Bull. Korean Math. Soc. **53** (2016), No. 3, pp. 681–698
http://dx.doi.org/10.4134/BKMS.b150233
pISSN: 1015-8634 / eISSN: 2234-3016

ON RELATIVE ESSENTIAL SPECTRA OF BLOCK OPERATOR MATRICES AND APPLICATION

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Abstract. In this paper, we investigate relative essential spectra of 2×2 block operator matrix using the Fredholm perturbation theory. Furthermore, an example for two-group transport equations is presented to illustrate the validity of the main results.

1. Introduction

Numerous mathematical and physical problems lead to operator pencils, $L-\lambda M$ (see for example [8, 18]). Recently, the spectral theory of operator pencils attracts the attention of many mathematicians. Moreover, the motivations for studying the M-essential spectra of block operator matrix are various and meaningful in transport theory.

In this paper, we are mainly concerned with the study of the spectral theory for pencils of the form

$$L_0 - \lambda M := \begin{pmatrix} A & B \\ C & D \end{pmatrix} - \lambda \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \tag{1}$$

considered on the product Banach space $X\times Y$, where M is a bounded operator. The operator A (resp. D) acts on the Banach space X (resp. Y) and has the domain $\mathcal{D}(A)$ (resp. $\mathcal{D}(D)$) and the intertwining operator B (resp. C) is defined on the domain $\mathcal{D}(B)$ (resp. $\mathcal{D}(C)$) and acts from Y into X (resp. from X into Y).

The block operator matrix of the form (1) is densely defined with domain given by

$$\begin{aligned}&\mathcal{D}(L_0-\lambda M)\\ &:= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in (\mathcal{D}(A)\cap\mathcal{D}(C)) \times (\mathcal{D}(B)\cap\mathcal{D}(D)) \text{ such that } \Gamma_X x = \Gamma_Y y \right\},\end{aligned}$$

Received April 4, 2015; Revised September 21, 2015.
2010 *Mathematics Subject Classification.* 47A10, 47A53, 34K08.
Key words and phrases. matrix operator, Fredholm perturbations, M-essential spectra, transport operator.

where Γ_X (resp. Γ_Y) is a linear operator acting from X (resp. Y) into a Banach space Z. The closure of the block operator matrix $L_0, L := \overline{L}_0$, is discussed in details in the paper of [25] under some assumptions on the entries components. The study of the block operator matrix is the subject of many authors under different assumptions. In this direction some issues may be found in the literature, we can quote for example [1, 2, 10, 16, 24]. Recently, an account research and a wide panorama of methods to investigate the spectral theory of block operator matrices is given in [3, 4, 5, 11, 14, 25]. More precisely, the description of various essential spectra of a block operator matrix L appears in [3, 5, 11, 14] to improve and generalize some results given by [1, 2, 24] for block operator matrices in Banach spaces under some compactness assumptions.

However, it should be noted that several results for the authors cited in the papers of [1, 2, 4, 14, 24] are aimed at providing methods for dealing with spectral theory for operator in the form $L_0 - \lambda M$ where $M = I$.

The purpose of this work consists principally in extending results given in [4] and we concern ourselves exclusively with the investigation of some M-essential spectra of unbounded 2×2 block operator matrices for pencils of the form $L_0 - \lambda M$, where M is a bounded operator defined on the product of two Banach spaces $X \times Y$ under a coupling condition between the two components of its elements.

To do this, we firstly establish some results on right and left-Fredholm perturbations theory (see Theorems 2.2 and 2.3). Eventually, we dispose different conditions in terms of the Fredholm, right and left-Fredholm perturbations to prove the Fredholmness perturbations of block 2×2 operator matrix having the form

$$(L - \lambda M)^{-1} - (L_{\lambda_0} - \lambda M)^{-1}$$

(see Section 3), which allows us to investigate the stability of their M-essential spectra in terms of Schur-complement whose M_4-essential spectra is easier to calculate. Moreover, the use of the M-resolvent, the Fredholm perturbations theory and the lower factorization allows us to formulate some supplements to many results presented in [4, 14] and to ameliorate the description of the M-essential spectra of two-group transport equations without knowing the totality of the relative essential spectra of the operator A, but only the relative essential spectra of its restriction which is more general than the one provided by [4, 14].

The present paper consists of four sections: In Section 2, we present some basic notations and auxiliary lemmas connected to the main body of the paper. We advise that Section 3 constitutes the main results. Section 4 is devoted to illustrate our abstract results to two group transport operator in Banach spaces.

2. Preliminary results

Let X and Y be two Banach spaces. Throughout this section, T denotes a linear operator from X into Y with domain $\mathcal{D}(T) \subset X$ and range $\mathcal{R}(T) \subset Y$. By $\mathcal{C}(X, Y)$ (resp. $\mathcal{L}(X, Y)$), we designate the set of all closed, densely defined linear operators (resp. the set of all bounded linear operators) from X into Y and by $\mathcal{K}(X,Y)$ the subset of all compact operators of $\mathcal{L}(X,Y)$. For $T \in \mathcal{C}(X,Y)$, $\mathcal{N}(T)$ denotes the null space of T. The nullity $\alpha(T)$ of T is defined as the dimension of $\mathcal{N}(T)$ and the deficient $\beta(T)$ of T is defined as the codimension of $\mathcal{R}(T)$ in Y.

Let S be a non null bounded operator from X into Y. For $T \in \mathcal{C}(X,Y)$, we define the S-resolvent set of T by:

$$\rho_S(T) := \{\lambda \in \mathbb{C} : T - \lambda S \text{ has a bounded inverse}\}$$

and the S-spectrum of T by:

$$\sigma_S(T) := \mathbb{C} \setminus \rho_S(T).$$

In what follows, we need to introduce some important classes of operators. The set of upper semi-Fredholm operators from X into Y is defined by:

$$\Phi_+(X,Y) = \{T \in \mathcal{C}(X,Y) : \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in } Y\},$$

and the set of lower semi-Fredholm operators from X into Y is defined by:

$$\Phi_-(X,Y) = \{T \in \mathcal{C}(X,Y) : \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in } Y\}.$$

$\Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_-(X,Y)$ (resp. $\Phi_\pm(X,Y) := \Phi_+(X,Y) \cup \Phi_-(X,Y)$) denotes the set of Fredholm (resp. semi-Fredholm) operators from X into Y. If $T \in \Phi(X,Y)$, the number $i(T) := \alpha(T) - \beta(T)$ is called the index of T. The set $\Phi_{T,S}$ is defined as:

$$\Phi_{T,S} = \{\lambda \in \mathbb{C} : T - \lambda S \in \Phi(X,Y)\}.$$

A complex number λ is in $\Phi_{+T,S}$, $\Phi_{-T,S}$ or $\Phi_{T,S}$ if $T - \lambda S$ is in $\Phi_+(X,Y)$, $\Phi_-(X,Y)$ or $\Phi(X,Y)$, respectively. If $X = Y$, then $\mathcal{L}(X,Y), \mathcal{C}(X,Y), \mathcal{K}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$ and $\Phi_-(X,Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$ and $\Phi_-(X)$ respectively.

Recall the following results established in [20].

Definition 2.1. Let X and Y be two Banach spaces. An operator $T \in \mathcal{L}(X,Y)$ is said to have a left (resp. a right) Fredholm inverse if there exists an operator $T_l \in \mathcal{L}(Y,X)$ (resp. $T_r \in \mathcal{L}(Y,X)$) such that $T_lT - I \in \mathcal{K}(X)$ (resp. $TT_r - I \in \mathcal{K}(Y)$). The operators T_l (resp. T_r) is called left (resp. right) Fredholm inverse of T.

We will denote by $\Phi_l(X,Y)$ (resp. $\Phi_r(X,Y)$) the set of operators which have left (resp. right) Fredholm inverse.

