# CONTINUED FRACTION AND DIOPHANTINE EQUATION 

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#### Abstract

Our paper is devoted to the study of certain diophantine equations on the ring of polynomials over a finite field, which are intimately related to algebraic formal power series which have partial quotients of unbounded degree in their continued fraction expansion. In particular it is shown that there are Pisot formal power series with degree greater than 2 , having infinitely many large partial quotients in their simple continued fraction expansions. This generalizes an earlier result of Baum and Sweet for algebraic formal power series.


## 1. Introduction

Let $\mathbb{F}_{q}$ be a finite field of characteristic $p$ with $q$ elements. We consider $\mathbb{F}_{q}[X], \mathbb{F}_{q}(X)$ and $\mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$ as analogues of $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ respectively. If $w=\sum_{n=n_{0}}^{+\infty} a_{n} X^{-n}$ is an element of $\mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$ with $a_{n_{0}} \neq 0$, we introduce the absolute value $|w|=q^{-n_{0}}$ and $|0|=0$. Diophantine approximation in the function field case was initiated by K. Mahler [10]. In the case of real numbers, it is well known that Liouville's theorem was the beginning of a long path, with the works of Thue, Siegel, Dyson and others, leading of the celebrated Roth's theorem which was established in 1955. This improvement can be transposed in fields of power series if the base field has characteristic zero, as shown by Uchiyama in 1960. But this is not the case in positive characteristic and consequently the study of rational approximation to algebraic elements becomes somewhat more complex.

Many examples can be studied. A special case is the one where the algebraic element $w$ satisfies an equation of the form $w=\frac{A w^{p^{s}}+B}{C w^{p^{s}}+D}$, where $A, B, C$ and $D$ belong to $\mathbb{F}_{q}[X]$, with $A D-B C \neq 0$, and $s$ is a positive integer. Those elements have been studied by Baum and Sweet, Mills and Robbins, Thakur, Voloch, de Mathan [3], [11], [12], [17], [18], and [19]. Pass [14] has shown that there are algebraic integers with infinitely many large partial quotients. In particular, his result read as follows:

Received April 7, 2015.
2010 Mathematics Subject Classification. 11A55, 11D72, 11J61, 11J68.
Key words and phrases. Pisot element, continued fraction, Laurent series in finite fields.

Theorem 1.1. Given $N>0$, there is an algebraic integer $\beta$ of degree $>2$, such that

$$
\begin{equation*}
|Q \beta-P|<\frac{1}{N Q} \tag{1.1}
\end{equation*}
$$

has infinitely many solutions $(P, Q) \in \mathbb{N} \times \mathbb{N}$.
This generalizes an earlier result of Davenport [5] for algebraic numbers.
In this paper, we consider the analogous problem for the partial quotients of Pisot formal power series. In particular, we shall improve the following result of Baum and Sweet [3].
Theorem 1.2. Let $d, n \in \mathbb{N} \backslash\{0\}$. Then there is an algebraic formal power series $w \in \mathbb{F}_{2}\left(\left(X^{-1}\right)\right)$ of degree $2^{n}+1$ such that the equation

$$
\left|w-\frac{P}{Q}\right|=\frac{2^{-d}}{|Q|^{2^{n}+1}}
$$

has infinitely many solutions $(P, Q) \in \mathbb{F}_{2}[X] \times \mathbb{F}_{2}[X]$.
Theorem 1.3. If $I$ and $J$ are polynomials in $\mathbb{F}_{2}[X]$ with $I^{2^{n}}+J \notin \mathbb{F}_{2}$, then the equation

$$
P^{2^{n}+1}+I Q P^{2^{n}}+J P Q^{2^{n}}+(I J+1) Q^{2^{n}+1}=1
$$

has infinitely many polynomial solutions $(P, Q) \in \mathbb{F}_{2}[X] \times \mathbb{F}_{2}[X]$.
The remainder of the paper is organised in the following way. We gather some definitions and theorems in Section 2. In Section 3, we generalize Baum and Sweet results (Theorem 6 and Corollary 7 in [3]). In particular, we give a new version of Davenport [5] and Pass [14] theorems in the case of formal power series (Theorem 1.1 and Theorem 1.2). The Section 4 is devoted to the study of a certain diophantine equations on $\mathbb{F}_{q}[X]$, which generalizes Baum and Sweet's theorem (Theorem 10 in [3]).

