Bull. Korean Math. Soc. **53** (2016), No. 3, pp. 651–656
http://dx.doi.org/10.4134/BKMS.b140564
pISSN: 1015-8634 / eISSN: 2234-3016

CHARACTERIZATION OF SUZUKI GROUP BY NSE AND ORDER OF GROUP

Ali Iranmanesh, Hosein Parvizi Mosaed, and Abolfazl Tehranian

Abstract. Let G be a finite group and $\mathrm{nse}(G)$ be the set of numbers of elements of G of the same order. In this paper, we prove that the simple group $Sz(2^{2m+1})$, where $2^{2m+1}-1$ is a prime number, is uniquely determined by $\mathrm{nse}(Sz(2^{2m+1}))$ and $|Sz(2^{2m+1})|$.

1. Introduction

For a finite group G, by $\pi(G)$ we denote the set of primes q such that G contains an element of order q and by $\pi_e(G)$ we mean the set of element orders of G. Let $i \in \pi_e(G)$ and $m_i = m_i(G)$ be the number of elements of order i in G. Then we denote the set of numbers of elements of G of the same order by $\mathrm{nse}(G)$, that is; $\mathrm{nse}(G) = \{m_i \mid i \in \pi_e(G)\}$. The prime graph of group G which is denoted by $\Gamma(G)$ is a graph with vertex-set $\pi(G)$, two distinct vertices u and v are adjacent if and only if $uv \in \pi_e(G)$. The number of connected components of $\Gamma(G)$ is denoted by $t(G)$ and the set of the connected components is denoted by $\{\pi_i \mid i = 1, 2, \ldots, t(G)\}$. If G is a group of even order, then we always assume that $2 \in \pi_1$.

The characterization by nse is one of the problems related to the Thompson's problem (Problem 12.37 of [10]) that posed by Shao and et al.. In [12], they proved that if G is a simple K_4-group, then G is characterizable by $\mathrm{nse}(G)$ and $|G|$. (The simple group G is called simple K_n-group if $|\pi(G)| = n$.) Following this result, in [5, 6, 7, 9, 11], it is proved that the groups A_{12}, A_{13}, $L_2(2^n)$ where either $2^n - 1$ or $2^n + 1$ is a prime number, symmetric group S_r where r is a prime number and sporadic groups are characterizable by nse and order of group. In this paper, we continue this work and we prove that the simple Suzuki group is characterizable by nse and order of group. In fact, we prove the following main theorem:

Received July 22, 2014; Revised February 6, 2015.
2010 *Mathematics Subject Classification.* Primary 20D60; Secondary 20D06.
Key words and phrases. set of the numbers of elements of the same order, Suzuki group.

Main Theorem. *Let G be a finite group such that* $\mathrm{nse}(G) = \mathrm{nse}(Sz(2^{2m+1}))$ *and* $|G| = |Sz(2^{2m+1})|$, *where* $2^{2m+1} - 1$ *is a prime number. Then* $G \cong Sz(2^{2m+1})$.

2. Notation and preliminaries

For the finite group G, a Sylow q-subgroup of G is denoted by G_q and the number of Sylow q-subgroups of G is denoted by $n_q = n_q(G)$. Also, the largest element order of G_q is denoted by $exp(G_q)$. Moreover, we denote by φ the Euler totient function, by $\pi(n)$ the set of all prime divisors of natural number n and by (a, b) the greatest common divisor of integers a and b.

In the following, we bring some useful lemmas which will be used in the proof of the main theorem.

Lemma 2.1 ([1, 4])**.** *Let G be a Frobenius group of even order with kernel K and complement H. Then*

(1) $t(G) = 2$, *$\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$.*
(2) *$|H|$ divides $|K| - 1$.*
(3) *K is nilpotent.*

A group G is a 2-Frobenius group if there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernel K/H and H respectively.