We denote the sets $\Phi_l^b(X,Y)$, $\Phi_r^b(X,Y)$, $\Phi^b(X,Y)$, $\Phi_+^b(X,Y)$ and $\Phi_-^b(X,Y)$ by $\Phi_l(X,Y) \cap \mathcal{L}(X,Y)$, $\Phi_r(X,Y) \cap \mathcal{L}(X,Y)$, $\Phi(X,Y) \cap \mathcal{L}(X,Y)$, $\Phi_+(X,Y) \cap \mathcal{L}(X,Y)$ and $\Phi_-(X,Y) \cap \mathcal{L}(X,Y)$ respectively.

Our concern in this paper is mainly the following S-essential spectra:

$$\sigma_{el,S}(T) := \{\lambda \in \mathbb{C} : T - \lambda S \notin \Phi_l(X,Y)\} = \mathbb{C} \setminus \Phi_{lT,S},$$
$$\sigma_{er,S}(T) := \{\lambda \in \mathbb{C} : T - \lambda S \notin \Phi_r(X,Y)\} = \mathbb{C} \setminus \Phi_{rT,S},$$

$$\sigma_{e_4,S}(T) := \{\lambda \in \mathbb{C} \ : T - \lambda S \notin \Phi(X,Y)\} = \mathbb{C} \setminus \Phi_{T,S},$$
$$\sigma_{e_5,S}(T) := \mathbb{C} \setminus \rho_{5,S}(T),$$
$$\sigma_{e_6,S}(T) := \mathbb{C} \setminus \rho_{6,S}(T),$$

where $\rho_{5,S}(T) := \{\lambda \in \Phi_T$ such that $i(T - \lambda S) = 0\}$ and $\rho_{6,S}(T)$ denotes the set of those $\lambda \in \rho_{5,S}(T)$ such that all scalars near of λ are in $\rho_S(T)$.

We mention that if $S = I$, we recover the usual definition of the essential spectra of a closed densely defined linear operator A, that is, the subsets $\sigma_{el,I}(\cdot)$ and $\sigma_{er,I}(\cdot)$ are respectively the left and right essential spectra [20], $\sigma_{e_4,I}(\cdot)$ is the Wolf essential spectrum [26], $\sigma_{e_5,I}(\cdot)$ is the Schechter essential spectrum [23] and $\sigma_{e_6,I}(\cdot)$ denotes the Browder essential spectrum [21].

We turn our attention to the following inclusions:

$$\sigma_{e_3,S}(T) = \sigma_{e_1,S}(T) \cap \sigma_{e_2,S}(T) \subseteq \sigma_{e_4,S}(T) \subseteq \sigma_{e_5,S}(T) \subseteq \sigma_{e_6,S}(T),$$

and

$$\sigma_{e1,S}(T) \subset \sigma_{el,S}(T) \subset \sigma_{e4,S}(T), \qquad \sigma_{e2,S}(T) \subset \sigma_{er,S}(T) \subset \sigma_{e4,S}(T), \tag{2}$$

where

$$\sigma_{e_1,S}(T) := \{\lambda \in \mathbb{C} : T - \lambda S \notin \Phi_+(X,Y)\} = \mathbb{C} \setminus \Phi_{+T,S},$$
$$\sigma_{e_2,S}(T) := \{\lambda \in \mathbb{C} : T - \lambda S \notin \Phi_-(X,Y)\} = \mathbb{C} \setminus \Phi_{-T,S},$$
$$\sigma_{e_3,S}(T) := \{\lambda \in \mathbb{C} : T - \lambda S \notin \Phi_\pm(X,Y)\} = \mathbb{C} \setminus \Phi_{\pm T,S}.$$

Definition 2.2. Let X and Y be two Banach spaces. An operator $T \in \mathcal{L}(X,Y)$ is said to be weakly compact if $T(M)$ is relatively weakly compact in Y for every bounded subset $M \subset X$.

The family of weakly compact operators from X into Y is denoted by $\mathcal{W}(X,Y)$. If $X = Y$ the family of weakly compact operators on X, $\mathcal{W}(X) := \mathcal{W}(X,X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$ (cf. [7]).

Definition 2.3. Let X and Y be two Banach spaces. An operator $S \in \mathcal{L}(X,Y)$ is said to be strictly singular if the restriction of S to any infinite-dimensional subspace of X is not an homeomorphism.

Let $S(X,Y)$ denote the set of strictly singular operators from X to Y.

The concept of strictly singular operators was introduced in the pioneering paper by T. Kato [15] as a generalization of the notion of compact operators. For a detailed study of the properties of strictly singular operators, we refer to [15]. Note that $S(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$. In general, if $X = Y, S(X) := S(X,X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$. If X is a Hilbert space, then $S(X) = \mathcal{K}(X)$. The class of weakly compact operators in L_1-spaces (resp. $\mathcal{C}(\Omega)$-spaces with Ω is a compact Hausdroff space) is nothing else than the family of strictly singular operators on L_1-spaces (resp. $\mathcal{C}(\Omega)$-spaces) (see [22, Theorem 1]).

In the following, we introduce some definitions on Fredholm perturbations:

Definition 2.4. Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X,Y)$,

(i) F is called a Fredholm perturbation if $U + F \in \Phi(X,Y)$ whenever $U \in \Phi(X,Y)$.

(ii) F is called a left (resp. right) Fredholm perturbation if $U+F \in \Phi_l(X,Y)$ (resp. $U + F \in \Phi_r(X,Y)$) whenever $U \in \Phi_l(X,Y)$ (resp. $U + F \in \Phi_r(X,Y)$).

We denote by $\mathcal{F}(X,Y)$ the set of Fredholm perturbations and by $\mathcal{F}_l(X,Y)$ (resp. $\mathcal{F}_r(X,Y)$) the set of left (resp. right) Fredholm perturbations.

If $X = Y$ we write $\mathcal{F}(X), \mathcal{F}_l(X)$ and $\mathcal{F}_r(X)$ for $\mathcal{F}(X,X), \mathcal{F}_l(X,X)$ and $\mathcal{F}_r(X,X)$ respectively.

Remark 2.1. Let $\Phi^b(X,Y), \Phi^b_l(X,Y)$ and $\Phi^b_r(X,Y)$ denote respectively the sets $\Phi(X,Y) \cap \mathcal{L}(X,Y)$, $\Phi_l(X,Y) \cap \mathcal{L}(X,Y)$ and $\Phi_r(X,Y) \cap \mathcal{L}(X,Y)$. If in Definition 2.4 we replace $\Phi(X,Y)$, $\Phi_l(X,Y)$ and $\Phi_r(X,Y)$ by $\Phi^b(X,Y)$, $\Phi^b_l(X,Y)$ and $\Phi^b_r(X,Y)$ we obtain the sets $\mathcal{F}^b(X,Y), \mathcal{F}^b_l(X,Y)$ and $\mathcal{F}^b_r(X,Y)$ respectively.

The set of Fredholm perturbations $\mathcal{F}^b(X,Y)$ was introduced and investigated in [9]. In particular, it is shown that $\mathcal{F}^b(X,Y)$ is a closed subset of $\mathcal{L}(X,Y)$ and if $X = Y$, then $\mathcal{F}^b(X) := \mathcal{F}^b(X,X)$ is a closed two-sided ideal of $\mathcal{L}(X)$.

In [13], it was proved that if $X = Y$, then $\mathcal{F}^b_l(X) := \mathcal{F}^b_l(X,X)$ and $\mathcal{F}^b_r(X) := \mathcal{F}^b_r(X,X)$ are two-sided ideals of $\mathcal{L}(X)$, satisfying:

$$\mathcal{K}(X,Y) \subseteq \mathcal{F}^b_l(X,Y) \subseteq \mathcal{F}^b(X,Y)$$

and

$$\mathcal{K}(X,Y) \subseteq \mathcal{F}^b_r(X,Y) \subseteq \mathcal{F}^b(X,Y).$$

Let us recall the following results on Fredholm perturbations theory of 2×2 block operator matrix introduced by [13].