## 2. Formal power series and Pisot element

For $w=\sum_{n=n_{0}}^{+\infty} a_{n} X^{-n} \in \mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$, we define the integer part $[w]$ of $w$ by

$$
[w]=\sum_{n=n_{0}}^{0} a_{n} X^{-n}
$$

and the fractional part of $w$ by

$$
\{w\}=w-[w]=\sum_{n=1}^{+\infty} a_{n} X^{-n}
$$

As in Sprindžuk [16], we have a non-archimedean absolute value $|\cdot|$ on $\mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$, that is, for any element $w \in \mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$ of the form

$$
w=\sum_{n=n_{0}}^{+\infty} a_{n} X^{-n}, \quad\left(a_{n} \in \mathbb{F}_{q}\right)
$$

We define $|w|=q^{-n_{0}}$ if $w \neq 0$ and $n_{0}$ is the smallest index with $a_{n_{0}} \neq 0$ and $|w|=0$ if $w=0$. We know that $\mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$ is complete and locally compact with respect to the metric defined by this absolute value.

We denote by $\overline{\mathbb{F}_{q}\left(\left(X^{-1}\right)\right)}$ an algebraic closure of $\mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$. We note that the absolute value has a unique extension to $\overline{\mathbb{F}_{q}\left(\left(X^{-1}\right)\right)}$. To denote this extended absolute value, we also use the symbol $|\cdot|$.

For any nonzero polynomial $P(Y)=\sum_{i=0}^{d} A_{i} Y^{i} \in \mathbb{F}_{q}[X, Y]$, we define the logarithmic height $\mathcal{H}(P)$ of $P$ by

$$
\mathcal{H}(P)=\log _{q} \max _{0 \leq i \leq d}\left|A_{i}\right|=\max _{0 \leq i \leq d} \operatorname{deg}\left(A_{i}\right)
$$

where $\log _{q} x$ means the logarithmic function with base $q$. For an algebraic element $w \in \mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$, we denote by $\mathcal{H}(w)$ the logarithmic height of its minimal polynomial.

A Pisot element $w \in \mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$ is an algebraic integer over $\mathbb{F}_{q}[X]$ with $|w|>1$ whose remaining conjugates in $\overline{\mathbb{F}_{q}\left(\left(X^{-1}\right)\right)}$ have absolute value strictly smaller than 1.

In 1962 Bateman and Duquette [2] introduced and characterized Pisot element in a field of formal power series. They obtained the following results:

Theorem 2.1. An element $w$ in $\mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$ is a Pisot element if and only if its minimal polynomial can be written as $P(Y)=Y^{s}+A_{s-1} Y^{s-1}+\cdots+A_{0}$, $A_{i} \in \mathbb{F}_{q}[X]$ for $i=0, \ldots, s-1$, with $\left|A_{s-1}\right|=|w|>1$ and $\left|A_{i}\right|<|w|$ for $i=0, \ldots, s-2$.

Theorem 2.2. An element $w \in \mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$ satisfying $|w|>1$ is a Pisot element if and only if there exists $a \lambda \in \mathbb{F}_{q}\left(\left(X^{-1}\right)\right) \backslash\{0\}$ such that $\lim _{n \rightarrow+\infty}\left\{\lambda w^{n}\right\}$ $=0 ;$ moreover $\lambda$ can be chosen to belong to $\mathbb{F}_{q}(X)(w)$.

The study of the set $\mathcal{P}$ of Pisot element was resumed in 1967 by GrandetHugot $[6,7]$. In particular she showed that $\mathcal{P}$ is dense in $\mathbb{F}_{q}\left(\left(X^{-1}\right)\right) \backslash\{w:$ $|w|<1\}$. For more information about Pisot element see [4].

## 3. Continued fractions of formal power series

Let us start with the definition of the continued fraction algorithm. Let $\mathcal{J}=\left\{\theta \in \mathbb{F}_{q}\left(\left(X^{-1}\right)\right):|\theta|<1\right\}$. We define the map $T: \mathcal{J} \rightarrow \mathcal{J}$ by

$$
T(w)=\left\{\begin{array}{lll}
\left\{\frac{1}{w}\right\} & \text { if } & w \neq 0 \\
0 & \text { if } & w=0
\end{array}\right.
$$

