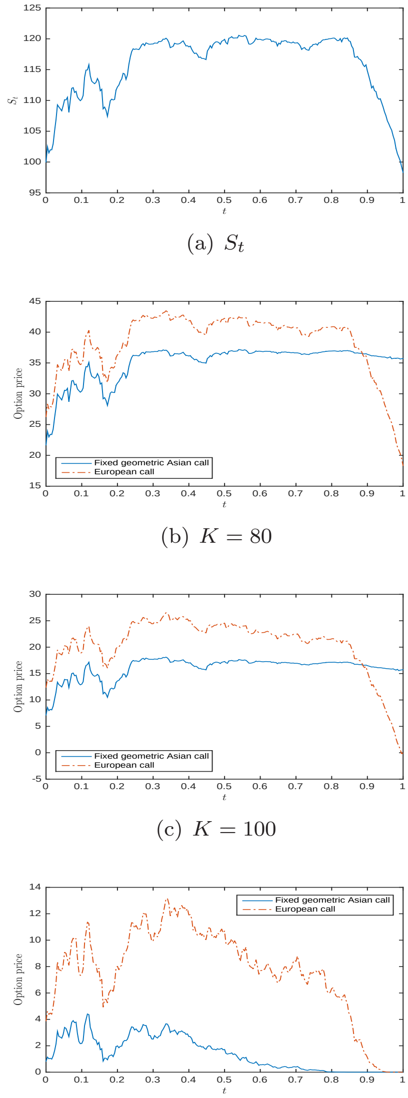
(a) S_t

(b) $K = 80$

(c) $K = 100$

(d) $K = 120$

FIGURE 1. Sample path with sharp rise.



(a) S_t

(b) $K = 80$

(c) $K = 100$

(d) $K = 120$

FIGURE 2. Sample path with sharp decline.

option prices stay quite stable near the maturity under both scenarios. On the other hand, European call option prices fluctuate to the same extent as the underlying asset prices near the maturity. These simulation results show evidently theoretical interest and practical importance of Asian options and our work.

All programming was done in MATLAB and the hardware used was an Intel(R) Core(TM) i3-3220 CPU 3.3 GHz PC. For the Monte Carlo simulation, we generate 10^5 replications for each result, with 1200 time points in each replication.

References

[1] G. S. Bakshi, C. Cao, and Z. W. Chen, *Empirical performance of alternative option pricing models*, J. Financ. **2** (1997), no. 2, 2003–2049.

[2] P. P. Boyle and A. Potapchikb, *Prices and sensitivities of Asian options*: *A survey*, Insurance Math. Econom. **42** (2008), no. 1,189–211.

[3] N. Cai and S. G. Kou, *Pricing Asian options under a hyper-exponential jump diffusion model*, Oper. Res. **60**, (2012), no. 1, 64–77.

[4] A. Černý and I. Kyriakou, *An improved convolution algorithm for discretely sampled Asian options*, Quant. Finance **11** (2011), no. 3, 381–389.

[5] D. Duffie, D. Filipović, and W. Schachermayer, *Affine processes and applications in finance*, Ann. Appl. Probab. **13** (2003), no. 3, 984–1053.

[6] D. Duffie, J. Pan, and K. Singleton, *Transform analysis and asset pricing for affine jump-diffusions*, Econometrica **68** (2000), no. 6, 1343–1376.

[7] G. Fusai and A. Meucci, *Pricing discretely monitored Asian options under Lévy processes*, J. Bank. Financ. **32** (2008), no. 10, 2076–2088.

[8] S. L. Heston, *A closed-form solution for options with stochastic volatility with applications to bonds and currency options*, Rev. Financ. Stud. **6** (1993), no. 2, 327–343.

[9] W. Kang and C. Kang, *Large deviations for affine diffusion processes on* $\mathbb{R}_+^m \times \mathbb{R}^n$, Stochastic Process. Appl. **124** (2014), no. 6, 2188–2227.

[10] J. Kim, B. Kim, K.-S. Moon, and I.-S. Wee, *Valuation of power options under Heston's stochastic volatility model*, J. Econom. Dynam. Control **36** (2012), no. 11, 1796–1813.

[11] B. Kim and I.-S. Wee, *Pricing of geometric Asian options under Heston's stochastic volatility model*, Quant. Financ. **14** (2014), 1795–1809.

[12] N. Tahani, *Exotic geometric average options pricing under stochastic volatility*, Appl. Math. Finance **20** (2013), no. 3, 229–245.

[13] Y. Umezawa and A. Yamazaki, *Pricing path-dependent options with discrete monitoring under time-changed Lévy processes*, Appl. Math. Finance **22** (2015), no. 2, 133–161.

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