

A SHORT NOTE ON BIHARMONIC SUBMANIFOLDS IN 3-DIMENSIONAL GENERALIZED (κ, μ) -MANIFOLDS

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ABSTRACT. We characterize proper biharmonic anti-invariant surfaces in 3-dimensional generalized (κ, μ) -manifolds with constant mean curvature by means of the scalar curvature of the ambient space and the mean curvature. In addition, we give a method for constructing infinity many examples of proper biharmonic submanifolds in a certain 3-dimensional generalized (κ, μ) -manifold. Moreover, we determine 3-dimensional generalized (κ, μ) -manifolds which admit a certain kind of proper biharmonic foliation.

1. Introduction

The notion of biharmonic submanifolds is a natural extension of the notion of minimal submanifolds from a variational point of view. Non-minimal biharmonic submanifolds are said to be proper. Considerable advancement has been made in the study of proper biharmonic submanifolds in manifolds with special metric properties (e.g., real space forms, complex space forms, Sasakian space forms, conformally flat spaces, etc.) since the beginning of this century.

When the dimension of the ambient space is three, it seems worthwhile and interesting to construct and classify proper biharmonic submanifolds in contact metric 3-manifolds. Some classification results for proper biharmonic submanifolds in Sasakian 3-manifolds have been obtained (see, for example, [3], [5], [6]).

As an extension of the notion of a Sasakian manifold, Koufogiorgos and Tsihlias [7] introduced the notion of a generalized (κ, μ) -manifold for two real functions $\kappa (\leq 1)$ and μ . If $\kappa \equiv 1$, then it is a Sasakian manifold. Markellos and Papantoniou [10] studied proper biharmonic Legendre curves and anti-invariant surfaces of 3-dimensional non-Sasakian generalized (κ, μ) -manifolds. However, they provided no examples of such submanifolds. As far as the author knows, there have been no examples of proper biharmonic submanifolds in non-Sasakian contact metric 3-manifolds.

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In this short note, we first give a method for constructing infinity many examples of proper biharmonic Legendre curves in a certain non-Sasakian generalized (κ, μ) -manifold. This result shows that Theorem 3.2 in [10] is incorrect. We then give a characterization of proper biharmonic anti-invariant surfaces in 3-dimensional non-Sasakian generalized (κ, μ) -manifolds with constant mean curvature, by means of the scalar curvature of the ambient space and the mean curvature. Applying the result, we obtain a method for constructing infinity many examples of proper biharmonic anti-invariant surfaces in a certain 3-dimensional non-Sasakian generalized (κ, μ) -manifold. Moreover, we determine 3-dimensional non-Sasakian generalized (κ, μ) -manifolds which admit a certain kind of proper biharmonic anti-invariant foliation.

2. Preliminaries

Let M^n be an n -dimensional submanifold of a Riemannian manifold \tilde{M} . Let us denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections on M^n and \tilde{M} , respectively. The Gauss and Weingarten formulas are respectively given by

$$(2.1) \quad \begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + B(X, Y), \\ \tilde{\nabla}_X N &= -A_N X + D_X N \end{aligned}$$

for tangent vector fields X, Y and normal vector field N , where B, A and D are the second fundamental form, the shape operator and the normal connection.

The mean curvature vector field H is defined by

$$(2.2) \quad H = (1/n)\text{trace } B.$$

The function $|H|$ is called the *mean curvature*. If it vanishes identically, then M is called a *minimal submanifold*.

3. Biharmonic submanifolds

Let $f : M \rightarrow N$ be a smooth map of an n -dimensional Riemannian manifold into another Riemannian manifold. The *tension field* $\tau(f)$ of f is a section of the induced vector bundle f^*TN defined by

$$\tau(f) = \sum_{i=1}^n \{ \nabla_{e_i}^f df(e_i) - df(\nabla_{e_i} e_i) \}$$

for a local orthonormal frame $\{e_i\}$ on M , where ∇^f and ∇ denote the induced connection and the Levi-Civita connection of M , respectively. If f is an isometric immersion, then we have

$$(3.1) \quad \tau(f) = nH.$$

A smooth map f is called a *harmonic map* if it is a critical point of the energy functional

$$E(f) = \int_{\Omega} |df|^2 dv_g$$

with respect to all variations with compact support, where dv_g is the volume form of M . A smooth map f is harmonic if and only if $\tau(f) = 0$ at each point on M .