Theorem 2.1 ([13, Theorems 3.1-3.2])**.** *Let X_1 and X_2 be two Banach spaces and $F := \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$ where $F_{ij} \in \mathcal{L}(X_j, X_i), i, j = 1, 2$. Then*

(i) *$F \in \mathcal{F}^b(X_1 \times X_2)$ if and only if $F_{ij} \in \mathcal{F}^b(X_j, X_i)$, $\forall i, j = 1, 2$.*

(ii) *$F \in \mathcal{F}^b_l(X_1 \times X_2)$ if and only if $F_{ij} \in \mathcal{F}^b_l(X_j, X_i)$, $\forall i, j = 1, 2$.*

(iii) *$F \in \mathcal{F}^b_r(X_1 \times X_2)$ if and only if $F_{ij} \in \mathcal{F}^b_r(X_j, X_i)$, $\forall i, j = 1, 2$.*

Theorem 2.2. *Let $A \in \mathcal{C}(X,Y)$ and $S \in \mathcal{L}(X,Y)$. Then*

(i) *If $F \in \mathcal{F}^b_r(X,Y)$, then $\sigma_{er,S}(F + A) = \sigma_{er,S}(A)$.*

(ii) *If $F \in \mathcal{F}^b_l(X,Y)$, then $\sigma_{el,S}(F + A) = \sigma_{el,S}(A)$.*

Proof. (i) Let $\lambda \in \mathbb{C}$ such that $\lambda S - A \in \Phi_r(X,Y)$. Since $F \in \mathcal{F}^b_r(X,Y)$, then $\lambda S - A \in \Phi^b_r(X,Y)$ if and only if $\lambda S - A - F \in \Phi^b_r(X,Y)$. Hence $\sigma_{er,S}(A) = \sigma_{er,S}(A + F)$.

Arguing as above we derive the item (ii). $\square$

To close this section, we state a straight forward, but useful result to provide the stability of the S-right and S-left spectra.

Theorem 2.3. *Let X be a Banach space, T_1, T_2 two closed densely defined linear operators on X and S an invertible operator on X.*

(i) *If for some $\lambda_0 \in \rho_S(T_1) \cap \rho_S(T_2)$, the operator*

$$(\lambda_0 S - T_1)^{-1} - (\lambda_0 S - T_2)^{-1} \in \mathcal{F}_r^b(X),$$

then

$$\sigma_{er,S}(T_1) = \sigma_{er,S}(T_2).$$

(ii) *If for some $\lambda_0 \in \rho_S(T_1) \cap \rho_S(T_2)$, the operator*

$$(\lambda_0 S - T_1)^{-1} - (\lambda_0 S - T_2)^{-1} \in \mathcal{F}_l^b(X),$$

then

$$\sigma_{el,S}(T_1) = \sigma_{el,S}(T_2).$$

Proof. Let $\lambda \in \mathbb{C} \setminus \{\lambda_0\}$. The proof of this theorem is based on the following relation

$$T_1 - \lambda S = (\lambda - \lambda_0) S \left[(\lambda - \lambda_0)^{-1} S^{-1} - (T_1 - \lambda_0 S)^{-1}\right] (T_1 - \lambda_0 S).$$

Since $T_1 - \lambda_0 S$ is one to one and onto, then

$$\begin{aligned} \alpha(T_1 - \lambda S) &= \alpha \left[(\lambda - \lambda_0)^{-1} S^{-1} - (T_1 - \lambda_0 S)^{-1}\right], \\ \mathcal{R}(T_1 - \lambda S) &= \mathcal{R} \left[(\lambda - \lambda_0)^{-1} S^{-1} - (T_1 - \lambda_0 S)^{-1}\right] \text{ and} \\ \beta(T_1 - \lambda S) &= \beta \left[(\lambda - \lambda_0)^{-1} S^{-1} - (T_1 - \lambda_0 S)^{-1}\right]. \end{aligned}$$

This shows that $\lambda \in \Phi_{T_1,S,r}$ (resp. $\lambda \in \Phi_{T_1,S,l}$) if and only if $(\lambda - \lambda_0)^{-1} \in \Phi_{(T_1 - \lambda_0 S)^{-1}, S^{-1}, r}$ (resp. $(\lambda - \lambda_0)^{-1} \in \Phi_{(T_1 - \lambda_0 S)^{-1}, S^{-1}, l}$).

Combining Theorem 2.2 and the fact that $(\lambda_0 S - T_1)^{-1} - (\lambda_0 S - T_2)^{-1} \in \mathcal{F}_r^b(X)$ (resp. $(\lambda_0 S - T_1)^{-1} - (\lambda_0 S - T_2)^{-1} \in \mathcal{F}_l^b(X)$), we get

$$\begin{aligned} \lambda \in \sigma_{er,S}(T_1) \Longleftrightarrow & (\lambda - \lambda_0)^{-1} \in \sigma_{er,S^{-1}}((T_1 - \lambda_0 S)^{-1}) \\ & = \sigma_{er,S^{-1}}((T_2 - \lambda_0 S)^{-1}) \\ \Longleftrightarrow & \lambda \in \sigma_{er,S}(T_2) \end{aligned}$$

$$\begin{aligned} (\text{resp. } \lambda \in \sigma_{el,S}(T_1) \Longleftrightarrow & (\lambda - \lambda_0)^{-1} \in \sigma_{el,S^{-1}}((T_1 - \lambda_0 S)^{-1}) \\ & = \sigma_{el,S^{-1}}((T_2 - \lambda_0 S)^{-1}) \\ \Longleftrightarrow & \lambda \in \sigma_{el,S}(T_2)). \end{aligned}$$

This achieves the proof of theorem. □

3. Fredholm perturbations of block operators matrices and stability of their M-essential spectra

Let X, Y and Z be three Banach spaces and Γ_X (resp. Γ_Y) be the linear operator from X (resp. Y) into Z. In the Banach space $X \times Y$, we consider the linear operator $L_0 - \lambda M$ given by the block operator matrix

$$L_0 - \lambda M := \begin{pmatrix} A & B \\ C & D \end{pmatrix} - \lambda \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},$$

where M is a bounded operator, A (resp. D) is a densely defined closable (resp. closed) linear operator in X (resp. Y) and B (resp. C) is a linear operator acts from Y (resp. X) into X (resp. Y). The domain of $L_0 - \lambda M$ is given by:

$$\mathcal{D}(L_0 - \lambda M)$$

$$:= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in (\mathcal{D}(A) \cap \mathcal{D}(C)) \times (\mathcal{D}(B) \cap \mathcal{D}(D)) \text{ such that } \Gamma_X x = \Gamma_Y y \right\}.$$

The main purpose of this section is to discuss the M-essential spectra of the closure of the matrix operator L_0 denoted by L and defined on the product of Banach spaces $X \times Y$. First of all, we shall make some hypotheses.

Let X, Y and Z three Banach spaces and assume that:

($H1$) A is a closable, densely defined linear operator.
It follows from this hypothesis that, $\mathcal{D}(A)$, equipped with the graph norm $\|x\|_A = \|x\| + \|Ax\|$ can be completed to a Banach space X_A which coincides with $\mathcal{D}(\overline{A})$ the domain of the closure of A.

($H2$) $\mathcal{D}(A) \subset \mathcal{D}(\Gamma_X) \subset X_A$ and Γ_X is bounded as a mapping from X_A into Z.

($H3$) The set $\mathcal{D}(A) \cap \mathcal{N}(\Gamma_X)$ is dense in X with $\rho_{M_1}(A_1) \neq \emptyset$ for $A_1 := A\,|_{\mathcal{D}(A) \cap \mathcal{N}(\Gamma_X)}$.

