Bull. Korean Math. Soc. **53** (2016), No. 3, pp. 711–722
http://dx.doi.org/10.4134/BKMS.b150263
pISSN: 1015-8634 / eISSN: 2234-3016

TOEPLITZ OPERATORS ON GENERALIZED FOCK SPACES

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Abstract. We study Toeplitz operators T_ν on generalized Fock spaces F^2_ϕ with a locally finite positive Borel measures ν as symbols. We characterize operator-theoretic properties (boundedness and compactness) of T_ν in terms of the Fock-Carleson measure and the Berezin transform $\widetilde{\nu}$.

1. Introduction

Let $\phi : \mathbb{C} \to \mathbb{R}$ be a (nonharmonic) subharmonic function whose Laplacian $\Delta\phi$ is a doubling measure. We consider the measure

$$d\mu_\phi(z) = e^{-2\phi(z)} dA(z)$$

on $\mathbb{C}$, where dA is a Lebesgue area measure. The generalized Fock space F^2_ϕ is defined by

$$F^2_\phi = \left\{ f \in H(\mathbb{C}) : \|f\|^2_\phi = \int_{\mathbb{C}} |f(z)|^2 \, d\mu_\phi(z) < +\infty \right\},$$

where $H(\mathbb{C})$ is the space of all entire functions in $\mathbb{C}$.

Let $K(z, \zeta)$ denote the reproducing kernel for F^2_ϕ. Then we get the orthogonal projection

$$P : L^2(\mathbb{C}, d\mu_\phi) \to F^2_\phi$$

which has the following integral representation

$$Pf(z) = \int_{\mathbb{C}} K(z, \zeta) f(\zeta) \, d\mu_\phi(\zeta), \quad z \in \mathbb{C}.$$

If ν is a Borel measure on $\mathbb{C}$, then we will define the Toeplitz operator T_ν by the formula

$$T_\nu(f)(z) = \int_{\mathbb{C}} f(\zeta) K(z, \zeta) e^{-2\phi(\zeta)} \, d\nu(\zeta), \quad z \in \mathbb{C}.$$

Received April 17, 2015.

2010 *Mathematics Subject Classification.* 30H20, 47B35.

Key words and phrases. generalized Fock space, Fock-Carleson measure, Toeplitz operator, Berezin transform.

The author was supported by NRF of Korea (NRF-2014R1A1A2056828) and RIBS of Pusan National University (RIBS-PNU-2014-108).

Note that if ν satisfies

$$\int_{\mathbb{C}} |K(z,\zeta)|^2 e^{-2\phi(\zeta)}\, d|\nu|(\zeta) < \infty \tag{1.1}$$

for every $z \in \mathbb{C}$, then the Toeplitz operator T_ν is densely defined on F^2_ϕ. The normalized reproducing kernel of F^2_ϕ is defined by

$$k_z(\zeta) = \frac{\overline{K(z,\zeta)}}{\|K(\cdot,z)\|} = \frac{K(\zeta,z)}{\sqrt{K(z,z)}}.$$

All measures used in the paper will be assumed to satisfy condition (1.1), so that all Toeplitz operators are well-defined. It also follows from condition (1.1) that we can define a function $\widetilde{\nu}$ on $\mathbb{C}$ as follows:

$$\widetilde{\nu}(z) = \int_{\mathbb{C}} |k_z(\zeta)|^2 e^{-2\phi(\zeta)}\, d\nu(\zeta), \quad z \in \mathbb{C}.$$

We call $\widetilde{\nu}$ the Berezin transform of ν. It is given by

$$\widetilde{\nu}(z) = \langle T_\nu k_z, k_z \rangle.$$

A positive measure ν on $\mathbb{C}$ is called a Fock-Carleson measure, if the inclusion map $i : F^2_\phi \hookrightarrow L^2(\mathbb{C}, e^{-2\phi} d\nu)$ is bounded, i.e., there exists a constant $C > 0$ such that

$$\|f\|_{L^2(\mathbb{C}, e^{-2\phi} d\nu)} \le C\|f\|_\phi \quad \text{for} \quad f \in F^2_\phi. \tag{1.2}$$

Also, we call ν a vanishing Fock-Carleson measure, if

$$\lim_{n\to\infty} \|f_n\|_{L^2(\mathbb{C}, e^{-2\phi} d\nu)} = 0$$

whenever $\{f_n\}$ is a bounded sequence in F^2_ϕ that converges to 0 uniformly on compact subsets.

