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2-ENGELIZER SUBGROUP OF A 2-ENGEL TRANSITIVE GROUPS

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Abstract. A general notion of χ-transitive groups was introduced by C. Delizia et al. in [6], where χ is a class of groups. In [5], Ciobanu, Fine and Rosenberger studied the relationship among the notions of conjugately separated abelian, commutative transitive and fully residually χ-groups.

In this article we study the concept of 2-Engel transitive groups and among other results, its relationship with conjugately separated 2-Engel and fully residually χ-groups are established. We also introduce the notion of 2-Engelizer of the element x in G and denote the set of all 2-Engelizers in G by $E^2(G)$. Then we construct the possible values of $|E^2(G)|$.

1. Introduction

An element x of a group G is called a *right Engel* element, if for every $y \in G$, there exists a natural number $n = n(x, y)$ such that $[x, {}_n y] = 1$. If n can be chosen independent of y, then x is called a *right n-Engel* element or simply a *bounded right Engel* element. We denote the sets of all right Engel elements and bounded right Engel elements of G by $R(G)$ and $\overline{R}(G)$, respectively.

An element x of G is called a *left Engel* element, if for every $y \in G$, there exists a natural number $n = n(x, y)$ such that $[y, {}_n x] = 1$. If n can be chosen independent of y, then x is called a *left n-Engel* element or simply a *bounded left Engel* element. We denote the sets of all left Engel elements and bounded left Engel elements of G by $L(G)$ and $\overline{L}(G)$, respectively. For any positive integer n, a group G is called an n-Engel group, if $[x, {}_n y] = [y, {}_n x] = 1$ for all $x, y \in G$.

A proper subset E of a group G is said to be *n-Engel set*, whenever $[x, {}_n y] = [y, {}_n x] = 1$ for all $x, y \in E$.

Let χ be a class of groups. Then a group G is *residually χ* if for every non-trivial element $g \in G$, there is a homomorphism $\phi : G \to H$, where H

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is a χ-group such that $\phi(g) \neq 1$. Also a group G is *fully residually* χ if for finitely many non-trivial elements $g_1, \ldots, g_n$ in G there exists a homomorphism $\phi : G \to H$ where H is a χ-group such that $\phi(g_i) \neq 1$ for all $i = 1, \ldots, n$.

Definition 1.1. A subgroup H of a group G is called *malnormal* or *conjugately separated*, if $H \cap H^x = 1$ for all $x \in G \backslash H$.

It is clear that the intersection of a family of malnormal subgroups of a given group G is again malnormal, which allows us to define the *malnormal closure* of a subgroup H of G. Clearly the intersection of all malnormal subgroups of G contains H is malnormal.

2. 2-Engel transitive groups

A group G is called a *conjugately separated* 2-*Engel* (henceforth CSE^2-group) if all of its maximal 2-Engel subgroups are malnormal. In the following, we discuss the notion of 2-Engel transitive group and then give its relationship with CSE^2-group and fully residually χ-groups.

Definition 2.1. (a) A group G is 2-*Engel transitive* (henceforth 2-ET), when $[x, y, y] = 1$ and $[y, z, z] = 1$ imply that $[x, z, z] = 1$ for every non-trivial elements x, y, z in G.

(b) For a given element x of G, we call

$$E^2_G(x) = \{y \in G : [x, y, y] = 1, \ [y, x, x] = 1\}$$

to be the set of 2-*Engelizer* of x in G. The family of all 2-Engelizers in G is denoted by $E^2(G)$ and $|E^2(G)|$ denotes the number of distinct 2-Engelizers in G.

As an example consider $Q_{16} = \langle a, b : a^8 = 1, \ a^4 = b^4, \ b^{-1}ab = a^{-1} \rangle$, the Quaternion group of order 16 and take the element b in Q_{16}. Then one can easily check that the 2-Engelizer set of b is as follows:

$$E^2_{Q_{16}}(b) = \{1, a^2, a^4, a^6, b, a^2b, a^4b, a^6b\}.$$

The following lemma is useful for our further investigations.