Eells and Lemaire introduced the notion of biharmonic maps as a natural generalization of the notion of harmonic maps from a variational point of view.

Definition 3.1 ([4]). A smooth map $f : M \rightarrow \tilde{M}$ is called a *biharmonic map* if it is a critical point of the bienergy functional defined by

$$E_2(f) = \int_{\Omega} |\tau(f)|^2 dv_g$$

with respect to all variations with compact support.

The Euler-Lagrange equation for E_2 is given by

$$(3.2) \quad \tau_2(f) := -\Delta_f(\tau(f)) + \text{trace}\tilde{R}(\tau(f), df)df = 0,$$

where $\Delta_f = -\text{trace}(\nabla^f \nabla^f - \nabla_{\nabla}^f)$ and \tilde{R} is the curvature tensor of \tilde{M} which is defined by

$$\tilde{R}(X, Y)Z = [\tilde{\nabla}_X, \tilde{\nabla}_Y]Z - \tilde{\nabla}_{[X, Y]}Z$$

for the Levi-Civita connection $\tilde{\nabla}$ of \tilde{M} .

If f is a biharmonic isometric immersion, then M or $f(M)$ is called a *biharmonic submanifold* in \tilde{M} . It follows from (3.1) and (3.2) that any minimal submanifold is biharmonic. However, the converse is not generally true. Non-minimal biharmonic submanifolds are called *proper* biharmonic submanifolds.

4. Generalized (κ, μ) -manifolds

A differentiable manifold \tilde{M} is called an *almost contact manifold* if it admits a unit vector field ξ , a one-form η and a $(1, 1)$ -tensor field ϕ satisfying

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi.$$

Every almost contact manifold admits a Riemannian metric g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

If, in addition, the condition

$$d\eta(X, Y) := (1/2)(X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])) = g(X, \phi Y)$$

holds, then $(\tilde{M}, \xi, \eta, \phi, g)$ is called a *contact metric manifold*. A contact metric manifold is called a *Sasakian manifold* if it satisfies

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ which is defined as $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi(X), \phi(Y)] - \phi[\phi(X), Y] - \phi[X, \phi(Y)]$.

Koufogiorgos and Tsihlias [7] introduced the notion of *generalized (κ, μ) -manifold*, which is defined as contact metric manifold whose curvature tensor \tilde{R} satisfies

$$\tilde{R}(X, Y)\xi = (\kappa I + \mu h)(\eta(Y)X - \eta(X)Y)$$

for any vector field X and Y , where $2h$ is the Lie differentiation of ϕ with respect to ξ , and κ, μ are smooth functions. If κ and μ are constant, then the manifold is simply called a (κ, μ) -manifold ([2]). Sasakian manifolds are (κ, μ) -manifolds with $\kappa = 1$ and $h = 0$.

In [7], it was proved that if the dimension of generalized (κ, μ) -manifold is greater than 3, then κ and μ must be constant, and if the dimension is 3, then there exist examples of generalized (κ, μ) -manifold with κ, μ non-constant smooth functions.

Perrone [12] proved that a contact metric 3-manifold \tilde{M}^3 is a generalized (κ, μ) -manifold on an everywhere dense open subset if and only if the vector field ξ defines a harmonic map from \tilde{M}^3 into its unit tangent bundle equipped with the Sasaki metric.

Any 3-dimensional generalized (κ, μ) -manifold \tilde{M}^3 with $\kappa < 1$ admits three mutually orthogonal distributions $D(0) = \text{Span}\{\xi\}$, $D(\lambda)$ and $D(-\lambda)$ determined by the eigenspaces of h , where $\lambda = \sqrt{1 - \kappa}$. Furthermore, on such a manifold the following relations hold (see Lemma 3.3 of [9] and its proof):

$$(4.1) \quad \tilde{R}(\phi X, X)X = \left(-\frac{\Delta\lambda}{2\lambda} - \frac{\|\text{grad } \lambda\|^2}{2\lambda^2} - \kappa - \mu \right) \phi X,$$

$$(4.2) \quad S = -\frac{\Delta\lambda}{\lambda} - \frac{\|\text{grad } \lambda\|^2}{\lambda^2} + 2(\kappa - \mu),$$

where X is a unit vector lying in $D(\pm\lambda)$, $\Delta\lambda = -\sum_{i=1}^3 \{e_i(e_i\lambda) - (\tilde{\nabla}_{e_i} e_i)\lambda\}$, and S denotes the scalar curvature.