We characterize operator-theoretic properties (boundedness and compactness) of T_ν in terms of the Fock-Carleson measure and the Berezin transform $\widetilde{\nu}$.

Theorem 1.1. *Let ν be a locally finite positive Borel measure on $\mathbb{C}$. Then the following conditions are equivalent.*

(a) *The Toeplitz operator T_ν is bounded on F^2_ϕ.*
(b) *ν is a Fock-Carleson measure for F^2_ϕ.*
(c) *$\widetilde{\nu}$ is a bounded function on $\mathbb{C}$.*

Theorem 1.2. *Let ν be a locally finite positive Borel measure on $\mathbb{C}$. Then the following conditions are equivalent.*

(a) *The Toeplitz operator T_ν is compact on F^2_ϕ.*
(b) *ν is a vanishing Fock-Carleson measure for F^2_ϕ.*
(c) *$\widetilde{\nu}(z) \to 0$ as $z \to \infty$.*

For the case of classical Fock spaces see ([4], [10]). In [3] Constantin and Peláez considered the Fock-Carleson measures on generalized Fock spaces of a wide class of weights. They used peak functions instead of reproducing kernels, because there are no precise estimates of reproducing kernels. Our results do not completely overlap with theirs. The multidimensional case of special weights was considered in [8] and [9].

2. Preliminaries

From now on we shall assume that ϕ is a (nonharmonic) subharmonic function on $\mathbb{C}$ such that $\Delta\phi$ is a doubling measure. Recall that when ϕ is subharmonic $\Delta\phi$ is a locally finite positive Borel measure.

Given $z \in \mathbb{C}$ and $r > 0$, we write

$$D(z,r) = \{\zeta \in \mathbb{C} : |z-\zeta| < r\}$$

for the Euclidean disc centered at z with radius r. We let $\nu = \Delta\phi$ and denote by $\rho(z)$ the positive radius for which we have $\nu(D(z,\rho(z))) = 1$, $z \in \mathbb{C}$. This is always well defined since for any doubling measure in $\mathbb{C}$, the measure of any circle is 0. Thus the function $r \to \nu(D(z,r))$ is continuous and strictly increasing.

If ϕ is subharmonic with $\Delta\phi$ doubling, then there exist $\psi \in C^\infty(\mathbb{C})$ subharmonic and $C > 0$ such that $|\phi - \psi| \le C$, $\Delta\psi$ is a doubling measure and $\Delta\psi \sim \rho^{-2}$ ([5], [6]). Thus the function ρ^{-2} is a regularized version of $\Delta\phi$, as described in [1].

A first observation about $\rho(z)$ is that $\rho(z)$ is a Lipschitz function (see [6]). More precisely

$$|\rho(z) - \rho(\zeta)| \le |z-\zeta|, \quad z,\zeta \in \mathbb{C}. \tag{2.1}$$

We denote $D^r(z) = D(z, r\rho(z))$ and for $r = 1$ we simply write $D(z)$ instead of $D^1(z)$.

Lemma 2.1 ([6]). *There exist $\eta, C > 0$ and $\beta \in (0,1)$ such that*

$$C^{-1}|z|^{-\eta} \le \rho(z) \le C|z|^{\beta}, \quad |z| > 1. \tag{2.2}$$

Lemma 2.2 ([6]). *For any $r > 0$ there exists $C > 0$ depending only on r and the doubling constant for $\Delta\phi$ such that*

$$C^{-1}\rho(\zeta) \le \rho(z) \le C\rho(\zeta) \quad \textit{for} \ \ \zeta \in D^r(z). \tag{2.3}$$

For $z,\zeta \in \mathbb{C}$, the distance d_ϕ induced by the metric $\rho^{-2}(z)dz \otimes d\bar{z}$ is given by

$$d_\phi(z,\zeta) = \inf_\gamma \int_0^1 \frac{|\gamma'(t)|}{\rho(\gamma(t))}\,dt,$$

where γ runs over the piecewise C^1 curves $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(0) = z$ and $\gamma(1) = \zeta$.

In view of the estimates of the Bergman kernel stated below it follows that $B(\frac{\partial}{\partial z}, z) \sim 1/\rho(z)$, where $B(\frac{\partial}{\partial z}, z)$ is the Bergman metric at the point z (see [2]).