Every $w \in \mathcal{J}$ has a unique continued fraction defined as follows

$$
w=\frac{1}{A_{1}(w)+\frac{1}{A_{2}(w)+\frac{1}{\ddots}}},
$$

where $A_{1}(w)=\left[\frac{1}{w}\right]$ and $A_{n}(w)=A_{1}\left(T^{n-1}(w)\right)$ for $n \geq 2$. For $w \in \mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$, we put $A_{0}(w)=[w]$ and we have

$$
\begin{equation*}
w=A_{0}(w)+\frac{1}{A_{1}(w)+\frac{1}{A_{2}(w)+\frac{1}{?}}} . \tag{3.1}
\end{equation*}
$$

As a shorthand for (3.1) we write

$$
w=\left[A_{0}(w) ; A_{1}(w), A_{2}(w), \ldots\right]
$$

As usual, we refer to $\frac{P_{n}}{Q_{n}}=\left[A_{0}(w) ; A_{1}(w), A_{2}(w), \ldots, A_{n}(w)\right](n \geq 0)$ as $n$-th convergent to $w$ and to $A_{n}(w)$ as partial quotient. Note that $\left|A_{n}(w)\right|>1$ if $n \neq 0$. Also we have $\left|Q_{n+1}\right|=\left|A_{n+1}(w) Q_{n}\right|$, so

$$
\left|Q_{n} w-P_{n}\right|=\left|Q_{n+1}\right|^{-1}=\left|A_{n+1}(w) Q_{n}\right|^{-1}<\left|Q_{n}\right|^{-1}
$$

We have the following characterization of convergent (see for example [3]): let $P, Q \in \mathbb{F}_{q}[X], Q \neq 0$. If $w \in \mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$ and $|Q w-P|<|Q|^{-1}$, then $\frac{P}{Q}$ is a convergent of $w$. The corresponding property in the real case is (see for instance [8]): let $p, q \in \mathbb{Z}, q \neq 0$, if $x \in \mathbb{R}$ and $|q x-p|<|2 q|^{-1}$, then $\frac{p}{q}$ is a convergent of $x$.

The aim of this section is to improve Baum and Sweet's theorem stated in Theorem 1.2.

Theorem 3.1. Let $\mu, n \in \mathbb{N} \backslash\{0\}, \lambda \in \mathbb{Z}$ and $\tau \in \mathbb{N}$ with $\tau \geq \lambda$. Then there is an algebraic formal power series $w \in \mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$ of degree $p^{n}+1$ satisfying $\mathcal{H}(w)=\mu+\tau$ and $\mu=\log _{q}|w|$ such that the equation

$$
\left|w-\frac{P}{Q}\right|=\frac{p^{-\left(\lambda+\mu p^{n}\right)}}{|Q|^{p^{n}+1}}
$$

has infinitely many solutions $(P, Q) \in \mathbb{F}_{q}[X] \times \mathbb{F}_{q}[X]$.
The proof uses the following lemmas.
Lemma 3.2. Let $P(Y)=A_{d} Y^{d}+\cdots+A_{0}$, with $A_{i} \in \mathbb{F}_{q}[X], A_{d} \neq 0$ and $\left|A_{d-1}\right|>\left|A_{i}\right|$ for all $i \neq d-1$. Then $P$ has only one root $w \in \mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$ satisfying $|w|>1$. Moreover $[w]=-\left[\frac{A_{d-1}}{A_{d}}\right]$ and all other roots of $P$ in $\overline{\mathbb{F}_{q}\left(\left(X^{-1}\right)\right)}$ have an absolute value strictly smaller than 1.

Proof. Let $P(Y)=A_{d} Y^{d}+\cdots+A_{0}$ such that $A_{i} \in \mathbb{F}_{q}[X]$ for each $i \in$ $\{0,1, \ldots, d\}, A_{d} \neq 0$ and $\left|A_{d-1}\right|>\left|A_{i}\right|$, for all $i \neq d-1$. Let $\lambda=-\frac{A_{d-1}}{A_{d}}$ and let $Z$ be defined by $Y=\lambda Z$. We have

$$
\begin{equation*}
\frac{-1}{A_{d-1} \lambda^{d-1}} P(\lambda Z)=Z^{d-1}(Z-1)+L(Z) \tag{3.2}
\end{equation*}
$$