Lemma 2.2. *Let G be a 2-ET group. Then 2-Engelizer of each non-trivial element of G is 2-Engel set.*

Proof. Let G be a 2-Engel transitive group, then $[x, y, y] = 1$ and $[y, z, z] = 1$ imply $[x, z, z] = 1$ for all non-trivial elements x, y, z in G. Clearly using the definition, for $y, z \in E^2_G(x)$, it follows that $[z, y, y] = 1$ and $[y, z, z] = 1$. Thus $E^2_G(x)$ is 2-Engel set. □

We remark that for the identity element e of G, we have $G = E^2_G(e)$ and hence $G \in E^2(G)$. Clearly in general, the 2-Engelizer of each non-trivial element of an arbitrary group G does not form a subgroup. The following example shows our claim.

Example 2.3. Let G be a finitely presented group of the following form:

$$G = \langle a_1, a_2, a_3, a_4 : a_3^3 = a_4^3 = 1, [a_1, a_2] = 1, [a_1, a_3] = a_4,$$
$$[a_1, a_4] = 1, [a_2, a_3] = 1, [a_2, a_4] = a_2, [a_3, a_4] = 1\rangle.$$

Using GAP [7] implies that G is an infinite group. One can easily check that G is not 2-ET, as $[a_2, a_1, a_1] = 1$ and $[a_1, a_4, a_4] = 1$, while $[a_2, a_4, a_4] = a_2$. Moreover, $E^2_G(a_1)$ is not a subgroup of G, since it is easily calculated that $a_2, a_3 \in E^2_G(a_1)$ but $a_2a_3 \notin E^2_G(a_1)$.

Here, we state an interesting property of 2-Engel transitive groups.

Proposition 2.4. *Let G be a 2-ET group. Then $x^{E^2_G(x)}$ is nilpotent of class at most 3, for every non-trivial element x of G.*

Proof. Note that $x^{E^2_G(x)} = \langle x^y : y \in E^2_G(x)\rangle$. Now, for every $y \in E^2_G(x)$;

$$[x^y, x] = [x[x, y], x] = [x, y, x] = 1.$$

On the other hand $[x^y, x, x] = 1$ and $[x, x^z, x^z] = 1$ imply that $[x^y, x^z, x^z] = 1$, as G is 2-ET. Hence $x^{E^2_G(x)}$ is 2-Engel group and so nilpotent of class at most 3. □

Now, we discuss the condition under which the 2-Engelizer of each non-trivial element of G is a subgroup.

Theorem 2.5. *Let G be an arbitrary group. Then the set of each 2-Engelizer of a non-trivial element in G forms a subgroup if and only if the group $x^{E^2_G(x)}$ is abelian for all non-trivial element x of G.*

Proof. Let $y \in E^2_G(x)$. Then one can easily see that

$$\begin{aligned}[y^{-1}, x, x] = [[x, y]^{y^{-1}}, x] &= [[x, y][x, y, y^{-1}], x]\\ &= [[x, y][x, y, y]^{-y^{-1}}, x]\\ &= [x, y, x] = 1,\end{aligned}$$

and also

$$\begin{aligned}[x, y^{-1}, y^{-1}] = [[y, x]^{y^{-1}}, y^{-1}] &= [[y, x][y, x, y^{-1}], y^{-1}]\\ &= [[y, x][y, x, y]^{-y^{-1}}, y^{-1}]\\ &= [y, x, y^{-1}] = [y, x, y]^{-y^{-1}} = 1.\end{aligned}$$

Thus $y^{-1} \in E^2_G(x)$.

Now, for every $y, z \in E^2_G(x)$ we have;

$$\begin{aligned}[yz, x, x] &= [[y, x]^z[z, x], x]\\ &= [[y, x]^z, x]^{[z,x]}[z, x, x]\\ &= [[y, x][y, x, z], x]^{[z,x]}\\ &= [y, x, z, x]^{[z,x]}.\end{aligned}$$

Clearly, using Witt identity and the same technique in the proof of Theorem 7.13 in [8], we may have $[y,x,z,x]=1$ if and only if $x^{E_G^2(x)}$ is abelian. Similarly, $[x,yz,yz]=1$ and the proof is complete. □

The proof of the following lemma is a routine argument by using Zorn's Lemma.