The following estimates for the distance d_ϕ hold:

Lemma 2.3. *There exists $\delta \in (0,1)$ such that for every $r > 0$ there exists $C_r > 0$ such that*

$$C_r^{-1} \frac{|z-\zeta|}{\rho(z)} \leq d_\phi(z,\zeta) \leq C_r \frac{|z-\zeta|}{\rho(z)} \quad \text{for } \zeta \in D^r(z), \tag{2.4}$$

and

$$C_r^{-1} \left(\frac{|z-\zeta|}{\rho(z)}\right)^{\delta} \leq d_\phi(z,\zeta) \leq C_r \left(\frac{|z-\zeta|}{\rho(z)}\right)^{2-\delta} \quad \text{for } \zeta \in D^r(z)^c. \tag{2.5}$$

We will use the following Cauchy-type estimates for the disc $D^r(z)$.

Lemma 2.4. *For any $r > 0$ there exists $C = C(r) > 0$ such that for any $f \in H(\mathbb{C})$ and $z \in \mathbb{C}$*

$$|f(z)|^2 e^{-2\phi(z)} \leq \frac{C}{|D^r(z)|} \int_{D^r(z)} |f(\zeta)|^2 \, d\mu_\phi(\zeta). \tag{2.6}$$

In [6], Marzo and Ortega-Cerdà proved the following estimates for the reproducing kernel $K(z,\zeta)$.

Theorem 2.5. *There exist positive constants c and σ (depending only on the doubling constant for $\Delta\phi$) such that for any $z, \zeta \in \mathbb{C}$,*

$$|K(z,\zeta)| \leq c \frac{1}{\rho(z)\rho(\zeta)} \frac{e^{\phi(z)+\phi(\zeta)}}{\exp d_\phi(z,\zeta)^\sigma}.$$

Moreover, there exists $r > 0$ such that

$$|K(z,\zeta)| \gtrsim \frac{e^{\phi(z)+\phi(\zeta)}}{\rho(z)\rho(\zeta)} \quad \text{for } \zeta \in D^r(z).$$

Proposition 2.6.

$$k_z \to 0 \quad \text{weakly as} \quad z \to \infty.$$

Proof. For $f \in F^2_\phi$ it follows that

$$\begin{aligned} |\langle f, k_z \rangle| &= \left| \left\langle f, \frac{K(\zeta,z)}{\sqrt{K(z,z)}} \right\rangle \right| \\ &= \frac{|f(z)|}{\sqrt{K(z,z)}} \\ &\lesssim |f(z)| e^{-\phi(z)} \rho(z). \end{aligned}$$

Since $f \in F^2_\phi$, $fe^{-\phi} \in L^2(\mathbb{C})$ and

$$\int_{|\zeta|>R} |f(\zeta)|^2 e^{-2\phi(\zeta)}\, dA(\zeta) \to 0, \quad R \to \infty.$$

By (2.2), it follows that $\rho(z) \leq C|z|$ for large $|z| > 1$. Choose $r > 0$ so that $rC < 1$. Take $R = (1 - rC)|z|$. Then $D^r(z) \subset \{|\zeta| > R\}$ for large $|z| > 1$. By (2.6),

$$\begin{aligned} |f(z)|^2 e^{-2\phi(z)} \rho(z)^2 &\lesssim \int_{D^r(z)} |f(\zeta)|^2 e^{-2\phi(\zeta)}\, dA(\zeta) \\ &\lesssim \int_{|\zeta|>R} |f(\zeta)|^2 e^{-2\phi(\zeta)}\, dA(\zeta) \to 0 \end{aligned}$$

as $|z| \to \infty$ if $f \in F^2_\phi$. Thus we have

$$|\langle f, k_z\rangle| \lesssim |f(z)| e^{-\phi(z)} \rho(z) \to 0, \quad |z| \to \infty.$$

This means that

$$k_z \to 0 \quad \text{weakly as} \quad z \to \infty. \qquad \square$$

3. Toeplitz operators on F^2_ϕ

For any $z \in \mathbb{C}$ and $r > 0$ we let

$$B(z, r) = \{\zeta \in \mathbb{C} : d_\phi(z, \zeta) < r\}$$

denote the ρ-metric disk centered at z with radius r.

Lemma 3.1. *Let $r > 0$.*

(a) *There exist two positive numbers $m = m(r)$ and $M = M(r)$ such that*

$$D^m(z) \subset B(z, r) \subset D^M(z)$$

for every z in $\mathbb{C}$.

(b) *There exist two positive numbers $m = m(r)$ and $M = M(r)$ such that*

$$B(z, m) \subset D^r(z) \subset B(z, M)$$

for every z in $\mathbb{C}$.

Proof. We prove only (a). The proof of (b) is similar to that of (a).