Remark 3.1. From ($H1$)-($H3$), one can easily check that $\Gamma_X(\mathcal{D}(A_1)) = \{0\}$ and that the operator A_1 is closed.

Now, let us recall the following lemma:

Lemma 3.1 ([25, Lemma 3.1])**.** *Assume that the hypotheses* ($H1$)-($H3$) *are satisfied. Then, for any* $\lambda \in \rho_{M_1}(A_1)$, *the following assertions hold*:

(i) $\mathcal{D}(A) := \mathcal{D}(A_1) \oplus \mathcal{N}(A_{\lambda,M_1})$, *where the operator* A_{λ,M_1} *is defined on* $\mathcal{D}(A)$ *by* $A_{\lambda,M_1} := (A - \lambda M_1)$.

(ii) *The restriction* $\Gamma_\lambda := \Gamma_X|_{\mathcal{N}(A_{\lambda,M_1})}$ *is injective.*

(iii) $\mathcal{R}(\Gamma_\lambda) = \Gamma_X(\mathcal{N}(A_{\lambda,M_1})) = \Gamma_X(\mathcal{D}(A))$ *does not depend on* λ.

As a direct consequence of the last lemma, we let, for $\lambda \in \rho_{M_1}(A_1)$, the following operator K_λ defined by:

$$K_\lambda := (\Gamma_\lambda)^{-1} := (\Gamma_X|_{\mathcal{N}(A_{\lambda,M_1})})^{-1} : \Gamma_X(\mathcal{D}(A)) \longrightarrow \mathcal{N}(A_{\lambda,M_1}).$$

In other words, $K_\lambda z = x$ means that $x \in \mathcal{D}(A), A_{\lambda,M_1} x = 0$ and $\Gamma_X x = z$.

Lemma 3.2 ([25, Lemma 3.2])**.** *For every* $\lambda, \mu \in \rho_{M_1}(A_1)$ *and under the assumptions* ($H1$)-($H3$), *we have*:

$$K_\lambda - K_\mu = (\lambda - \mu)(A_1 - \lambda M_1)^{-1} M_1 K_\mu. \tag{3}$$

If K_λ *is closable for some* $\lambda \in \rho_{M_1}(A_1)$, *then it is closable for all* λ, *with closure satisfying*

$$\overline{K}_\lambda - \overline{K}_\mu = (\lambda - \mu)(A_1 - \lambda M_1)^{-1} M_1 \overline{K}_\mu.$$

Now, we suppose that:

($H4$) $\mathcal{D}(A) \subset \mathcal{D}(C) \subset X_A$ and $C(A_1 - \lambda M_1)^{-1}$ is a bounded operator from X_A into Y.

Remark 3.2. (i) Combining the closed graph theorem with the above assumption, we infer that, for $\lambda \in \rho_{M_1}(A_1)$, the operator $F(\lambda) := (C - \lambda M_3)(A_1 - \lambda M_1)^{-1}$ is bounded from X into Y.

(ii) If the assumptions ($H1$)-($H3$) are satisfied, then for $\lambda \in \rho_{M_1}(A_1)$ and $x \in \mathcal{D}(A)$, we have

$$(A - \lambda M_1)x = (A_1 - \lambda M_1)(I - K_\lambda \Gamma_X)x.$$

In addition, we will assume that:

($H5$) For some (hence for all) $\lambda \in \rho_{M_1}(A_1)$, K_λ is a bounded operator from $\Gamma_X(\mathcal{D}(A))$ into X, its extension by continuity to $\overline{\Gamma_X(\mathcal{D}(A))}$ is denoted by $\overline{K}_\lambda$.

($H6$) $D \in \mathcal{C}(Y)$ with $\rho_{M_4}(D) \neq \emptyset$.

($H7$) $\mathcal{D}(B) \cap \mathcal{D}(D) \subset \mathcal{D}(\Gamma_Y)$, the set

$$Y_1 := \{y \in \mathcal{D}(B) \cap \mathcal{D}(D) \text{ such that } \Gamma_Y y \in \Gamma_X(\mathcal{D}(A))\}$$

is dense in Y. We denote by $\overline{\Gamma}_Y^0$ the continuous extension of $\Gamma_Y \mid_{Y_1}$ on the all space Y.

($H8$) The operator B is densely defined and for some (hence for all) $\lambda \in \rho_{M_1}(A_1)$, the operator $(A_1 - \lambda M_1)^{-1}B$ is bounded on its domain.

For $\lambda \in \rho_{M_1}(A_1)$, the operator $S_\lambda := D + (C - \lambda M_3)[K_\lambda \Gamma_Y - (A_1 - \lambda M_1)^{-1}(B - \lambda M_2)]$ is defined on the set Y_1, which is dense in Y according to ($H7$).

We also introduce the following assumptions:

($H9$) For some (hence for all) $\lambda \in \rho_{M_1}(A_1)$, the operator $C[-K_\lambda \Gamma_Y + (A_1 - \lambda M_1)^{-1}B]$ is bounded on Y_2 where

$$Y_2 := \{y \in \mathcal{D}(B) \cap \mathcal{D}(\Gamma_Y) \text{ such that } \Gamma_Y y \in \Gamma_X(\mathcal{D}(A))\}$$

is dense in Y such that the restriction of Γ_Y to this set is bounded as an operator from Y into Z.

($H10$) The set Y_1 is a core of D.

Having formulate the above assumptions, the following theorem holds:

Theorem 3.1. *Under the assumptions* ($H1$)-($H10$)*, the operator* L_0 *is closable with closure*

(4)
$$L = \lambda M + \begin{pmatrix} I & 0 \\ F(\lambda) & I \end{pmatrix} \begin{pmatrix} A_1 - \lambda M_1 & 0 \\ 0 & D + \overline{R}_\lambda - \lambda M_4 \end{pmatrix} \begin{pmatrix} I & G(\lambda) \\ 0 & I \end{pmatrix},$$

where

$$G(\lambda) = -\overline{K}_\lambda \overline{\Gamma}_Y^0 + \overline{(A_1 - \lambda M_1)^{-1}(B - \lambda M_2)},$$

and

$$R_\lambda = -(C - \lambda M_3)[-K_\lambda \Gamma_Y + (A_1 - \lambda M_1)^{-1}(B - \lambda M_2)].$$

Proof. Taking account of assumptions $(H3)$ and $(H6)$, we can easily check that A_1 and D are closed operators.

Obviously, combining the fact that $C[-K_\lambda\Gamma_Y+(A_1-\lambda M_1)^{-1}B]$ is bounded and densely defined on Y_2 with the boundedness of the operators M_2 and M_3, we deduce that, for $\lambda \in \rho_{M_1}(A_1)$, the operator

$$\overline{(C-\lambda M_3)[-K_\lambda\Gamma_Y+(A_1-\lambda M_1)^{-1}(B-\lambda M_2)]}$$

is bounded, everywhere defined and hence it is bounded on the dense set Y_2.