where $L(Z)=-\sum_{j=0}^{d-2} \frac{A_{j}}{\lambda^{d-j-1} A_{d-1}} Z^{j}$. We remark that the coefficients of $L(Z)$ have absolute value strictly smaller than 1 . Using Hensel's lemma [1], we get
that $P(Y)$ has a unique root $w$ in $\overline{\mathbb{F}_{q}\left(\left(X^{-1}\right)\right)}$ such that $w=\lambda z$ and $|z-1|<1$, moreover $w \in \mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$. From $|z-1|<1$, we have $|z|=1$, and since the coefficients of $L(Z)$ have absolute value strictly smaller than $\frac{1}{|\lambda|}$, then $|z-1|=$ $|L(z)|<\frac{1}{|\lambda|}$. Thus the unique root $w$ of $P$ verifies $|w-\lambda|<1$ and therefore

$$
[w]=[\lambda]=-\left[\frac{A_{d-1}}{A_{d}}\right] .
$$

The other roots $z_{i}$ of the polynomial $P(\lambda Z)$ in $\overline{\mathbb{F}_{q}\left(\left(X^{-1}\right)\right)}$ satisfying $\left|z_{i}-1\right| \geq 1$. It is clear that $\left|z_{i}\right|<1$ because otherwise from (3.2) we get

$$
\left|z_{i}\right|^{d-1} \leq\left|z_{i}\right|^{d-1}\left|z_{i}-1\right|=\left|L\left(z_{i}\right)\right|<\frac{1}{|\lambda|}\left|z_{i}\right|^{d-2}
$$

consequently $\left|z_{i}\right|<\frac{1}{|\lambda|}<1$, a contradiction.
Then the other roots $w_{i}=\lambda z_{i}$ satisfy $\left|w_{i}\right|<|\lambda|$, which implies that $\left|w_{i}\right|<$ 1. If $1 \leq\left|w_{i}\right|<|\lambda|$, we obtain for any $k \in\{0,1, \ldots, d-1\},\left|A_{k}\right|\left|w_{i}\right|^{k}<$ $\left|A_{d-1} \|\right|^{d-1}$ and $\left|A_{d}\left\|\left.w_{i}\right|^{d}<\left|A_{d-1} \| w_{i}\right|^{d-1}\right.\right.$, hence $\left|P\left(w_{i}\right)\left\|A_{d-1}\right\| w_{i}\right|^{d-1}$. This leads to a contradiction with $P\left(w_{i}\right)=0$.

Remark 3.3. Let $A, B, C \in \mathbb{F}_{q}[X]$ with $\operatorname{deg} B>0$ and $A C \neq 0$ then by Lemma 3.2 the polynomial

$$
\begin{equation*}
L(Y)=A Y^{p^{n}+1}+A B C Y^{p^{n}}+C \tag{3.3}
\end{equation*}
$$

has a unique root $w \in \mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$ such that $|w|>1$.
We next describe a class of algebraic elements with explicitly given continued fraction expressions, a special case of which is the unique root $w$ in $\mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$ of

$$
Y^{p^{n}+1}+X Y^{p^{n}}+1=0
$$

whose continued fraction is

$$
w=\left[-X, X^{p^{n}},-X^{p^{2 n}}, X^{p^{3 n}},-X^{p^{4 n}}, \ldots\right] .
$$

Theorem 3.4 (See [13]). Let $\alpha \in \mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$ be an algebraic element of degree $m$ over $\mathbb{F}_{q}(X)$ such that $[\alpha] \neq 0$ and $A_{m} \alpha^{m}+\cdots+A_{1} \alpha+A_{0}=0$, with $\operatorname{deg} A_{k}<\operatorname{deg} A_{m-1}$ for all $k \in\{0, \ldots, m\} \backslash\{m-1\}$. Then $[\alpha]=-\left[A_{m-1} / A_{m}\right]$ and the formal power series $h=1 /(\alpha-[\alpha])$ is of type $(I)$.

The next corollary is an application of:
Corollary 3.5. Let $n \in \mathbb{N} \backslash\{0\}$, $A, B, C \in \mathbb{F}_{q}[X]$ with $\operatorname{deg} B>0, A C \neq 0$, $\operatorname{gcd}(A, C)=1$ and $w \in \mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$ satisfying $|w|>1$ and $A w^{p^{n}+1}+A B C w^{p^{n}}+$ $C=0$. Then

$$
w=\left[a_{0} ; a_{1}, \ldots\right],
$$

where $a_{s}=(-1)^{s+1} A^{\frac{p^{s n}-(-1)^{s}}{p^{n}+1}} B^{p^{s n}} C^{\frac{p^{(s+1) n}+(-1)^{s}}{p^{n}+1}}$.