By (2.4) and (2.5), there exist $\delta \in (0, 1)$ and $C > 0$ such that

$$d_\phi(z, \zeta) \leq C \left[\frac{|z - \zeta|}{\rho(z)} + \left(\frac{|z - \zeta|}{\rho(z)} \right)^{2-\delta} \right]$$

for $z, \zeta \in \mathbb{C}$. Thus

$$d_\phi(z, \zeta) \leq C(m + m^{2-\delta}) < r \quad \text{for} \quad \zeta \in D^m(z),$$

if m is sufficiently small. Hence $D^m(z) \subset B(z, r)$.

By (2.4) and (2.5), there exist $\delta \in (0,1)$ and $C > 0$ such that

$$d_\phi(z,\zeta) \geq C^{-1}\frac{|z-\zeta|}{\rho(z)} \quad \text{or} \quad \geq C^{-1}\left(\frac{|z-\zeta|}{\rho(z)}\right)^\delta \quad \text{for } \ z,\zeta \in \mathbb{C}.$$

Hence

$$\begin{aligned}\frac{|z-\zeta|}{\rho(z)} &\leq Cd_\phi(z,\zeta) + (Cd_\phi(z,\zeta))^{\frac{1}{\delta}} \\ &\leq Cr + (Cr)^{\frac{1}{\delta}} = M \quad \text{for } \ \zeta \in B(z,r).\end{aligned}$$

Thus $B(z,r) \subset D^M(z)$. □

By Lemma 3.1, it follows that $|B(z,r)| \sim \rho(z)^2$. For a locally finite positive Borel measure ν we write the average function of a measure ν as

$$\widehat{\nu}_r(z) = \frac{\nu(B(z,r))}{\rho(z)^2}.$$

We will use the following Cauchy-type estimates for the disc $B(z,r)$.

Lemma 3.2. *For any $r > 0$ there exists $C = C(r) > 0$ such that for any $f \in H(\mathbb{C})$ and $z \in \mathbb{C}$*

$$|f(z)|^2 e^{-2\phi(z)} \leq \frac{C}{|B(z,r)|}\int_{B(z,r)} |f(\zeta)|^2\, d\mu_\phi(\zeta). \tag{3.1}$$

Proof. We choose $m > 0$ such that $D^m(z) \subset B(z,r)$. By (2.6), we have

$$|f(z)|^2 e^{-2\phi(z)} \leq \frac{C}{|D^m(z)|}\int_{D^m(z)} |f(\zeta)|^2\, d\mu_\phi(\zeta).$$

Since $|D^m(z)| \sim \rho(z)^2 \sim |B(z,r)|$, we get the result. □

Lemma 3.3. *Let ν be a positive Borel measure on $\mathbb{C}$. Then for $f \in H(\mathbb{C})$ and any $r > 0$*

$$\int_{\mathbb{C}} |f(z)|^2 e^{-2\phi(z)}\, d\nu(z) \lesssim \int_{\mathbb{C}} |f(\zeta)|^2 \widehat{\nu}_r(\zeta)\, d\mu_\phi(\zeta). \tag{3.2}$$

Proof. Let

$$I(f) = \int_{\mathbb{C}} |f(z)|^2 e^{-2\phi(z)}\, d\nu(z).$$

By (3.1), we have

$$I(f) \lesssim \int_{\mathbb{C}}\left(\int_{B(z,r)} |f(\zeta)|^2\, \frac{d\mu_\phi(\zeta)}{\rho(\zeta)^2}\right) d\nu(z).$$

We know that

$$\chi_{B(z,r)}(\zeta) = \chi_{B(\zeta,r)}(z) \quad \text{for } \ z,\zeta \in \mathbb{C}.$$

By Fubini's theorem, we have

$$\begin{aligned} I(f) &\lesssim \int_{\mathbb{C}} \left(\int_{\zeta\in\mathbb{C}} \chi_{B(z,r)}(\zeta)|f(\zeta)|^2 \frac{d\mu_\phi(\zeta)}{\rho(\zeta)^2} \right) d\nu(z) \\ &= \int_{\mathbb{C}} \left(\int_{\zeta\in\mathbb{C}} \chi_{B(\zeta,r)}(z)|f(\zeta)|^2 \frac{d\mu_\phi(\zeta)}{\rho(\zeta)^2} \right) d\nu(z) \\ &\lesssim \int_{\mathbb{C}} \nu(B(\zeta,r))\rho(\zeta)^{-2}|f(\zeta)|^2\, d\mu_\phi(\zeta) \\ &= \int_{\mathbb{C}} |f(\zeta)|^2 \widehat{\nu}_r(\zeta)\, d\mu_\phi(\zeta). \end{aligned}$$

Thus we get the result. □

By conditions (2.1) and (2.2), there exist constants $0 < c < \infty$ and $R > 1$ such that

(a) $\rho(z) \le c|z|$, $|z| > R$,
(b) $|\rho(z) - \rho(w)| \le |z - w|$, $z, w \in \mathbb{C}$.