Below, we shall exhibit examples of 3-dimensional generalized (κ, μ) -manifolds which are not (κ, μ) -manifolds.

Example 4.1 ([9]). Let $\lambda : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a non-constant *positive* function defined on an open interval I . We put $\lambda' = \frac{d\lambda}{dz}$. Consider a manifold $\tilde{M}^3 = \{(x, y, z) \in \mathbb{R}^2 \times I \subset \mathbb{R}^3\}$. The vector fields

$$(4.3) \quad e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = (\pm 2y + f(z))\frac{\partial}{\partial x} + \left(2\lambda x - \frac{\lambda'}{2\lambda}y + h(z) \right)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

are linearly independent at each point of \tilde{M}^3 , where $f(z)$ and $h(z)$ are arbitrary functions of z . Let g be the Riemannian metric defined by $g(e_i, e_j) = \delta_{ij}$, $i, j = 1, 2, 3$, and η the dual 1 form to e_1 . If we define the $(1, 1)$ -tensor field ϕ by $\phi e_1 = 0$, $\phi e_2 = \pm e_3$ and $\phi e_3 = \mp e_2$, then the manifold $(\tilde{M}^3, \phi, e_1, \eta, g)$ is a generalized (κ, μ) -manifold with

$$(4.4) \quad \kappa = 1 - \lambda^2, \quad \mu = 2(1 \pm \lambda),$$

where double signs correspond to $\pm 2y$ in (4.3). Let us denote the manifold by $\tilde{M}(1 - \lambda^2, 2(1 \pm \lambda))$.

The following classification result was obtained by Koufogiorgos and Tsihlias.

Theorem 4.1 ([9]). *Let \tilde{M}^3 be a 3-dimensional generalized (κ, μ) -manifold which is not a (κ, μ) -manifold. If \tilde{M}^3 satisfies $\xi\mu = 0$, then it is locally given by $\tilde{M}(1 - \lambda^2, 2(1 \pm \lambda))$.*

Remark 4.1. Every generalized (κ, μ) -manifold satisfies $\xi\kappa = 0$ (see [8]).

Remark 4.2. Hereafter, except (6.1), all double signs correspond to that of $\tilde{M}(1 - \lambda^2, 2(1 \pm \lambda))$.

On $\tilde{M}(1 - \lambda^2, 2(1 \pm \lambda))$, the following relations are true (see the proof of Theorem 4.2 in [9]):

$$(4.5) \quad \tilde{\nabla}_{e_1} e_1 = \tilde{\nabla}_{e_3} e_3 = 0,$$

$$(4.6) \quad \tilde{\nabla}_{e_2} e_2 = \frac{\lambda'}{2\lambda} e_3,$$

$$(4.7) \quad \tilde{\nabla}_{e_2} e_3 = -\frac{\lambda'}{2\lambda} e_2 + (\lambda \pm 1)e_1,$$

$$(4.8) \quad h e_2 = \pm \lambda e_2.$$

5. Biharmonic Legendre curves in 3-dimensional generalized (κ, μ) -manifolds

Let \tilde{M}^3 be a contact metric 3-manifold. A curve $\gamma : I \subset \mathbb{R} \rightarrow \tilde{M}^3$ parametrized by arclength is called a *Legendre curve* if $\eta(\gamma'(s)) = 0$ holds identically.

For Legendre curves in 3-dimensional (κ, μ) -manifolds, we have:

Theorem 5.1 ([10]). *Let γ be a non-geodesic Legendre curve in a 3-dimensional (κ, μ) -manifold with $\kappa < 1$. Then γ is biharmonic if and only if either $\nabla_{\gamma'} \gamma' \parallel \phi \gamma'$ and in this case γ is a helix satisfying $k_g^2 + \tau_g^2 = -(\kappa + \mu)$ or $\nabla_{\gamma'} \gamma' \parallel \xi$ and γ is a helix satisfying $k_g^2 + \tau_g^2 = \kappa + \delta\mu$, where k_g (resp. τ_g) denotes the geodesic curvature (resp. the geodesic torsion), and $\delta := g(h\gamma', \gamma')$ is constant along γ .*

On the other hand, in [10, Theorem 3.2] the authors claimed the following:

Let γ be a Legendre curve satisfying $\nabla_{\gamma'} \gamma' \parallel \phi \gamma'$ in a 3-dimensional generalized (κ, μ) -manifold which is not a (κ, μ) -manifold. Then, γ is biharmonic if and only if it is a geodesic.