Lemma 2.2 ([1])**.** *Let G be a 2-Frobenius group of even order. Then*

(1) $t(G) = 2$, *$\pi(G/K) \cup \pi(H) = \pi_1$ and $\pi(K/H) = \pi_2$.*
(2) *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|\mathrm{Aut}(K/H)|$.*

Lemma 2.3 ([13])**.** *Let G be a finite group with $t(G) > 1$. Then one of the following statements holds*:

(1) *G is a Frobenius group.*
(2) *G is a 2-Frobenius group.*
(3) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|\mathrm{Out}(K/H)|$. Moreover, any odd order component of G is also an order component of K/H.*

Lemma 2.4 ([3])**.** *Let p, q be prime numbers and m, n be natural numbers such that $p^m - q^n = 1$. Then one of the following statements holds*:

(1) *If $m = 1$, then $p = 2^{2^t} + 1$, where $t \geq 0$ is a integer number.*
(2) *If $n = 1$, then $q = 2^{p_0} - 1$, where p_0 is a prime number.*
(3) *If $m, n > 1$, then $(p, q, m, n) = (3, 2, 2, 3)$.*

Lemma 2.5 ([14])**.** *Let q, k, l be natural numbers. Then*

(1) $(q^k - 1, q^l - 1) = q^{(k,l)} - 1$.

(2) $(q^k+1, q^l+1) = \begin{cases} q^{(k,l)}+1 & \textit{if both } \frac{k}{(k,l)} \textit{ and } \frac{l}{(k,l)} \textit{ are odd,} \\ (2, q+1) & \textit{otherwise.} \end{cases}$

(3) $(q^k-1, q^l+1) = \begin{cases} q^{(k,l)}+1 & \textit{if } \frac{k}{(k,l)} \textit{ is even and } \frac{l}{(k,l)} \textit{ is odd,} \\ (2, q+1) & \textit{otherwise.} \end{cases}$

In particular, for every $q \geq 2$ and $k \geq 1$ the inequality $(q^k-1, q^k+1) \leq 2$ holds.

Lemma 2.6 ([2])**.** *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \mid g^m = 1\}$, then $m \mid |L_m(G)|$.*

Lemma 2.7. *Let G be a finite group. Then for every $i \in \pi_e(G)$,*

$$\begin{cases} \varphi(i) \mid m_i \\ i \mid \sum_{d|i} m_d \end{cases}$$

and if $i > 2$, then m_i is even.

Proof. According to Lemma 2.6, the proof is straightforward. □

Lemma 2.8. *Let S be Suzuki group $Sz(2^{2m+1})$, where $p = 2^{2m+1} - 1$ is a prime number. Then $m_p = (p-1)(p+1)^2((p+1)^2+1)/2$ and $p \mid m_r$ for every $r \in \pi_e(S) \setminus \{1, p\}$.*

Proof. Since $|S_p| = p$, we deduce that S_p is a cyclic group of order p and hence Lemma 2.7 implies that $m_p = \varphi(p)n_p = (p-1)n_p$. Now it is enough to show that $n_p = (p+1)^2((p+1)^2+1)/2$. Since S_p is a cyclic group of order p, we deduce that $|N_S(S_p)/C_S(S_p)| \mid (p-1)$ and since according to Table 3 in [8], p is an isolated point of $\Gamma(S)$, we deduce that $|C_S(S_p)| = p$ and hence $|N_S(S_p)| = xp$, where $x \mid (p-1)$. On the other hand, $N_S(S_p) \leq S$ implies that $x \mid (p+1)^2((p+1)^2+1)$. Thus $x \mid (p+1)^2((p+1)^2+1) = ((p^3+5p^2+12p+18)(p-1)+20)$. So we deduce that $x \mid 20$. But by Sylow's theorem, $n_p \equiv 1 \pmod p$ and hence $(p+1)^2((p+1)^2+1)/x \equiv 1 \pmod p$. Thus $p \mid (2-x)$. Since $x \mid 20$ and $p = 2^{2m+1}-1$ is prime, we deduce that $x = 2$ and $n_p = (p+1)^2((p+1)^2+1)/2$, as desired.