The following lemma on coverings is due to Oleinik (see [7]).

Lemma 3.4. *There exists a sequence of points $\{a_j\} \subset \mathbb{C}$ such that*

(1) $|a_j| > R/2$ *for all* j.
(2) $a_j \notin D^r(a_k)$, $j \ne k$.
(3) $\{|z| > R\} \subset \bigcup_j D^r(a_j)$.
(4) $\widetilde{D}^r(a_j) \subset D^{3r}(a_j)$, *where* $\widetilde{D}^r(a_j) = \bigcup_{z\in D^r(a_j)} D^r(z)$, $j = 1, 2, \ldots$.
(5) $\{D^{3r}(a_j)\}$ *is a covering of* $\{|z| > R\}$ *of finite multiplicity* N.

Theorem 3.5. *Let ν be a locally finite positive Borel measure on $\mathbb{C}$. Then the following conditions are equivalent.*

(a) *The measure ν is a Fock-Carleson measure.*
(b) $\widehat{\nu}_r(z) \lesssim 1$ *for any* $r > 0$.

Proof. (a) $\Longrightarrow$ (b): Since ν is a Fock-Carleson measure we have

$$\|f\|_{L^2(\mathbb{C}, e^{-2\phi}d\nu)} \le C\|f\|_\phi \quad \text{for} \quad f \in F^2_\phi.$$

For any fixed $z \in \mathbb{C}$, we take $f_z(\zeta) = K(z, \zeta)$. Then

$$\|f_z\|^2_\phi \lesssim \frac{e^{2\phi(z)}}{\rho(z)^2} \tag{3.3}$$

and

$$\begin{aligned} \|f_z\|^2_{L^2(\mathbb{C}, e^{-2\phi}d\nu)} &\gtrsim \int_{B(z,r)} |K(z,\zeta)|^2 e^{-2\phi(\zeta)}\, d\nu(\zeta) \\ &\gtrsim \frac{e^{2\phi(z)}}{\rho(z)^2} \widehat{\nu}_r(z). \end{aligned} \tag{3.4}$$

By (1.2), (3.3), and (3.4), we have $\widehat{\mu}_r(z) \lesssim 1$.

(b) $\Longrightarrow$ (a): Since ν is a locally finite measure, by (2.6), we have

$$\begin{aligned}\int_{|z|<R} |f(z)|2e^{-2\phi(z)}\,d\nu(z) &\le \nu\left(\{|z|<R\}\right)\sup_{|z|<R}\left(|f(z)|^2e^{-2\phi(z)}\right)\\ &\lesssim \left(\sup_{|z|<R}\frac{1}{\rho(z)^2}\right)\|f\|_\phi^2.\end{aligned} \tag{3.5}$$

If we use the covering in Lemma 3.4, then

$$\int_{|z|>R} |f(z)|^2e^{-2\phi(z)}\,d\nu(z) \le \sum_j \int_{D^r(a_j)} |f(z)|^2e^{-2\phi(z)}\,d\nu(z). \tag{3.6}$$

If $z\in D^r(a_j)$, then $D^r(z)\subset D^{3r}(a_j)$. Thus

$$\begin{aligned}|f(z)|^2e^{-2\phi(z)} &\le C\frac{1}{\rho(z)^2}\left(\int_{D^r(z)}|f(\zeta)|^2e^{-2\phi(\zeta)}\,dA(\zeta)\right)\\ &\lesssim \frac{1}{\rho(a_j)^2}\left(\int_{D^{3r}(a_j)}|f(\zeta)|^2e^{-2\phi(\zeta)}\,dA(\zeta)\right).\end{aligned}$$