Lemma 2.6. *Every* 2*-Engel subgroup* H *of a given group* G *is contained in a maximal* 2*-Engel subgroup.*

The following fact is needed in proving our main result.

Proposition 2.7. *Let* G *be a* CSE^2*-group. Then every non-trivial* 2*-Engel normal subgroup of* G *is maximal.*

Proof. Let G be a CSE^2-group and K a non-trivial 2-Engel normal subgroup of G. Then by Lemma 2.6, K is contained in a maximal 2-Engel subgroup M of G. Let $1 \neq k \in K$, then for each $x \in G$ we have $k^x \in M$. Since G is CSE^2, it follows that M is malnormal and therefore $x \in M$. Thus $G = M$, which implies that K is maximal. □

Using the above proposition, we obtain the following useful result.

Corollary 2.8. *Let* G *be a* CSE^2*-group. Then every* 2*-Engel normal subgroup of* G *is equal to the second centre of* G*.*

Lemma 2.9. *Let* χ *be a class of groups such that each non-*2*-Engel group* $H \in \chi$ *is* CSE^2*-group. Let* N *be a* 2*-Engel normal subgroup of a non-*2*-Engel residually* χ*-group* G*. Then* N *is contained in the second centre of* G*.*

Proof. Let $G \in \chi$, then by the assumption G is CSE^2 and therefore by Corollary 2.8, every 2-Engel normal subgroup of G is equal to the second centre of G. Now let N be a 2-Engel normal subgroup of a non-2-Engel residually χ-group G so that N is not contained in the second centre of G. Then there exist elements $n \in N$ and $g_1, g_2 \in G$ such that $[n, g_1, g_2] = x \neq 1$, say. Since G is residually χ, there exists a normal subgroup N_x of G such that $G/N_x \in \chi$ and $x \notin N_x$. Clearly NN_x/N_x is a non-trivial 2-Engel normal subgroup of G/N_x. Then $NN_x/N_x = Z_2(G/N_x)$ and this contradicts that $x \notin N_x$. Therefore N is contained in the second centre of G. □

Remark 2.10. Let G be a 2-ET and non 2-Engel group, then it is clear that $Z_2(G) = 1$. So it follows from the above lemma that any normal 2-Engel subgroup of G must be trivial.

Now we study the relationship between the non 2-Engel CSE^2, 2-ET and fully residually χ-groups.

Theorem 2.11. *Let* χ *be a class of groups such that each non* 2*-Engel* χ*-group is* CSE^2 *and* G *be a non* 2*-Engel and residually* χ*-group. Then*

(i) *G is a* CSE^2*.*

(ii) *If G is a* 2*-Engel transitive, then G is fully residually* χ*-group.*

Proof. (i) Let G be a non 2-Engel group. Then there exist $x, y \in G$ such that $[x, y, y] \neq 1$. On the other hand, there is a normal subgroup N of G, for which $[x, y, y] \notin N$ and $G/N \in \chi$, as G is residually χ. Clearly, $x, y \notin N$ and G/N is non 2-Engel. Hence G/N is CSE^2 and so every maximal 2-Engel subgroup in G/N is malnormal. Suppose M/N is a maximal 2-Engel subgroup of G/N. Then $\frac{M}{N} \cap (\frac{M}{N})^{gN} = N$, for all $gN \in \frac{G}{N} \backslash \frac{M}{N}$. This implies that $M \cap M^g = 1$ for every $g \in G \backslash M$, and hence G is CSE^2.

(ii) Let G be a 2-ET, non 2-Engel and residually χ-group. Then we show that G is fully residually χ. In order to do this, we prove that for given non-trivial elements $g_1, \ldots, g_n$ in G there is a normal subgroup N such that $g_1, \ldots, g_n$ are not in N and $G/N \in \chi$. This is equivalent to showing that given non-trivial elements $g_1, \ldots, g_n \in G$ there exists a non-trivial element $g \in G$ such that for any normal subgroup N of G if $g \notin N$, then $g_i \notin N$ for $i = 1, \ldots, n$. We proceed by induction on n. This is true for $n = 1$, as G is residually χ. Now assume the result holds for $n - 1$, if $[g_n^x, g, g] = 1 = [g, g_n^x, g_n^x]$ for any $x \in G$. Then by 2-Engel transitivity, the normal closure g_n^G is 2-Engel and hence by Remark 2.11 it is trivial, but g_n is in g_n^G, which is non-trivial. Therefore either $[g_n^x, g, g] \neq 1$ or $[g, g_n^x, g_n^x] \neq 1$, for some $x \in G$. Then either of the latest commutators is not in some normal subgroup N of G. This follows that $g_1, \ldots, g_n \notin N$, which gives the proof. □