Proof. Let $\alpha_{0}=\alpha, a_{0}=\left[\alpha_{0}\right], P_{0}=A, Q_{0}=A B C, R_{0}=0, S_{0}=C$ and $\alpha_{s+1}=1 /\left(\alpha_{s}-\left[\alpha_{s}\right]\right)$ then using Theorem 3.4, $\alpha_{s}$ is of type (I) and satisfying the equation $P_{s} \alpha^{p^{n}+1}+Q_{s} \alpha^{p^{n}}+R_{s} \alpha_{s}+S_{s}=0$ with

$$
\begin{gathered}
P_{s+1}=P_{s} a^{p^{n}+1}+Q_{s} a^{p^{n}}+R_{s} a_{s}+S_{s}, \quad Q_{s+1}=P_{s} a^{p^{n}} \\
R_{s+1}=Q_{s}+a_{s} P_{s}, \quad S_{s+1}=P_{s}, \quad a_{s+1}=-\left[\frac{Q_{s+1}}{P_{s+1}}\right] .
\end{gathered}
$$

Using a simple induction on $s$ we prove that

$$
\begin{aligned}
& P_{s}=A^{\frac{1+(-1)^{s}}{2}} C^{\frac{1-(-1)^{s}}{2}}, \\
& Q_{s}=(-1)^{s} A^{\frac{1-(-1)^{s}}{2}} A^{\frac{p^{s n}+(-1)^{s} p^{n}}{p^{n}+1}} B^{p^{s n}} C^{\frac{1+(-1)^{s}}{2}} C^{\frac{p^{(s+1) n}-(-1)^{s} p^{n}}{p^{n}+1}}, \\
& R_{s}=0, S_{s}=A^{\frac{1-(-1)^{s}}{2}} C^{\frac{1+(-1)^{s}}{2}}
\end{aligned}
$$

and as required

$$
a_{s}=(-1)^{s+1} A^{\frac{p^{s n}-(-1)^{s}}{p^{n}+1}} B^{p^{s n}} C^{\frac{p^{(s+1) n}+(-1)^{s}}{p^{n}+1}} .
$$

The case $A=C=1$ in Corollary 3.5 over $\mathbb{F}_{2}[X]$ leads us to the well-known result due to Baum and Sweet [3].

Corollary 3.6 (Theorem $6,[3])$. Let $n \in \mathbb{N}, P, Q \in \mathbb{F}_{2}[X]$ with $P+Q^{2^{n}} \notin \mathbb{F}_{2}$.
Then the equation

$$
Y^{2^{n}+1}+Q Y^{2^{n}}+P Y+P Q+1=0
$$

has a unique root $w$ in $\mathbb{F}_{2}\left(\left(X^{-1}\right)\right)$, and

$$
w=\left[Q ; Q^{2^{2 n}}+P^{2^{n}}, Q^{2^{3 n}}+P^{2^{2 n}}, \ldots\right] .
$$

Proof. To prove Corollary 3.6, let $g=\frac{1}{w-Q}$ and observe that

$$
g^{2^{n}+1}+\left(Q^{2^{n}}+P\right) g^{2^{n}}+1=0
$$

the desired result now follows from Corollary 3.5.
Proof of Theorem 3.1. Let $w$ the unique root of (3.3) satisfying $|w|>1$ and $w=\left[a_{0} ; a_{1}, \ldots\right]$, then from Corollary 3.5

$$
a_{s}=(-1)^{s+1} A^{\frac{p^{s n}-(-1)^{s}}{p^{n}+1}} B^{p^{s n}} C^{\frac{p^{(s+1) n}+(-1)^{s}}{p^{n}+1}} .
$$