Since $\{D^{3r}(a_j)\}$ is a covering of $\{|z|>R\}$ of finite multiplicity N, we have

$$\begin{aligned}&\sum_{|a_j|>R/2}\int_{D^r(a_j)}|f(z)|^2e^{-2\phi(z)}\,d\nu(z)\\ &\lesssim \sum_{|a_j|>R/2}\int_{D^r(a_j)}\frac{d\nu(z)}{\rho(a_j)^2}\left(\int_{D^{3r}(a_j)}|f(\zeta)|^2e^{-2\phi(\zeta)}\,dA(\zeta)\right)\\ &\lesssim \sum_{|a_j|>R/2}\int_{B(a_j,M)}\frac{d\nu(z)}{\rho(a_j)^2}\left(\int_{D^{3r}(a_j)}|f(\zeta)|^2e^{-2\phi(\zeta)}\,dA(\zeta)\right)\\ &= \sum_{|a_j|>R/2}\widehat{\nu}_M(a_j)\left(\int_{D^{3r}(a_j)}|f(\zeta)|^2e^{-2\phi(\zeta)}\,dA(\zeta)\right)\\ &\lesssim \sum_{|a_j|>R/2}\left(\int_{D^{3r}(a_j)}|f(\zeta)|^2e^{-2\phi(\zeta)}\,dA(\zeta)\right)\\ &\lesssim \left(\sum_{|a_j|>R/2}\int_{D^{3r}(a_j)}|f(\zeta)|^2e^{-2\phi(\zeta)}\,dA(\zeta)\right)\\ &\lesssim N\|f\|_\phi^2.\end{aligned} \tag{3.7}$$

By (3.5), (3.6), and (3.7), we have

$$\int_{\mathbb{C}}|f(z)|^2e^{-2\phi(z)}\,d\nu(z)\lesssim \|f\|_\phi^2.$$

□

Theorem 3.6. *Let ν be a locally finite positive Borel measure on $\mathbb{C}$. Then the following conditions are equivalent.*

(a) *The measure ν is a vanishing Fock-Carleson measure.*
(b) $\lim_{z\to\infty} \widehat{\nu}_r(z) = 0$ *for any $r > 0$.*

Proof. (a) $\implies$ (b): For any sequence $z_n \to \infty$ we let

$$f_n(\zeta) = k_{z_n}(\zeta), \quad \zeta \in \mathbb{C}.$$

Then $\|f_n\|_\phi = 1$ and

$$\begin{aligned}\sup_{|\zeta|<R} |f_n(\zeta)| &= \sup_{|\zeta|<R} |k_{z_n}(\zeta)| \\ &\lesssim \sup_{|\zeta|<R} \frac{e^{\phi(\zeta)}}{\rho(\zeta)} \frac{1}{\exp d_\phi(z_n,\zeta)^\sigma}.\end{aligned}$$

Let $|\zeta| < R$. By Lemma 2.4, there exists $\delta \in (0,1)$ such that

$$\frac{|z_n - \zeta|}{\rho(z_n)} \lesssim d_\phi(z_n,\zeta) \quad \text{or} \quad \left(\frac{|z_n-\zeta|}{\rho(z_n)}\right)^\delta \lesssim d_\phi(z_n,\zeta).$$

By Lemma 2.2, there exists $\beta \in (0,1)$ such that $\rho(z_n) \lesssim |z_n|^\beta$ for large n. For sufficiently large n it follows that $\frac{1}{2}|z_n| < |z_n - \zeta|$ for $|\zeta| < R$. Hence

$$|z_n|^{1-\beta} \lesssim d_\phi(z_n,\zeta) \quad \text{or} \quad |z_n|^{\delta(1-\beta)} \lesssim d_\phi(z_n,\zeta)$$

for sufficiently large n. This means that

$$|z_n|^{\delta(1-\beta)} \lesssim d_\phi(z_n,\zeta), \quad n \to \infty. \tag{3.8}$$

Thus we have

$$\sup_{|\zeta|<R} |f_n(\zeta)| \lesssim \frac{1}{e^{c|z_n|^{\sigma\delta(1-\beta)}}} \to 0 \quad \text{as} \quad n \to \infty.$$

This means that $\{f_n\}$ is a bounded sequence in F^2_ϕ that converges to 0 uniformly on compact subsets. Since ν is a vanishing Fock-Carleson measure, we have

$$\|f_n\|^2_{L^2(\mathbb{C},e^{-2\phi}d\nu)} \to 0, \quad n \to \infty.$$

Now

$$\begin{aligned}\|f_n\|^2_{L^2(\mathbb{C},e^{-2\phi}d\nu)} &\gtrsim \int_{B(z,r)} |k_{z_n}(\zeta)|^2 e^{-2\phi(\zeta)}\, d\nu(\zeta) \\ &\gtrsim \widehat{\nu}_r(z).\end{aligned}$$

Thus we get the result.