This together with the fact that Y_1 is a core of D, we conclude that S_λ is closable for every $\lambda \in \rho_{M_1}(A_1)$ with closure

$$\begin{aligned}\overline{S}_\lambda &= D-\overline{(C-\lambda M_3)[-K_\lambda\Gamma_Y+(A_1-\lambda M_1)^{-1}(B-\lambda M_2)]}\\ &= D+\overline{R}_\lambda.\end{aligned} \tag{5}$$

Now, applying [25, Theorem 3.1], we deduce that the operator L_0 is closable and its closure $L := \overline{L}_0$ is given by (4). □

Remark 3.3. Under the assumptions $(H1)$-$(H10)$, Eq. (5) allows us to write, for $\lambda \in \rho_{M_1}(A_1)\cap\rho_{M_4}(D)\cap\rho_{M_4}(\overline{S}_\lambda)$, the M_4-resolvent of the operator $\overline{S}_\lambda$ as:

$$\begin{aligned}(\overline{S}_\lambda-\lambda M_4)^{-1} &= (D-\lambda M_4)^{-1}+(\overline{S}_\lambda-\lambda M_4)^{-1}-(D-\lambda M_4)^{-1}\\ &= (D-\lambda M_4)^{-1}+(\overline{S}_\lambda-\lambda M_4)^{-1}[D-\overline{S}_\lambda](D-\lambda M_4)^{-1}\\ &= (D-\lambda M_4)^{-1}-(\overline{S}_\lambda-\lambda M_4)^{-1}\overline{R}_\lambda(D-\lambda M_4)^{-1}\end{aligned}$$

or

$$\begin{aligned}(\overline{S}_\lambda-\lambda M_4)^{-1} &= (D-\lambda M_4)^{-1}-(D-\lambda M_4)^{-1}(\overline{S}_\lambda-D)(\overline{S}_\lambda-\lambda M_4)^{-1}\\ &= (D-\lambda M_4)^{-1}-(D-\lambda M_4)^{-1}\overline{R}_\lambda(\overline{S}_\lambda-\lambda M_4)^{-1}.\end{aligned}$$

Now, we are in the position to express the main result of this section based on Fredholm perturbations theory to describe the M-essential spectra of a block operator matrix L. To do this, we introduce for an arbitrary fixed $\lambda_0 \in \rho_{M_1}(A_1)$, the following block diagonal matrix L_{λ_0}:

$$L_{\lambda_0} := \begin{pmatrix} A_1 & 0 \\ 0 & D+\overline{R}_{\lambda_0}\end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & \overline{S}_{\lambda_0}\end{pmatrix}.$$

Theorem 3.2. *Assume that the assumptions* $(H1)$-$(H10)$ *are fulfilled. Then, if for some (hence for all)* $\lambda_0 \in \rho_{M_1}(A_1)\cap\rho_{M_4}(D)$, *we have*:

(i) M_2, M_3, $(D-\lambda_0 M_4)^{-1}C(A_1-\lambda_0 M_1)^{-1}$ *and* $[-\overline{K}_{\lambda_0}\overline{\Gamma}_Y^0+\overline{(A_1-\lambda_0 M_1)^{-1}B}](D-\lambda_0 M_4)^{-1}$ *are Fredholm perturbations, then, for* $\lambda \in \rho_M(L)\cap\rho_M(L_{\lambda_0})$,

$$(L-\lambda M)^{-1}-(L_{\lambda_0}-\lambda M)^{-1} \in \mathcal{F}^b(X\times Y),$$

in particular,

$$\begin{aligned}\sigma_{e4,M}(L) &= \sigma_{e4,M_1}(A_1)\cup\sigma_{e4,M_4}(D+\overline{R}_{\lambda_0}),\\ \sigma_{e5,M}(L) &\subseteq \sigma_{e5,M_1}(A_1)\cup\sigma_{e5,M_4}(D+\overline{R}_{\lambda_0}).\end{aligned}$$

If $C\sigma_{e4,M_1}(A_1)$ is connected, then

$$\sigma_{e5,M}(L) = \sigma_{e5,M_1}(A_1) \cup \sigma_{e5,M_4}(D + \overline{R}_{\lambda_0}).$$

Moreover, if $C\sigma_{e5,M}(L)$ and $C\sigma_{e5,M_4}(D + \overline{R}_{\lambda_0})$ are connected, then

$$\sigma_{e6,M}(L) = \sigma_{e6,M_1}(A_1) \cup \sigma_{e6,M_4}(D + \overline{R}_{\lambda_0}).$$

(ii) *$M_2, M_3, (D - \lambda_0 M_4)^{-1}C(A_1 - \lambda_0 M_1)^{-1}$ and $[-\overline{K}_{\lambda_0}\overline{\Gamma}_Y^0 + \overline{(A_1 - \lambda_0 M_1)^{-1}B}](D - \lambda_0 M_4)^{-1}$ are right-Fredholm perturbations, then, for $\lambda \in \rho_M(L) \cap \rho_M(L_{\lambda_0})$,*

$$(L - \lambda M)^{-1} - (L_{\lambda_0} - \lambda M)^{-1} \in \mathcal{F}_r^b(X \times Y),$$

in particular,

$$\sigma_{er,M}(L) = \sigma_{er,M_1}(A_1) \cup \sigma_{er,M_4}(D + \overline{R}_{\lambda_0}).$$

(iii) *$M_2, M_3, (D - \lambda_0 M_4)^{-1}C(A_1 - \lambda_0 M_1)^{-1}$ and $[-\overline{K}_{\lambda_0}\overline{\Gamma}_Y^0 + \overline{(A_1 - \lambda_0 M_1)^{-1}B}](D - \lambda_0 M_4)^{-1}$ are left-Fredholm perturbations, then, for $\lambda \in \rho_M(L) \cap \rho_M(L_{\lambda_0})$,*

$$(L - \lambda M)^{-1} - (L_{\lambda_0} - \lambda M)^{-1} \in \mathcal{F}_l^b(X \times Y),$$

in particular,

$$\sigma_{el,M}(L) = \sigma_{el,M_1}(A_1) \cup \sigma_{el,M_4}(D + \overline{R}_{\lambda_0}).$$

Proof. Let $\lambda_0 \in \rho_{M_1}(A_1)$ and $\lambda \in \mathbb{C}$ such that $\lambda \in \rho_M(L) \cap \rho_M(L_{\lambda_0})$. According to Eq. (4) and Remark 3.3, the representation of $(L - \lambda M)^{-1} - (L_{\lambda_0} - \lambda M)^{-1}$ can be written as:

$$(6) \quad (L - \lambda M)^{-1} - (L_{\lambda_0} - \lambda M)^{-1}$$
$$= \begin{pmatrix} \begin{matrix} G(\lambda)(D - \lambda M_4)^{-1}F(\lambda) \\ -G(\lambda)(\overline{S}_\lambda - \lambda M_4)^{-1}\overline{R}_\lambda(D - \lambda M_4)^{-1}F(\lambda) \end{matrix} & \begin{matrix} -G(\lambda)(D - \lambda M_4)^{-1} \\ +G(\lambda)(D - \lambda M_4)^{-1}\overline{R}_\lambda(\overline{S}_\lambda - \lambda M_4)^{-1} \end{matrix} \\ & \\ \begin{matrix} -(D - \lambda M_4)^{-1}F(\lambda) \\ -(\overline{S}_\lambda - \lambda M_4)^{-1}\overline{R}_\lambda(D - \lambda M_4)^{-1}F(\lambda) \end{matrix} & (\overline{S}_\lambda - \lambda M_4)^{-1} - (\overline{S}_{\lambda_0} - \lambda M_4)^{-1} \end{pmatrix}.$$

Based on Theorems 2.1 and 2.3, we will prove the Fredholmness perturbation of $(L - \lambda M)^{-1} - (L_{\lambda_0} - \lambda M)^{-1}$, hence it remains to show that all entries of this block operator matrix are Fredholm perturbations.