However, *this claim is incorrect*. In fact, there exist infinity many non-geodesic biharmonic Legendre curves in $\tilde{M}(1 - \lambda^2, 2(1 \pm \lambda))$, as shown below.

For two constants b and c , we consider the Legendre curve in $\tilde{M}(1 - \lambda^2, 2(1 \pm \lambda))$ given by

$$(5.1) \quad \gamma_{\{b,c\}}(s) = (b, s, c).$$

It follows from (4.6) that the curve (5.1) satisfies $\nabla_{\gamma'} \gamma' \parallel \phi \gamma'$.

Proposition 5.1. *Let $\gamma(s)$ be a Legendre curve in $\tilde{M}(1-\lambda^2, 2(1\pm\lambda))$ given by (5.1). Then γ is a proper biharmonic curve if and only if c satisfies $\lambda'(c) \neq 0$ and*

$$(5.2) \quad \lambda\lambda'' - 2(\lambda')^2 - 8\lambda^2(1\pm\lambda)|_{z=c} = 0.$$

Proof. The unit tangent vector field of γ is given by e_2 . Equation (4.6) yields that γ is a geodesic if and only if c satisfies $\lambda'(c) = 0$. Equation (3.2) can be written as

$$(5.3) \quad \tilde{\nabla}_{e_2}\tilde{\nabla}_{e_2}\tilde{\nabla}_{e_2}e_2 + \tilde{R}(\tilde{\nabla}_{e_2}e_2, e_2)e_2 = 0.$$

Since (4.8) is satisfied, we can put $X = e_2$ in (4.1). Then, using (4.5)-(4.7) and (4.1), by a straightforward computation we see that (5.3) is equivalent to (5.2). This finishes the proof. \square

Example 5.1. In Proposition 5.1, we put $\lambda = z^{-n}$, where $z > 0$ and $n \in (0, 1)$. Then $\lambda > 0$, $\lambda' \neq 0$ and (5.2) becomes

$$8c^2(1 \pm c^{-n}) = n(1 - n).$$

Given any $n \in (0, 1)$, this equation for c has a positive solution, because $\lim_{c \rightarrow 0} c^2(1 \pm c^{-n}) = 0$ and $\lim_{c \rightarrow \infty} c^2(1 \pm c^{-n}) = \infty$.

6. Biharmonic anti-invariant surfaces in 3-dimensional generalized (κ, μ) -manifolds

A submanifold M in a contact metric manifold is said to be *anti-invariant* if it is tangent to ξ and satisfies $g(X, \phi Y) = 0$ for all tangent vectors X and Y of M . For proper biharmonic anti-invariant surfaces in 3-dimensional (κ, μ) -manifolds, we have the following result.

Theorem 6.1 ([1]). *Let $f : M^2 \rightarrow \tilde{M}^3$ be an anti-invariant isometric immersion of a Riemannian 2-manifold into a 3-dimensional (κ, μ) -manifold. Then f is proper biharmonic if and only if $\kappa = 1$; that is, \tilde{M}^3 is a Sasakian manifold, and moreover, $|H|^2 = (1/8)(S - 6) = \text{constant} (\neq 0)$ on M^2 .*

This motivates us to study proper biharmonic anti-invariant surfaces in 3-dimensional generalized (κ, μ) -manifolds which are not (κ, μ) -manifolds. The following theorem characterizes such surfaces with constant mean curvature by means of S and H .

Theorem 6.2. *Let $f : M^2 \rightarrow \tilde{M}^3$ be an anti-invariant immersion of a Riemannian 2-manifold into a 3-dimensional generalized (κ, μ) -manifold which is not a (κ, μ) -manifold. Assume that M^2 has non-zero constant mean curvature. Then f is biharmonic if and only if S is constant on M^2 and the following conditions hold at any point of M^2 :*

$$(6.1) \quad |H|^2 \leq S/8, \quad h(\phi H) = \left(\pm \sqrt{\frac{S - 8|H|^2}{6}} - 1 \right) \phi H \neq 0.$$