(b) $\implies$ (a): Let $\{f_n\}$ be a bounded sequence in F^2_ϕ that converges to 0 uniformly on every compact subset. We suppose that

$$\sup_n \|f_n\|_\phi \le K.$$

To complete the proof, it suffices to show

$$\lim_{n\to\infty} \|f_n\|_{L^2(\mathbb{C},e^{-2\phi}d\nu)} = 0.$$

Let $\epsilon > 0$ be given. Choose $R > 2r$ such that $\nu(D^r(z))\rho(z)^{-2} < \epsilon$ whenever $|z| > R/2$. Since $f_n \to 0$ uniformly on compact sets, we can choose n_0 such that

$$\int_{|\zeta|<R} |f_n(\zeta)|^2 \, e^{-2\phi(\zeta)} d\nu(\zeta) < \epsilon$$

whenever $n > n_0$.

By Lemma 3.4, we can choose a sequence $\{a_j\}$ such that $|a_j| > R/2$ and $\{D^r(a_j)\}$ covers $\{z : |z| > R\}$. If $n > n_0$, then

$$\begin{aligned} \|f_n\|^2_{L^2(\mathbb{C},e^{-2\phi}d\nu)} &= \int_{|\zeta|<R} |f_n(\zeta)|^2 \, e^{-2\phi(\zeta)} d\nu(\zeta) + \int_{|\zeta|>R} |f_n(\zeta)|^q \, e^{-2\phi(\zeta)} d\nu(\zeta) \\ &\le \epsilon + \sum_j \int_{D^r(a_j)} |f_n(\zeta)|^2 \, e^{-2\phi(\zeta)} d\nu(\zeta). \end{aligned}$$

By Lemma 3.1, there exists a constant C such that

$$|f_n(\zeta)|^2 \, e^{-2\phi(\zeta)} \le C \frac{1}{\rho(a_j)^2} \int_{D^{3r}(a_j)} |f_n(z)|^2 \, d\mu_\phi(z)$$

for every $\zeta \in D^r(a_j)$. Therefore,

$$\begin{aligned} \|f_n\|^2_{L^2(\mathbb{C},e^{-2\phi}d\nu)} &\le \epsilon + C \sum_j \frac{\nu(D^r(a_j))}{\rho(a_j)^2} \int_{D^{3r}(a_j)} |f_n(z)|^2 \, d\mu_\phi(z) \\ &\le \epsilon + C \sum_j \frac{\nu(B(a_j, M))}{\rho(a_j)^2} \int_{D^{3r}(a_j)} |f_n(z)|^2 \, d\mu_\phi(z). \end{aligned}$$

Since $\nu(B(a_j, M))\rho(z)^{-2} < \epsilon$, for a suitable constant C, we have

$$\|f_n\|^2_{L^2(\mathbb{C},e^{-2\phi}d\nu)} \le \epsilon + C\epsilon \sum_j \int_{D^{3r}(a_j)} |f_n(z)|^2 \, d\mu_\phi(z).$$

From the local finiteness of the covering $\{D^{3r}(a_j)\}$, there exists a positive integer N such that

$$\begin{aligned} \sum_j \int_{D^{3r}(a_j)} |f_n(z)|^2 \, d\mu_\phi(z) &\le N \int_{\mathbb{C}} |f_n(z)|^2 \, d\mu_\phi(z) \\ &= N\|f_n\|^2_\phi \le NK^2. \end{aligned}$$

Therefore, we have

$$\|f_n\|^2_{L^2(\mathbb{C},e^{-2\phi}d\nu)} \le \epsilon + \epsilon CNK^2$$

whenever $n > n_0$. Since ϵ is arbitrary, this shows that

$$\lim_{n\to\infty} \|f_n\|_{L^2(\mathbb{C},e^{-2\phi}d\nu)} = 0.$$

□

Lemma 3.7. *Let ν be a locally finite positive Borel measure on $\mathbb{C}$. Then*

(a) *$\widehat{\nu}_r(z) \lesssim \widetilde{\nu}(z)$ for any $r > 0$ and $z \in \mathbb{C}$.*
(b) *If $\widehat{\nu}_r \in L^\infty(\mathbb{C})$ for any $r > 0$, then $\widetilde{\nu} \in L^\infty(\mathbb{C})$.*
(c) *If $\lim_{z\to\infty} \widehat{\nu}_r(z) = 0$ for any $r > 0$, then $\lim_{z\to\infty} \widetilde{\nu}(z) = 0$.*