(i) For details the proof of this assertion, we infer from the assumptions that:

$$\begin{aligned}(D - \lambda M_4)^{-1}F(\lambda) = {} & (D - \lambda M_4)^{-1}C(A_1 - \lambda M_1)^{-1} \\ & - \lambda(D - \lambda M_4)^{-1}M_3(A_1 - \lambda M_1)^{-1}\end{aligned}$$

and

$$\begin{aligned} & G(\lambda)(D - \lambda M_4)^{-1} \\ = {} & [-\overline{K}_\lambda\overline{\Gamma}_Y^0 + \overline{(A_1 - \lambda M_1)^{-1}(B - \lambda M_2)}](D - \lambda M_4)^{-1} \\ = {} & [-\overline{K}_\lambda\overline{\Gamma}_Y^0 + \overline{(A_1 - \lambda M_1)^{-1}B} - \lambda(A_1 - \lambda M_1)^{-1}M_2](D - \lambda M_4)^{-1}\end{aligned}$$

$$\begin{aligned}&= [-\overline{K}_\lambda \overline{\Gamma}_Y^0 + \overline{(A_1 - \lambda M_1)^{-1}B}](D - \lambda M_4)^{-1} \\ &\quad - \lambda(A_1 - \lambda M_1)^{-1} M_2 (D - \lambda M_4)^{-1}\end{aligned}$$

are Fredholm perturbations. According the boundedness property of the operators $G(\lambda), (\overline{S}_\lambda - \lambda M_4)^{-1}$ and $\overline{R}_\lambda$ with Proposition 2 in [9], we get the Fredholmness perturbations of each operators $G(\lambda)(D - \lambda M_4)^{-1}F(\lambda)$, $G(\lambda)(\overline{S}_\lambda - \lambda M_4)^{-1}\overline{R}_\lambda(D - \lambda M_4)^{-1}F(\lambda)$, $G(\lambda)(D - \lambda M_4)^{-1}\overline{R}_\lambda(\overline{S}_\lambda - \lambda M_4)^{-1}$ and $(\overline{S}_\lambda - \lambda M_4)^{-1}\overline{R}_\lambda(D - \lambda M_4)^{-1}F(\lambda)$.

Analogously, one can see that the following right lower corner

$$\begin{aligned}
&(\overline{S}_\lambda - \lambda M_4)^{-1} - (\overline{S}_{\lambda_0} - \lambda M_4)^{-1} \\
&= (\overline{S}_\lambda - \lambda M_4)^{-1}(\overline{S}_{\lambda_0} - \overline{S}_\lambda)(\overline{S}_{\lambda_0} - \lambda M_4)^{-1} \\
&= (\overline{S}_\lambda - \lambda M_4)^{-1}(\overline{R}_{\lambda_0} - \overline{R}_\lambda)(\overline{S}_{\lambda_0} - \lambda M_4)^{-1} \\
&= (\lambda - \lambda_0)(\overline{S}_\lambda - \lambda M_4)^{-1}F(\lambda)M_1G(\lambda_0)(\overline{S}_{\lambda_0} - \lambda M_4)^{-1} \\
&\quad - (\lambda - \lambda_0)(\overline{S}_\lambda - \lambda M_4)^{-1}M_3G(\lambda_0)(\overline{S}_{\lambda_0} - \lambda M_4)^{-1} \\
&\quad + (\lambda_0 - \lambda)(\overline{S}_\lambda - \lambda M_4)^{-1}F(\lambda)M_2(\overline{S}_{\lambda_0} - \lambda M_4)^{-1} \\
&= (\lambda - \lambda_0)(D - \lambda M_4)^{-1}F(\lambda)M_1G(\lambda_0)(\overline{S}_{\lambda_0} - \lambda M_4)^{-1} \\
&\quad - (\lambda - \lambda_0)(\overline{S}_\lambda - \lambda M_4)^{-1}\overline{R}_\lambda(D - \lambda M_4)^{-1}F(\lambda)M_1G(\lambda_0)(\overline{S}_{\lambda_0} - \lambda M_4)^{-1} \\
&\quad - (\lambda - \lambda_0)(\overline{S}_\lambda - \lambda M_4)^{-1}M_3G(\lambda_0)(D - \lambda_0 M_4)^{-1} \\
&\quad + (\lambda - \lambda_0)(\overline{S}_\lambda - \lambda M_4)^{-1}M_3G(\lambda_0)(D - \lambda_0 M_4)^{-1}\overline{R}_{\lambda_0}(\overline{S}_{\lambda_0} - \lambda M_4)^{-1} \\
&\quad + (\lambda_0 - \lambda)(D - \lambda M_4)^{-1}F(\lambda)M_2(\overline{S}_{\lambda_0} - \lambda M_4)^{-1} \\
&\quad - (\lambda_0 - \lambda)(\overline{S}_\lambda - \lambda M_4)^{-1}\overline{R}_\lambda(D - \lambda M_4)^{-1}F(\lambda)M_2(\overline{S}_{\lambda_0} - \lambda M_4)^{-1}
\end{aligned}$$

is a Fredholm perturbation since it is the product of bounded operators and the Fredholm perturbation operators $(D - \lambda M_4)^{-1}F(\lambda)$ and $G(\lambda_0)(D - \lambda_0 M_4)^{-1}$. Hence, according to [12, Theorem 2.2], we get

$$\sigma_{e4,M}(L) = \sigma_{e4,M}(L_{\lambda_0}) = \sigma_{e4,M_1}(A_1) \cup \sigma_{e4,M_4}(D + \overline{R}_{\lambda_0}),$$

with

$$i(L - \lambda M) = i(A_1 - \lambda M_1) + i(D + \overline{R}_{\lambda_0} - \lambda M_4) = 0.$$

Hence, from these two equalities, we have

$$\sigma_{e5,M}(L) \subseteq \sigma_{e5,M_1}(A_1) \cup \sigma_{e5,M_4}(D + \overline{R}_{\lambda_0}),$$

According to [12, Lemma 2.1], we get

$$\sigma_{e5,M}(L) = \sigma_{e5,M}(L_{\lambda_0}) = \sigma_{e5,M_1}(A_1) \cup \sigma_{e5,M_4}(D + \overline{R}_{\lambda_0}),$$

and

$$\sigma_{e6,M}(L) = \sigma_{e6,M}(L_{\lambda_0}) = \sigma_{e6,M_1}(A_1) \cup \sigma_{e6,M_4}(D + \overline{R}_{\lambda_0}).$$

The use of Theorems 2.1, 2.2 and 2.3 allows us to reach the results of assertions (ii) and (iii) in a similar ways as in (i). □

Remark 3.4. It is noted that, in the paper [14] the authors supposed that the operators $-\overline{K}_\lambda\overline{\Gamma}^0_Y + \overline{(A_1-\lambda)^{-1}B}$ and $C(A_1-\lambda)^{-1}$ are Fredholm perturbations. But in our case, we consider a weaker condition and we suppose only that $[-\overline{K}_\lambda\overline{\Gamma}^0_Y + \overline{(A_1-\lambda M_1)^{-1}(B-\lambda M_2)}](D-\lambda M_4)^{-1}$ and $(D-\lambda M_4)^{-1}C(A_1-\lambda M_1)^{-1}$ are Fredholm perturbations in order to investigate the M-essential spectra of the operator L in term of its Schur-complement whose M_4-essential spectrum is easier to calculate. So, Theorem 3.2 may be regarded as an extension of [14, Theorem 3.3] to a larger class of operators.

The notion of Fredholm perturbations theory plays a crucial role in spectral theory. This notion is tested for two-group transport equations and is applicable to propose an abstract framework for the computation of the M-essential spectra of a one-dimensional problem of transport operator.