Proof. Let M^2 be an anti-invariant surface in a 3-dimensional generalized (κ, μ) -manifold which is not a (κ, μ) -manifold. Let $\{\xi, e_2\}$ be a local orthonormal frame of M^2 such that $H = \alpha\phi e_2$. We put

$$(6.2) \quad \beta = 1 + g(he_2, e_2), \quad \gamma = g(he_2, \phi e_2).$$

Then, the following relations are true (see Lemma 4.1 of [10]):

$$(6.3) \quad \xi(\gamma) = (2\beta - \mu)(\beta - 1),$$

$$(6.4) \quad (\beta - 1)^2 + \gamma^2 = 1 - \kappa > 0.$$

Note that if $\gamma = 0$, then (6.2)-(6.4) lead to

$$(6.5) \quad h(\phi H) = (\beta - 1)\phi H \neq 0,$$

$$(6.6) \quad \kappa = 2\beta - \beta^2,$$

$$(6.7) \quad \mu = 2\beta \neq 2.$$

Assume that the mean curvature is non-zero constant. Then, it follows from Proposition 4.1 in [10] that M^2 is biharmonic if and only if the following system of partial differential equations holds:

$$(6.8) \quad -4\alpha^2 - 2\beta^2 + (S/2) - \kappa - \mu(\beta - 1) = 0,$$

$$(6.9) \quad 2\beta\gamma + \xi(\beta) + \mu\gamma = 0,$$

$$(6.10) \quad e_2(\beta) + 2\alpha\gamma = 0.$$

We shall prove that (6.8)-(6.10) hold if and only if S is constant on M^2 and (6.1) holds at each point.

First, suppose that (6.8)-(6.10) are satisfied. Then, the proof of Theorem 4.3 of [10] shows that $\gamma = 0$, and β is constant on M^2 . Substituting (6.6) and (6.7) into (6.8), we obtain

$$(6.11) \quad -4\alpha^2 - 3\beta^2 + (S/2) = 0,$$

which shows that S is constant on M^2 . Combining (6.5) and (6.11) leads to (6.1).

Conversely, suppose that S is constant on M^2 and (6.1) is satisfied at each point. Then, (6.2) implies that $\gamma = 0$, $\beta^2 = (S - 8\alpha^2)/6$ and β is constant on M^2 . Hence, (6.9) and (6.10) are satisfied. Using (6.6) and (6.7) we obtain (6.8). The proof is finished. \square

For a constant c , we consider the immersion

$$(6.12) \quad f_c(x, y) = (x, y, c)$$

of \mathbb{R}^2 into $\tilde{M}(1 - \lambda^2, 2(1 \pm \lambda))$. Suppose that \mathbb{R}^2 is equipped with the induced metric f_c^*g . Then, $\{f_c^*e_1, f_c^*e_2\}$ forms an orthonormal frame of f_c . It is clear that f_c is an anti-invariant immersion whose unit normal vector field N is $\phi(f_c^*e_2) = f_c^*e_3$, and $\{f_c(\mathbb{R}^2) | c \in I\}$ defines a codimension one foliation on $\tilde{M}(1 - \lambda^2, 2(1 \pm \lambda))$.

By (2.1), (2.2) and (4.6), the squared mean curvature of $f_c(\mathbb{R}^2)$ is a constant given by

$$(6.13) \quad |H|^2 = (\lambda')^2 / (16\lambda^2)|_{z=c}.$$

Thus, the surface $f_c(\mathbb{R}^2)$ is minimal if and only if $\lambda'(c) = 0$.

Applying Theorem 6.2, we obtain a method for constructing of infinity many examples of proper biharmonic anti-invariant surfaces in 3-dimensional generalized (κ, μ) -manifolds which are not (κ, μ) -manifolds, as shown below.

Corollary 6.1. *Let $f_c : \mathbb{R}^2 \rightarrow \tilde{M}(1-\lambda^2, 2(1\pm\lambda))$ be an anti-invariant isometric immersion given by (6.12). Then f_c is proper biharmonic if and only if c satisfies $\lambda'(c) \neq 0$ and*

$$(6.14) \quad \lambda\lambda'' - 2(\lambda')^2 - 8\lambda^2(1 \pm \lambda)^2|_{z=c} = 0.$$

Proof. It follows from (4.4)-(4.6) that (4.2) can be rewritten as

$$(6.15) \quad S = \frac{\lambda''}{\lambda} - \frac{3(\lambda')^2}{2\lambda^2} - 2(1 \pm \lambda)^2,$$

which is constant on \mathbb{R}^2 . By using (6.13), (6.15) and (4.8), we can show that (6.1) is equivalent to (6.14). The proof is finished. \square

Example 6.1. In Corollary 6.1, we put $\lambda(z) = z^{-n}$, where $z > 0$ and $n \in (0, 1)$. Then $\lambda > 0$, $\lambda' \neq 0$ and (6.14) becomes

$$8c^2(1 \pm c^{-n})^2 = n(1 - n).$$

Similarly to Example 5.1, given any $n \in (0, 1)$, this equation for c has a positive solution.