In 1967, B. Baumslag [3] introduced the notion of fully residually free groups and proved that a residually free group is fully residually free if and only if it is commutative transitive. A group G is commutative transitive, if $[x, y] = 1$ and $[y, z] = 1$ implies that $[x, z] = 1$ for nontrivial elements x, y, z in G.

Here we show that Baumslag's theorem is also true in the case of 2-Engel transitive groups.

Theorem 2.12. *Let G be a residually free group. Then G is fully residually free if and only if G is 2-Engel transitive.*

Proof. Let G be a fully residually free group. Assume $[x, y, y] = [y, z, z] = 1$, for every non-trivial elements $x, y, z \in G$. We must show that $[x, z, z] = 1$. If $[x, z, z] \neq 1$, there exists a homomorphism $\phi : G \to F$, where F is a free group and

$$\phi([x, z, z]) = [\phi(x), \phi(z), \phi(z)] \neq 1, \ \phi(x) \neq 1, \ \phi(y) \neq 1, \ \phi(z) \neq 1.$$

Hence, $\phi([x, y, y]) \neq 1$ and $\phi([y, z, z]) \neq 1$ in F, which contradict the assumptions that $[x, y, y] = 1$ and $[y, z, z] = 1$ in G. Thus G is 2-ET.

Conversely, without loss of generality we may assume that G is non-abelian. Also if G is non 2-Engel and residually free, then the result holds by Theorem 2.11(ii). Now, let G be a non-abelian 2-Engel residually free group. Then $[x, y, y] = 1$ for all x, y in G and for some non-trivial elements $x_0, y_0 \in G$, we have $[x_0, y_0] \neq 1$. Hence there is a homomorphism $\phi : G \to F$, where F is a free group such that $\phi([x_0, y_0]) \neq 1$. Thus

$$\phi([x_0, y_0]) = [\phi(x_0), \phi(y_0)] \neq 1 \Rightarrow \phi(x_0) \neq 1,\ \phi(y_0) \neq 1.$$

On the other hand, since F is free we must have $\phi([x_0, y_0, y_0]) \neq 1$ in F, which contradicts that $[x_0, y_0, y_0] = 1$ in G. Therefore G is not 2-Engel and the required result is obtained from Theorem 2.11(ii), when we take χ to be the class of all free groups. □

3. The number of 2-Engelizers

As in the previous section, $E^2(G)$ denotes the set of all 2-Engelizers in the group G. Now for a given group G, one may ask about the size of $E^2(G)$. So our goal in this section is to study the possible values of $|E^2(G)|$. Note that in this section, we assume that the 2-Engelizer of each element of G is a subgroup. Indeed $x^{E^2_G(x)}$ is abelian, for every non-trivial element x of G.

One can easily check that G is 2-Engel group if and only if $|E^2(G)| = 1$. Moreover, $Z_2(G) \subseteq \cap_{x\in G} E^2_G(x)$.

Lemma 3.1. *A group G is the union of 2-Engelizers of all elements of $G\backslash Z_2(G)$, that is to say $G = \cup_{x\in G\backslash Z_2(G)} E^2_G(x)$.*

Proof. Clearly, $\cup_{x\in G\backslash Z_2(G)} E^2_G(x) \subseteq G$. By the definition, if $g \in Z_2(G)$, then $g \in E^2_G(x)$ for every $x \in G$ and hence $g \in \cup_{x\in G\backslash Z_2(G)} E^2_G(x)$. In the case $g \in G\backslash Z_2(G)$, then clearly $g \in E^2_G(g)$ and so

$$g \in \cup_{x\in G\backslash Z_2(G)} E^2_G(x).$$

Therefore $G \subseteq \cup_{x\in G\backslash Z_2(G)} E^2_G(x)$ and the proof is complete. □