4. Application to two-group transport equations

In this section, we will apply our main results to study the M-essential spectra of a problem of transport equations acting in the space

$$X \times X := L_1([-a,a]\times[-1,1];dxdv) \times L_1([-a,a]\times[-1,1];dxdv),\ \ a>0,$$

and given by the following matrix of two-group transport operators:

$$L-\lambda M := \begin{pmatrix} T_1-\lambda M_1 & K_{12}-\lambda M_2 \\ K_{21}-\lambda M_3 & T_2^H + K_{22}-\lambda M_4 \end{pmatrix}.$$

The operator T_1 is the closed linear operator defined by:

$$\begin{cases} T_1 : \mathcal{D}(T_1) \subseteq X \longrightarrow X \\ \qquad\qquad \psi \longrightarrow T_1\psi(x,v) = -v\dfrac{\partial\psi}{\partial x}(x,v) - \sigma_1(v)\psi(x,v) \\ \mathcal{D}(T_1) := \mathcal{W} := \{\psi \in X : v\dfrac{d\psi}{dx} \in X\} \end{cases}$$

and T_2^H is the steaming operator:

$$\begin{cases} T_2^H : \mathcal{D}(T_2^H) \subseteq X \longrightarrow X \\ \qquad\qquad \psi \longrightarrow T_2^H\psi(x,v) = -v\dfrac{\partial\psi}{\partial x}(x,v) - \sigma_2(v)\psi(x,v) \\ \mathcal{D}(T_2^H) = \big\{\psi \in \mathcal{W} : \psi^i = H\psi^o\big\}. \end{cases}$$

The collision frequency $\sigma_j(\cdot) \in \mathcal{L}^\infty(-1,1)$, ψ^o and ψ^i represent respectively the outgoing and the incoming fluxes related by the boundary operator H. ψ^o and ψ^i belong respectively to the spaces

$$X^o := L_1(\{-a\}\times[-1,0],|v|dv) \times L_1(\{a\}\times[0,1],|v|dv) = X_1^o \times X_2^o$$

and

$$X^i := L_1(\{-a\}\times[0,1],|v|dv) \times L_1(\{a\}\times[-1,0],|v|dv) = X_1^i \times X_2^i$$

(see [6] for more details). The bounded operators $K_{ij}, (i,j) \in \{(1,2),(2,1),(2,2)\}$ are defined on X by:

$$(7) \qquad \begin{cases} K_{ij} : X \longrightarrow X \\ \qquad \psi \longrightarrow K_{ij}\,\psi(x,v) = \displaystyle\int_{-1}^{1} \kappa_{ij}(x,v,v')\,\psi(x,v')\,dv', \end{cases}$$

with kernels κ_{ij} assumed to be measurable and the coefficients M_i are defined by:

$$\begin{cases} M_i : X \longrightarrow X \\ \qquad \psi \longrightarrow M_i\psi(x,v) = \eta_i(v)\psi(x,v), \qquad i = 1,4 \end{cases}$$

where $\eta_i(\cdot) \in \mathcal{L}^\infty(-1,1)$ and M_2, M_3 are bounded operators on X.

We define

$$\lambda_j^* := \inf_{v\in(-1,1)} \sigma_j(v),\ j = 1,2$$

and

$$\mu_j^* := \inf_{v\in(-1,1)} \eta_j(v),\ j = 1,4$$

and we assume that $\mu_j^* > 0,\ j = 1,4.$

To verify the hypotheses of Theorem 3.2, we shall define the operator $L-\lambda M$ on the domain:

$$D(L-\lambda M) := \left\{ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathcal{W}\times\mathcal{D}(T_2^H) : \psi_1^i = \psi_2^i \right\}$$

and we introduce the boundary operators Γ_X and Γ_Y as follows:

$$\begin{cases} \Gamma_X : \mathcal{W} \longrightarrow X^i \\ \qquad \psi_1 \longmapsto \psi_1^i, \end{cases} \quad \text{and} \quad \begin{cases} \Gamma_Y : \mathcal{W} \longrightarrow X^i \\ \qquad \psi_2 \longmapsto \psi_2^i = H\psi_2^o. \end{cases}$$

Let A_1 be the closed, densely defined linear operator with a non empty M_1-resolvent set defined as:

$$\begin{cases} A_1 := T_1, \\ \mathcal{D}(A_1) = \{\psi_1 \in \mathcal{D}(T_1) : \psi_1^i = 0\}. \end{cases}$$

In order to verify assumption $(H5)$, we will determine the solution of the equation:

$$(T_1 - \lambda M_1)\psi_1 = 0 \quad \text{for } \psi_1 \in \mathcal{W}.$$

A short computation shows that the operator K_λ is bounded by $(\mu_1^*\mathrm{Re}\lambda)^{-1}$ and is defined on X^i by:

$$\begin{cases} K_\lambda : X^i \longrightarrow X, K_\lambda u := \chi_{(0,1)}(v)K_\lambda^+ u + \chi_{(-1,0)}(v)K_\lambda^- u \quad \text{with} \\ (K_\lambda^- u)(x,v) := u(a,v)\,e^{-\frac{(\sigma_1(v)+\mu_1\lambda)|a-x|}{|v|}}, \quad v\in(-1,0) \\ (K_\lambda^+ u)(x,v) := u(-a,v)e^{-\frac{(\sigma_1(v)+\mu_1\lambda)|a+x|}{|v|}}, \quad v \in (0,1). \end{cases}$$

Consider the Schur-complement of the matrix $L-\lambda M$, which is formally given by the following expression:

$$S_\lambda := T_2^H + K_{22} - (K_{21} - \lambda M_3)[-K_\lambda\Gamma_Y + (T_1 - \lambda M_1)^{-1}(K_{12} - \lambda M_2)]$$

for $\lambda \in \rho_{M_1}(T_1)$.

Remark 4.1. It is easy to see that $\mathcal{D}(S_\lambda)$ is a core for $T_2^H + K_{22}$ since $T_2^H + K_{22}$ is a closed, densely defined operator with a nonempty M_4-resolvent set.

In view of the previous remark, it is not difficult to see that S_λ can be written for $\lambda \in \rho_{M_1}(T_1) \cap \rho_{M_4}(T_{H_2} + K_{22}) \cap \rho_{M_4}(S_\lambda)$, in the two ways:

(8)
$$(S_\lambda - \lambda M_4)^{-1} = (T_{H_2} + K_{22} - \lambda M_4)^{-1} - (S_\lambda - \lambda M_4)^{-1} R_\lambda (T_{H_2} + K_{22} - \lambda M_4)^{-1}$$

or

(9)
$$(S_\lambda - \lambda M_4)^{-1} = (T_{H_2} + K_{22} - \lambda M_4)^{-1} - (T_{H_2} + K_{22} - \lambda M_4)^{-1} R_\lambda (S_\lambda - \lambda M_4)^{-1},$$

where

$$R_\lambda := -(K_{21} - \lambda M_3)[-K_\lambda \Gamma_Y + (T_1 - \lambda M_1)^{-1}(K_{12} - \lambda M_2)].$$

Notice that the defined collision operators K_{12}, K_{21} and K_{22} act only on the velocity v', so x may be seen, simply, as a parameter in $[-a, a]$. Then, we will consider each of these operators as a function

$$K_{ij}(\cdot) : x \in [-a, a] \longrightarrow K(x) \in \mathcal{L}(L_1([-1, 1], dv)).$$

Definition 4.1 ([19])**.** A collision operator K_{ij} in the form (7), is said to be regular if it satisfies the following conditions:

$$\begin{cases} - \text{ the function } K_{ij}(\cdot) \text{ is mesurable,} \\ - \text{ there exists a compact subset } \mathcal{C} \subset \mathcal{L}(L_1([-1,1], dv)) \text{ such that :} \\ \quad K_{ij}(x) \in \mathcal{C} \text{ a.e. on } [-a, a], \\ - \ K_{ij}(x) \in \mathcal{K}(L_1([-1,1], dv)) \text{ a.e. on } [-a, a] \end{cases}$$

where $\mathcal{K}(L_1([-1, 1], dv))$ is the set of compact operators on $L_1([-1, 1], dv)$.

We recall the following lemma established in [12].

Lemma 4.1. *Let* $\lambda \in \rho_{M_1}(T_1)$.

(i) *If* $\frac{\kappa_{21}(x,v,v')}{|v'|}$ *defines a regular operator, then the operator*

$$K_{21}(T_1 - \lambda M_1)^{-1}$$

is a weakly compact operator on X.

(ii) *If* $K_{12}(x, v, v')$ *defines a regular operator, then the operator*

$$(T_1 - \lambda M_1)^{-1} K_{12}$$

is weakly compact on X.

As a consequence for the previous lemma, the following result holds:

Lemma 4.2. *Let* $\lambda \in \rho_{M_1}(T_1)$.