If $\frac{P_{s}}{Q_{s}}$ is the $s$-th convergent of $w$, then

$$
\begin{aligned}
\left|Q_{2 s}\right| & =\prod_{i=1}^{2 s}\left|a_{i}\right| \\
& =|A|^{\frac{p^{(2 s+1) n}-p^{n}}{\left(p^{n}+1\right)\left(p^{n}-1\right)}}|B|^{\frac{p^{(2 s+1) n}-p^{n}}{p^{n}-1}}|C|^{\frac{\left.p^{(2 s+2) n}-p^{2 n}\right)}{\left(p^{n}+1\right)\left(p^{n}-1\right)}},
\end{aligned}
$$

and

$$
\begin{aligned}
\left|Q_{2 s+1}\right| & =\prod_{i=1}^{2 s+1}\left|a_{i}\right| \\
& =|A|^{\frac{p^{(2 s+2) n}\left(p^{n}+1\right)\left(p^{n-1}\right)}{}}|B|^{\frac{p^{(2 s+2) n}-p^{n}}{p^{n}-1}}|C|^{\frac{p^{(2 s+3) n}-p^{2 n}-p^{n}}{\left(p^{n}+1\right)\left(p^{n n}-1\right)}} .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\left|a_{2 s+1}\right| & =|A|^{\frac{p^{(2 s+1) n}}{p^{n}+1}}|B|^{p^{(2 s+1) n}}|C|^{\frac{p^{(2 s+2) n}-1}{p^{n}+1}} \\
& =\left.|A||B|^{p^{n}}|C|^{p^{n}-1}\left|Q_{2 s}\right|\right|^{p^{n}-1} .
\end{aligned}
$$

Hence,

$$
\begin{align*}
\left|w-\frac{P_{2 s}}{Q_{2 s}}\right| & =\frac{1}{\left|a_{2 s+1}\right|\left|Q_{2 s}\right|^{2}} \\
& =|A|^{-1}|B|^{-p^{n}}|C|^{1-p^{n}}\left|Q_{2 s}\right|^{-\left(p^{n}+1\right)}  \tag{3.4}\\
& =\frac{p^{-\left(\lambda+\mu p^{n}\right)}}{\left|Q_{2 s}\right|^{p^{n}+1}},
\end{align*}
$$

with $\tau=\operatorname{deg}(A), \lambda=\operatorname{deg}(A)-\operatorname{deg}(C)$ and $\mu=\operatorname{deg}(B)+\operatorname{deg}(C)$,

$$
\begin{align*}
\left|w-\frac{P_{2 s+1}}{Q_{2 s+1}}\right| & =\frac{1}{\left|a_{2 s+2}\right|\left|Q_{2 s+1}\right|^{2}} \\
& =|B C|^{-p^{n}}\left|Q_{2 s+1}\right|^{-\left(p^{n}+1\right)}  \tag{3.5}\\
& =\frac{p^{-\mu p^{n}}}{\left|Q_{s}\right|^{p^{n}+1}}
\end{align*}
$$

with $\mu=\operatorname{deg}(B)+\operatorname{deg}(C)$ and $\lambda=\tau=0$.
We next prove that the algebraic element $w$ is of degree $p^{n}+1$, we will need the following lemmas. Liouville's theorem for algebraic numbers has the following analogue in $\mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$ :

Lemma 3.7. If $w \in \mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$ is algebraic of degree $D$ over $\mathbb{F}_{q}(X)$, then there is a constant $c>0$ such that

$$
\left|w-\frac{P}{Q}\right| \geq \frac{c}{|Q|^{D}}
$$

for all $P, Q \in \mathbb{F}_{q}[X]$.
The proof of Lemma 3.7 can be found in Mahler's paper [10] and in [3].
Lemma 3.8. Let $A, B$ and $C \in \mathbb{F}_{q}[X]$ with $A C \neq 0$ and $\operatorname{deg} B \geq 1$. Then the polynomial

$$
L(Y)=A Y^{p^{n}+1}+A B C Y^{p^{n}}+C
$$

is irreducible over $\mathbb{F}_{q}(X)$.

Proof. Let $w \in \mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$ the unique root of $L$ such that $|w|>1$. If $w$ has degree $D<p^{n}+1$ and $\frac{P}{Q}$ is a convergent of $w$, then by (3.4), (3.5) and Lemma 3.7 , there are two constants $c, c^{\prime}>0$ such that

$$
\frac{c}{|Q|^{D}} \leq\left|w-\frac{P}{Q}\right|=\frac{c^{\prime}}{|Q|^{p^{n}+1}}
$$

for arbitrarily large $|Q|$, which is a contradiction.
This completes the proof of Theorem 3.1.
Corollary 3.9. Let $\mu, n \in \mathbb{N} \backslash\{0\}$ and $\lambda \in \mathbb{N}$. Then there is a Pisot element $w \in \mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$ of degree $p^{n}+1$ satisfying $\mathcal{H}(w)=\mu=\log _{q}|w|$ such that

$$
\left|w-\frac{P}{Q}\right|=\frac{p^{-\left(\lambda+\mu p^{n}\right)}}{|Q|^{p^{n}+1}}
$$

has infinitely many solutions $(P, Q) \in \mathbb{F}_{q}[X] \times \mathbb{F}_{q}[X]$.
Proof. We observe that (3.3) has a unique Pisot element as solution if $A=$ 1.