Proof. (a) For any $r > 0$ it follows that

$$\begin{aligned}\widetilde{\nu}(z) &= \int_{\mathbb{C}} |k_z(\zeta)|^2 e^{-2\phi(\zeta)}\, d\nu(\zeta)\\ &\geq \int_{B(z,r)} \frac{|K(z,\zeta)|^2}{|K(z,z)|} e^{-2\phi(\zeta)}\, d\nu(\zeta)\\ &\gtrsim \int_{B(z,r)} \frac{1}{\rho(\zeta)^2} d\nu(\zeta) \sim \widehat{\nu}_r(z).\end{aligned}$$

(b) By (3.2), we have

$$\begin{aligned}\widetilde{\nu}(z) &= \int_{\mathbb{C}} |k_z(\zeta)|^2 e^{-2\phi(\zeta)}\, d\nu(\zeta)\\ &\lesssim \int_{\mathbb{C}} |k_z(\zeta)|^2 \widehat{\nu}_r(\zeta)\, d\mu_\phi(\zeta)\\ &\lesssim 1,\end{aligned}$$

since $\widehat{\nu}_r \in L^\infty(\mathbb{C})$ and $\|k_z\|_\phi = 1$.

(c) We have

$$\begin{aligned}\widetilde{\nu}(z) &= \int_{\mathbb{C}} |k_z(\zeta)|^2 e^{-2\phi(\zeta)}\, d\nu(\zeta)\\ &= \int_{|\zeta|<R} + \int_{|\zeta|>R} |k_z(\zeta)|^2 e^{-2\phi(\zeta)}\, d\nu(\zeta).\end{aligned}$$

Let $\{D^r(a_j)\}$ be the covering in Lemma 3.4. Since $|a_j| \to \infty$ as $j \to \infty$, $\lim_{j\to\infty} \widehat{\nu}_r(a_j) = 0$ for any $r > 0$. Given $\epsilon > 0$, there is a positive integer N_0 such that

$$\widehat{\nu}_r(a_j) < \epsilon, \quad j \geq N_0.$$

We choose $R_0 > 0$ such that

$$\{|\zeta| < R\} \cup \left(\cup_{j=1}^{N_0-1} D^r(a_j)\right) \subset \{|\zeta| < R_0\}.$$

Then

$$\begin{aligned}\widetilde{\nu}(z) &= \int_{\mathbb{C}} |k_z(\zeta)|^2 e^{-2\phi(\zeta)}\, d\nu(\zeta)\\ &\leq \int_{|\zeta|<R_0} + \sum_{j=N_0}^{\infty} \int_{D^r(a_j)} |k_z(\zeta)|^2 e^{-2\phi(\zeta)}\, d\nu(\zeta).\end{aligned}$$

By (3.8), for $|\zeta| < R_0$,

$$|z|^{\delta(1-\beta)} \lesssim d_\phi(z,\zeta), \quad |z| \to \infty.$$

Since ν is a locally finite measure, we have

$$\int_{|\zeta|<R_0} |k_z(\zeta)|^2 e^{-2\phi(\zeta)}\, d\nu(\zeta) \leq \nu(\{|\zeta| < R_0\}) \sup_{|\zeta|<R_0} \frac{1}{\rho(\zeta)^2 \exp(2d_\phi(z,\zeta)^\sigma)}$$

$$\lesssim \frac{1}{e^{c|z|^{\sigma\delta(1-\beta)}}} \int_{|\zeta|<R_0} \frac{1}{\rho(\zeta)^2}$$
$$\lesssim \frac{1}{e^{c|z|^{\sigma\delta(1-\beta)}}} \to 0 \quad \text{as} \quad z \to \infty.$$

Also, as in (3.7), it follows that

$$\sum_{j=N_0}^{\infty} \int_{D^r(a_j)} |k_z(\zeta)|^2 e^{-2\phi(\zeta)}\, d\nu(\zeta)$$
$$\lesssim \sum_{j=N_0}^{\infty} \widehat{\nu}_M(a_j) \left(\int_{D^{3r}(a_j)} |k_z(\zeta)|^2 e^{-2\phi(\zeta)}\, dA(\zeta) \right)$$
$$\lesssim \epsilon N \|k_z\|_\phi^2 = \epsilon N, \quad z \in \mathbb{C}.$$

Since ϵ is arbitrary, we have

$$\lim_{z\to\infty} \widetilde{\nu}(z) = 0.$$

Thus we get the result. □

By Theorem 3.5, Theorem 3.6, and Lemma 3.7, we get Theorem 1.1 and Theorem 1.2 (see [4] and [10]).

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