(i) *If* M_3 *is a Fredholm perturbation on* X *with the kernel* $\frac{\kappa_{21}(x,v,v')}{|v'|}$ *defines a regular operator, then* $(T_{H_2} + K_{22} - \lambda M_4)^{-1} F(\lambda)$ *is a Fredholm perturbation on* X.

(ii) *If M_2 is a Fredholm perturbation on X and K_{12} is a regular operator, then the operator* $(T_1-\lambda M_1)^{-1}(K_{12}-\lambda M_2)(T_{H_2}+K_{22}-\lambda M_4)^{-1}$ *is a Fredholm perturbation on X.*

Remark 4.2. It follows from Theorem 3.1 in [22] that $\mathcal{W}(X)=\mathcal{S}(X)$.

If $1<p<\infty$, X_p is reflexive and then $\mathcal{L}(X_p)=\mathcal{W}(X_p)$. On the other hand, it follows from [9, Theorem 5.2] that $\mathcal{K}(X_p)\subsetneq\mathcal{S}(X_p)\subsetneq\mathcal{W}(X_p)\subsetneq\mathcal{F}(X_p)$ with $p\neq 2$. For $p=2$ we have $\mathcal{K}(X_p)=\mathcal{S}(X_p)=\mathcal{W}(X_p)=\mathcal{F}(X_p)$.

Now, let us denote by:

$$L_{\lambda_0}:=\begin{pmatrix} T_1 & 0 \\ 0 & T_{H_2}+K_{22}+R_{\lambda_0}\end{pmatrix}.$$

The Fredholm perturbation theory is an important tool to describe the M-essential spectra and especially the M-essential spectra of an transport operator matrix L. In order to describe these subsets, for $\lambda\in\rho_M(L)$ and $\lambda\in\rho_M(L_{\lambda_0})$, we let:

$$\begin{aligned}&(L-\lambda M)^{-1}-(L_{\lambda_0}-\lambda M)^{-1}\\ &=\begin{pmatrix} G(\lambda)[S_\lambda-\lambda M_4]^{-1}F(\lambda) & -G(\lambda)[S_\lambda-\lambda M_4]^{-1} \\ & \\ -[S_\lambda-\lambda M_4]^{-1}F(\lambda) & [S_\lambda-\lambda M_4]^{-1}-[S_{\lambda_0}-\lambda M_4]^{-1}\end{pmatrix}\end{aligned}$$

where

$$\begin{cases} G(\lambda)=-K_\lambda\Gamma_Y+(T_1-\lambda M_1)^{-1}(K_{12}-\lambda M_2)\\ F(\lambda)=(K_{21}-\lambda M_3)(T_1-\lambda M_1)^{-1}.\end{cases}$$

The M-essential spectra of two-group transport operators can be described in the next theorem under additive Fredholm perturbations.

Theorem 4.1. *If the operators H, M_2, M_3 are Fredholm perturbations, K_{12}, K_{21}, K_{22} are regular operators and if $\frac{\kappa_{21}(x,v,v')}{|v'|}$ is regular, then*

$$(L-\lambda M)^{-1}-(L_{\lambda_0}-\lambda M)^{-1}\in\mathcal{F}^b(X\times X)$$

in particular,

$$\sigma_{ek,M}(L)=\{\lambda\in\mathbb{C}:\mathrm{Re}\lambda\leq-\min(\frac{\lambda_1^*}{\mu_1^*},\frac{\lambda_2^*}{\mu_4^*})\},\quad 4\leq k\leq 6, r, l.$$

Proof. According to Theorem 2.1, to characterize the Fredholm perturbations of the block operator matrix $(L-\lambda M)^{-1}-(L_{\lambda_0}-\lambda M)^{-1}$, it remains to provide the same property for all entries of this block operator matrix. To do this, for $\lambda\in\rho_{M_1}(T_1)\cap\rho_{M_4}(T_2^H+K_{22})\cap\rho_{M_4}(S_\lambda)$, we consider the operator $(S_\lambda-\lambda M_4)^{-1}F(\lambda)$ which can be expressed from Eq. (8) as:

$$\begin{aligned}(S_\lambda-\lambda M_4)^{-1}F(\lambda)&:=(T_2^H+K_{22}-\lambda M_4)^{-1}F(\lambda)\\ &\quad-(S_\lambda-\lambda M_4)^{-1}R_\lambda(T_2^H+K_{22}-\lambda M_4)^{-1}F(\lambda).\end{aligned}\tag{10}$$

The use of Lemma 4.2 and Proposition 2 in [9] implies that

$$(S_\lambda-\lambda M_4)^{-1}R_\lambda(T_2^H+K_{22}-\lambda M_4)^{-1}F(\lambda)$$

is a Fredholm perturbation on X. Now, the fact that $\mathcal{F}^b(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ allows us to deduce from Eq. (10) that $(S_\lambda - \lambda M_4)^{-1}F(\lambda)$ is also a Fredholm perturbation.

Since the operator H is a Fredholm perturbation on X, then Γ_Y has also this property. This together with Lemma 4.2-(ii), Proposition 2 in [9] and Eq. (9), make us conclude that

$$\begin{aligned} G(\lambda)(S_\lambda - \lambda M_4)^{-1} := {} & G(\lambda)(T_2^H + K_{22} - \lambda M_4)^{-1} \\ & - G(\lambda)(T_2^H + K_{22} - \lambda M_4)^{-1} R_\lambda (S_\lambda - \lambda M_4)^{-1} \end{aligned} \tag{11}$$

is a Fredholm perturbation on X, for $\lambda \in \rho_{M_1}(T_1) \cap \rho_{M_4}(T_2^H + K_{22}) \cap \rho_{M_4}(S_\lambda)$. In what follows, it easy to show from Eqs. (10) and (11) with Proposition 2 in [9], that the operators $G(\lambda)(S_\lambda - \lambda M_4)^{-1}F(\lambda)$ and $(S_\lambda - \lambda M_4)^{-1} - (S_{\lambda_0} - \lambda M_4)^{-1}$ are Fredholm perturbations on X.

For all claims cited above and from Theorem 2.1, we get

$$(L - \lambda M)^{-1} - (L_{\lambda_0} - \lambda M)^{-1} \in \mathcal{F}^b(X \times X).$$

Therefore, by combining Remark 4.2, Theorems 2.2 in [12] and 2.3, we have

$$\sigma_{ek,M}(L) = \sigma_{ek,M}(L_{\lambda_0}) = \sigma_{ek,M_1}(T_1) \cup \sigma_{ek,M_4}(S_\lambda), \quad 4 \leq k \leq 6, r, l.$$

If we combine the information about $\sigma_{ei,M_1}(T_1)$ and $\sigma_{ei,M_4}(S_\lambda)$, for $i = 1, \ldots, 6$ (see Section 4 in [25] for more details) with Eq. (2), we obtain the following result for the M_j-essential right and left spectra for $j = 1, 4$ of the operators T_1 and S_λ as:

$$\sigma_{er,M_1}(T_1) = \sigma_{el,M_1}(T_1) = \sigma_{ek,M_1}(T_1) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq -\frac{\lambda_1^*}{\mu_1^*}, \}, \quad 4 \leq k \leq 6.$$

$$\sigma_{er,M_4}(S_\lambda) = \sigma_{el,M_4}(S_\lambda) = \sigma_{ek,M_4}(S_\lambda) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq -\frac{\lambda_2^*}{\mu_4^*}\}, \quad 4 \leq k \leq 6$$

which ends this proof. □

Conclusion: In this paper, we provide some general results on right and left Fredholm perturbations. More specific perturbations results are stated until the paper where they are used to describe the Fredholm, right and left Fredholm perturbations of the difference between the resolvents of two block operator matrices which ensure the stability on their M-essential spectra under weaker conditions than proved in the papers of [4, 14, 25]. All the results are new and are not yet investigate considerably.

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