The last corollary improves Pass's theorem (Theorem 1.1) in the case of formal power series.

## 4. Diophantine equation

We prove the following result:
Theorem 4.1. If $A, B$ and $C$ are fixed polynomials in $\mathbb{F}_{q}[X]$ with $A C \neq$ $0, \operatorname{gcd}(A, C)=1$ and $\operatorname{deg} B \geq 1$, then

$$
\begin{equation*}
A P^{p^{n}+1}+A B C Q P^{p^{n}}+C Q^{p^{n}+1}=A \tag{1}
\end{equation*}
$$

and
$\left(E_{2}\right)$

$$
A P^{p^{n}+1}+A B C Q P^{p^{n}}+C Q^{p^{n}+1}=C
$$

have infinitely many solutions $(P, Q) \in \mathbb{F}_{q}[X] \times \mathbb{F}_{q}[X]$. Moreover if $\frac{P}{Q}$ and $\frac{P^{\prime}}{Q^{\prime}}$ are two consecutive convergent of the element $w$ in Corollary 3.5, then $(P, Q)$ is a solution of $E_{i}$ and $\left(P^{\prime}, Q^{\prime}\right)$ is a solution of $E_{j}$, where $i, j \in\{1,2\}$ and $i \neq j$.

We will need the following lemmas.
Lemma 4.2. Let $w$ the unique root of (3.3) satisfying $|w|>1$. If $\frac{P_{s}}{Q_{s}}$ is the $s$-th convergent of $w$, then for all $s \in \mathbb{N}, A \mid Q_{2 s+1}$ and $C \mid P_{2 s}$.

Proof. We have $P_{0}=-B C$ and $Q_{1}=A B^{p^{n}} C^{p^{n-1}}$ then we see that $C \mid P_{0}$ and $A \mid Q_{1}$ by using an induction on $s$ and the relation

$$
\begin{equation*}
Q_{s+2}=a_{s+2} Q_{s+1}+Q_{s} \text { for all } s \in \mathbb{N} \tag{4.1}
\end{equation*}
$$

This proves the lemma.

Lemma 4.3. Let $n \in \mathbb{N}^{*}, A, B, C \in \mathbb{F}_{q}[X]$ with $\operatorname{deg} B>0, A C \neq 0$, $\operatorname{gcd}(A, C)=1$ and

$$
H(Y, Z)=A Y^{p^{n}+1}+A B C Y^{p^{n}} Z+C Z^{p^{n}+1}
$$

Then for all $s \in \mathbb{N}$, there exists a polynomial $\Lambda_{s} \in \mathbb{F}_{q}[X]$ such that

$$
H\left(P_{2 s}, Q_{2 s}\right)=a_{2 s} \Lambda_{2 s}+C \text { and } H\left(P_{2 s+1}, Q_{2 s+1}\right)=a_{2 s+1} \Lambda_{2 s+1}+A,
$$

where $\frac{P_{s}}{Q_{s}}$ is an s-th convergent of $w$ the unique root of (3.3) satisfying $|w|>1$.
Proof. We have $P_{0}=-B C, P_{1}=A B^{p^{n}+1} C^{p^{n}}+1, Q_{0}=1$ and $Q_{1}=$ $A B^{p^{n}} C^{p^{n-1}}$, then

$$
H\left(P_{0}, Q_{0}\right)=C \text { and } H\left(P_{1}, Q_{1}\right)=A,
$$

using an induction on $s$ and (4.1), this prove the lemma.
Proof of Theorem 4.1. Let

$$
H(Y, Z)=A Y^{p^{n}+1}+A B C Y^{p^{n}} Z+C Z^{p^{n}+1}
$$

and $w$ the unique root in $\mathbb{F}_{q}\left(\left(X^{-1}\right)\right)$ of $L(Y)=H(Y, 1)$ satisfying $|w|>1$, then by writing $H(Y, 1)$ in the form

$$
H(Y, 1)=A w^{p^{n}}(Y-w)+A(Y+B C)(Y-w)^{p^{n}}
$$

we see that we can conclude from $[w]=-B C$ that

$$
\left|H\left(\frac{P}{Q}, 1\right)\right|=|A||B C|^{p^{n}}\left|w-\frac{P}{Q}\right| \quad \text { for }\left|w-\frac{P}{Q}\right|<1 .
$$