There exist proper biharmonic anti-invariant surfaces satisfying the equality case of the first relation in (6.1) at each point. In fact, using (6.14)-(6.13), we get:

Corollary 6.2. *Let $f_c : \mathbb{R}^2 \rightarrow \tilde{M}(1-\lambda^2, 2(1-\lambda))$ be an anti-invariant isometric immersion given by (6.12). Then f is a proper biharmonic immersion satisfying $|H|^2 = S/8$ on \mathbb{R}^2 if and only if c satisfies $\lambda'(c) \neq 0$ and*

$$(6.16) \quad \begin{cases} \lambda(c) = 1, \\ \lambda'' - 2(\lambda')^2|_{z=c} = 0. \end{cases}$$

Example 6.2. The general solution of the differential equation $\lambda''(z) - 2(\lambda'(z))^2 = 0$ is given by $\lambda(z) = -(1/2)(\ln|z - c_1|) + c_2$, where c_1 and c_2 are arbitrary constants. Therefore, if we choose such a function in Corollary 6.2, then $c = \pm \exp(2c_2 - 2) + c_1$ is a solution of (6.16) with $\lambda'(c) \neq 0$.

Remark 6.1. If an immersion of \mathbb{R}^2 into $\tilde{M}(1 - \lambda^2, 2(1 + \lambda))$ given by (6.12) is proper biharmonic, it does not have any points satisfying $|H|^2 = S/8$.

Remark 6.2. A generalized (κ, μ) -manifold with $\|\text{grad } \kappa\| = a$ (constant) $\neq 0$ is locally given by $\tilde{M}(1-\lambda^2, 2(1+\lambda))$ or $\tilde{M}(1-\lambda^2, 2(1-\lambda))$, where $\lambda = \sqrt{1-az-b}$ for some constant b (see [7, Proposition 4.4] and [8, Theorem 5]). In this case, since λ does not satisfy (6.14) for any c , there exists no anti-invariant surface $f_c(\mathbb{R}^2)$ which is proper biharmonic. This implies that anti-invariant surfaces described in [10, Example 4.1] is *not* proper biharmonic. Hence, Corollary 6.1 provides the first examples of proper biharmonic surfaces in non-Sasakian contact metric 3-manifolds.

The following theorem provides us with many examples of 3-dimensional generalized (κ, μ) -manifolds foliated by proper biharmonic surfaces.

Corollary 6.3. *Let \tilde{M}^3 be a 3-dimensional generalized (κ, μ) -manifold which is not a (κ, μ) -manifold. If the distribution $D(0) \oplus D(\pm\lambda)$ is integrable and its leaves are proper biharmonic surfaces, then \tilde{M}^3 is locally given by $\tilde{M}(1-\lambda^2, 2(1\pm\lambda))$, where λ is a positive solution of*

$$(6.17) \quad (\lambda')^2 = \beta\lambda^4 + 16\lambda^4 \ln \lambda \mp 32\lambda^3 - 8\lambda^2 \neq 0,$$

where β is some constant.

Proof. Let $\{\xi, X, \phi X\}$ be a local orthonormal frame such that

$$hX = \lambda X, \quad h\phi X = -\lambda\phi X.$$

Then, by Lemma 3.1 and 3.2 of [9] we have

$$[\xi, X] = (1 + \lambda - \mu/2)\phi X, \quad [\xi, \phi X] = (\lambda - 1 + \mu/2)X.$$

Hence, $D(0) \oplus D(1+\lambda)$ or $D(0) \oplus D(1-\lambda)$ is integrable if and only if $\mu = 2(1+\lambda)$ or $\mu = 2(1-\lambda)$, respectively. Applying Theorem 4.1 and Remark 4.1 shows that if $D(0) \oplus D(\lambda)$ or $D(0) \oplus D(-\lambda)$ is integrable, then \tilde{M}^3 is locally given by $\tilde{M}(1-\lambda^2, 2(1+\lambda))$ or $\tilde{M}(1-\lambda^2, 2(1-\lambda))$, respectively. In this case, each leaf is given by f_c . All leaves of a foliation $\{f_c(\mathbb{R}^2) | c \in I\}$ are biharmonic surfaces if and only if (6.14) is satisfied for all $c \in I$. By solving the ODE, we have (6.17). \square

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