Lemma 3.2. *A group G can not be written as the union of two proper subgroups of G.*

Proof. Suppose H and K are two proper subgroups of G such that $G = H \cup K$. Let $x \in H\backslash K$ and $y \in K\backslash H$. If $xy \in H$, then $x^{-1}xy = y \in H$, which gives a contradiction. Similarly, xy can not be in K and hence the claim is proved. □

Using the above lemmas we prove the following:

Theorem 3.3. *Let G be any group. Then $|E^2(G)| \geq 4$.*

Proof. Using Lemma 3.1, the group G is the union of its proper 2-Engelizers, i.e., $G = \cup_{x\in G\backslash Z_2(G)} E^2_G(x)$. If $|E^2(G)| = 1$, then G is 2-Engel, which contradicts the assumption. If $|E^2(G)| = 2$, then G is the proper subgroup of itself, which is impossible. Assume $|E^2(G)| = 3$. Then $E^2(G) = \{G, E^2_G(x), E^2_G(y)\}$, where $E^2_G(x)$ and $E^2_G(y)$ are proper 2-Engelizers of G. Therefore $G = E^2_G(x) \cup E^2_G(y)$, which contradicts Lemma 3.2. Hence $|E^2(G)| \geq 4$ and this completes the proof. □

Part (i) of the following example shows that the lower bound obtained in the above theorem is attained. Also one notes that the number of 2-Engelizers of a given group is always less than or equal to the number of centralizers.

Example 3.4. (i) Consider $D_{16} = \langle a, b : a^8 = b^2 = 1, bab = a^{-1}\rangle$, the dihedral group of order 16. It can be easily calculated that all 2-Engelizers of D_{16} are precisely as follows:

$$D_{16},\ E^2_{D_{16}}(a) = \langle a\rangle,\ E^2_{D_{16}}(b) = \{1, a^2, a^4, a^6, b, a^2b, a^4b, a^6b\},$$
$$E^2_{D_{16}}(ab) = \{1, a^2, a^4, a^6, ab, a^3b, a^5b, a^7b\}.$$

Hence $|E^2(D_{16})| = 4$.

(ii) All 2-Engelizers of the symmetric group $S_3 = \langle a, b : b^3 = a^2 = 1, aba^{-1} = b^{-1}\rangle$ are as follows:

$$S_3,\ E^2_{S_3}(a) = \{1, a\},\ E^2_{S_3}(b) = \{1, b, b^2\},\ E^2_{S_3}(ab) = \{1, ab\},\ E^2_{S_3}(ab^2) = \{1, ab^2\}.$$

Therefore $|E^2(S_3)| = 5$.

Lemma 3.5. *Let $|E^2_{G/Z_2(G)}(xZ_2(G))| = p$ for some non second central element x of a group G and p be an any prime number. For all $y \in G\backslash Z_2(G)$, if $E^2_{G/Z_2(G)}(xZ_2(G)) = E^2_{G/Z_2(G)}(yZ_2(G))$, then*

$$E^2_G(x) = E^2_G(y).$$

Proof. Clearly,

$$E^2_G(x)/Z_2(G) \leq E^2_{G/Z_2(G)}(xZ_2(G)).$$

Assume that $E^2_G(x)/Z_2(G) < E^2_{G/Z_2(G)}(xZ_2(G))$. As $|E^2_{G/Z_2(G)}(xZ_2(G))| = p$ and $|E^2_G(x)/Z_2(G)|$ divides $|E^2_{G/Z_2(G)}(xZ_2(G))|$, we get $|E^2_G(x)/Z_2(G)| = 1$ and so $E^2_G(x) = Z_2(G)$. Thus $x \in Z_2(G)$ which is a contradiction. Therefore $E^2_G(x)/Z_2(G) = E^2_{G/Z_2(G)}(xZ_2(G))$. Clearly for all $y \in G\backslash Z_2(G)$,