It follows from (3.4) and (3.5) that

$$
\left|H\left(\frac{P_{2 s}}{Q_{2 s}}, 1\right)\right|=|A||B C|^{p^{n}}\left|w-\frac{P_{2 s}}{Q_{2 s}}\right|=\frac{|C|}{\left|Q_{2 s}\right|^{p^{n}+1}}
$$

and

$$
\left|H\left(\frac{P_{2 s+1}}{Q_{2 s+1}}, 1\right)\right|=|A||B C|^{p^{n}}\left|w-\frac{P_{2 s+1}}{Q_{2 s+1}}\right|=\frac{|A|}{\left|Q_{2 s+1}\right|^{p^{n}+1}} .
$$

Since

$$
H(P, Q)=Q^{p^{n}+1} H\left(\frac{P}{Q}, 1\right)
$$

we obtain

$$
\begin{equation*}
\left|H\left(P_{2 s}, Q_{2 s}\right)\right|=|C| \text { and } \quad\left|H\left(P_{2 s+1}, Q_{2 s+1}\right)\right|=|A|, \tag{4.2}
\end{equation*}
$$

and from Lemma 4.2, we have

$$
\begin{equation*}
C \mid H\left(P_{2 s}, Q_{2 s}\right) \text { and } \quad A \mid H\left(P_{2 s+1}, Q_{2 s+1}\right) . \tag{4.3}
\end{equation*}
$$

The statements (4.2) and (4.3) now imply that there exists an $\alpha_{s} \in \mathbb{F}_{q}$ such that

$$
\begin{equation*}
H\left(P_{2 s}, Q_{2 s}\right)=\alpha_{2 s} C \text { and } \quad H\left(P_{2 s+1}, Q_{2 s+1}\right)=\alpha_{2 s+1} A . \tag{4.4}
\end{equation*}
$$

Since $\operatorname{deg}\left(a_{2 s}\right)>\operatorname{deg}(C)$ and $\operatorname{deg}\left(a_{2 s+1}\right)>\operatorname{deg}(A)$, using Lemma 4.3 and (4.4) we arrive at

$$
\Lambda_{2 s}=\Lambda_{2 s+1}=0 \text { and } \alpha_{2 s}=\alpha_{2 s+1}=1
$$

This completes the proof of Theorem 4.1.
Corollary 4.4 (Theorem 1.3). If $I$ and $J$ are polynomials in $\mathbb{F}_{2}[X]$ with $I^{2^{n}}+$ $J \notin \mathbb{F}_{2}$, then

$$
P^{2^{n}+1}+I Q P^{2^{n}}+J P Q^{2^{n}}+(I J+1) Q^{2^{n}+1}=1
$$

has infinitely many solutions $(P, Q) \in \mathbb{F}_{2}[X] \times \mathbb{F}_{2}[X]$.
Proof. Let $K(Y)=Y^{2^{n}+1}+I Y^{2^{n}}+J Y+(I J+1)$ be a polynomial, with $I^{2^{n}}+J \notin \mathbb{F}_{2}$, and $w$ the unique root in $\mathbb{F}_{2}\left(\left(X^{-1}\right)\right)$ with $|w|>1$ of the polynomial $L(Y)=H(Y, 1)=Y^{2^{n}+1}+\left(I^{2^{n}}+J\right) Y^{2^{n}}+1$. Observe that $v=I+\frac{1}{w}$ is a root of $K$. If $\frac{P}{Q}$ is a convergent of $v$, then $\frac{Q}{P-I Q}$ is a convergent of $w$. From Theorem 4.1, $(Q, P-I Q)$ is a solution of $\left(E_{1}\right)$ and $\left(E_{2}\right)$. Therefore the equation

$$
P^{2^{n}+1}+I Q P^{2^{n}}+J P Q^{2^{n}}+(I J+1) Q^{2^{n}+1}=1
$$

has infinitely many solutions $(P, Q) \in \mathbb{F}_{2}[X] \times \mathbb{F}_{2}[X]$.

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