$$E^2_G(y)/Z_2(G) \leq E^2_{G/Z_2(G)}(yZ_2(G)) = E^2_{G/Z_2(G)}(xZ_2(G)).$$

Hence $|E^2_{G/Z_2(G)}(xZ_2(G))| = |E^2_G(y)/Z_2(G)|$ and so

$$E^2_G(y)/Z_2(G) = E^2_G(x)/Z_2(G).$$

Thus

$$\frac{E^2_G(x)}{Z_2(G)} = \frac{E^2_G(y)}{Z_2(G)} = \{Z_2(G), x_1Z_2(G), x_2Z_2(G), \ldots, x_{p-1}Z_2(G)\},$$

where $\{x_1, \ldots, x_{p-1}\} \in E^2_G(x) \cap E^2_G(y)\backslash Z_2(G)$. So $E^2_G(x) = E^2_G(y)$. □

Characterization of finite groups in terms of the number of distinct centralizers has been an interesting topic of research in recent years (see [1, 2, 4]). In [4] Belcastro and Sherman proved that G is 4-centralizer if and only if $G/Z(G) \cong C_2 \times C_2$ and G is 5-centralizer if and only if $G/Z(G) \cong C_3 \times C_3$ or S_3. Here we calculate $|E^2(G)|$ in the case of $G/Z_2(G) \cong C_p \times C_p$ for any prime number p.

Theorem 3.6. *Let G be a group such that $G/Z_2(G) \cong C_p \times C_p$ for any prime number p. Then $|E^2(G)| = p + 2$.*

Proof. Suppose that $G/Z_2(G) \cong C_p \times C_p$, and hence

$$\frac{G}{Z_2(G)} = \langle xZ_2(G), yZ_2(G) : x^p, y^p, [x, y] \in Z_2(G)\rangle.$$

Clearly any non-trivial proper subgroup $H/Z_2(G)$ of $G/Z_2(G)$ has order p. Therefore $H = Z_2(G) \cup h_1Z_2(G) \cup h_2Z_2(G) \cup \cdots \cup h_{p-1}Z_2(G)$, where $h_i \in H\backslash Z_2(G)$ for all $1 \leq i \leq p-1$. Thus the proper subgroups of G properly containing $Z_2(G)$ are one of the following forms:

$$Z_2(G) \cup xZ_2(G) \cup x^2Z_2(G) \cup \cdots \cup x^{p-1}Z_2(G);$$

$$Z_2(G) \cup yZ_2(G) \cup y^2Z_2(G) \cup \cdots \cup y^{p-1}Z_2(G) \text{ or}$$

$Z_2(G) \cup x^iy^jZ_2(G)$, where $1 \leq i, j \leq p-1$. Note that, for all $1 \leq i, j \leq p-1$, it is easy to see that $x^iy^jZ_2(G) = x^jy^iZ_2(G)$ since $[x, y] \in Z_2(G)$. Hence we have only $p-1$ proper subgroups of G of latest type. For simplicity, we denote all the above subgroups by $H_1, H_2, \ldots, H_{p+1}$, respectively. Now we show that $H_1, H_2, \ldots, H_{p+1}$ are the only proper 2-Engelizers of G. Let $a \in G\backslash Z_2(G)$, then $aZ_2(G) = bZ_2(G)$ for some

$$b \in \{x, \ldots, x^{p-1}, y, \ldots, y^{p-1}, xy, xy^2, \ldots, xy^{p-1}, \ldots, x^{p-1}y, \ldots, x^{p-1}y^{p-1}\}.$$

Therefore $E^2_{G/Z_2(G)}(aZ_2(G)) = E^2_{G/Z_2(G)}(bZ_2(G))$ and Lemma 3.5 implies that $E^2_G(a) = E^2_G(b)$. Again let $b \in H_i\backslash Z_2(G)$ then $E^2_G(b) \subseteq \cup_{j=1}^{p+1} H_j$, as $H_1, \ldots,$ H_{p+1} are the only proper subgroups of G. Also $b \in E^2_G(b)$, and hence $E^2_G(b) \neq H_j$, for $1 \leq i \neq j \leq p+1$. Therefore $E^2_G(b) = H_i$, and $H_1, H_2, \ldots, H_{p+1}$ are the only proper 2-Engelizers of G and so $|E^2(G)| = p + 2$. □

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