Let $r \in \pi_e(S) \setminus \{1, p\}$. Since p is an isolated point of $\Gamma(S)$, we deduce that $p \nmid r$ and $pr \notin \pi_e(S)$. Thus S_p acts fixed point freely on the set of elements of order r by conjugation and hence $|S_p| \mid m_r$. So we conclude that $p \mid m_r$. □

3. Proof of Main Theorem

From now on, we suppose that $p := 2^{2m+1} - 1$ is a prime number. Also we denote $Sz(2^{2m+1})$ by S. Thus $\mathrm{nse}(G) = \mathrm{nse}(S)$ and $|G| = |S|$. We prove the main theorem in a sequence of lemmas.

Lemma 3.1. *$m_2(G) = m_2(S)$, $m_p(G) = m_p(S)$, $n_p(G) = n_p(S)$ and p is an isolated point of $\Gamma(G)$.*

Proof. According to Lemma 2.7, if H is a group of even order and $r \in \pi_e(H) \setminus \{1\}$ such that m_r is odd, then $r = 2$. Thus $m_2(G) = m_2(S)$. By Lemma 2.7, $(m_p(G), p) = 1$. Thus $p \nmid m_p(G)$ and hence by Lemma 2.8, $m_p(G) = m_p(S)$. Since G_p and S_p are cyclic groups of order p, we deduce that $m_p(G) = \varphi(p)n_p(G) = \varphi(p)n_p(S) = m_p(S)$. So $n_p(G) = n_p(S)$. If p is not an isolated point of $\Gamma(G)$, then there is $t \in \pi(G) - \{p\}$ such that $tp \in \pi_e(G)$. Thus $m_{tp}(G) = \varphi(tp)n_p(G)k$, where k is the number of cyclic subgroups of order t in $C_G(G_p)$. By the proof of Lemma 2.8, we have $n_p(S) = |S|/2p$ and since $n_p(S) = n_p(G)$ and $|S| = |G|$, we deduce that $n_p(G) = |G|/2p$. Thus $(t-1)(p-1)|G|/2p = \varphi(tp)n_p(G) \mid m_{tp}(G)$ and hence $(t-1)(p-1)|G|/2p \leq m_{tp}(G)$. So $(t-1)(p-1)/2p \leq 1$ because $|G| \geq m_{tp}(G)$. This implies that $(t-1) \leq 2p/(p-1) < 3$. So we deduce that $t \in \{2, 3\}$. If $t = 2$, then by Lemma 2.8, $p \mid m_{2p}(G)$. So $(p-1)|G|/2 \mid m_{2p}(G)$ which implies that $m_{2p}(G) > |G|$, a contradiction. Similar argument leads us to a contradiction for the case $t = 3$. □

Lemma 3.2. *G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. Let G be a Frobenius group with kernel K and complement H. Then by Lemma 2.1, $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$. Since p is an isolated point of $\Gamma(G)$, we deduce that $|K| = p$, $|H| = (p+1)^2((p+1)^2+1)$ or $|H| = p$, $|K| = (p+1)^2((p+1)^2+1)$ and hence $(p+1)^2((p+1)^2+1) \mid (p-1)$ or $p \mid ((p+1)^2((p+1)^2+1)-1)$. This is a contradiction. So G is not a Frobenius group. Let G be a 2-Frobenius group. Then by Lemma 2.2, $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernel K/H and H respectively, $\pi(G/K) \cup \pi(H) = \pi_1$, $\pi(K/H) = \pi_2$ and $|G/K|$ divides $|\mathrm{Aut}(K/H)|$. Let $|G/K| = x$. According to Lemma 3.1, p is an isolated point of $\Gamma(G)$. Thus $|K/H| = p$ and $x|H| = (p+1)^2((p+1)^2+1)$ and hence $x \mid (p+1)^2((p+1)^2+1)$. On the other hand, $|G/K|$ divides $|\mathrm{Aut}(K/H)|$. Thus $x \mid (p-1)$. So we deduce that $x \mid ((p+1)^2((p+1)^2+1) - (p^3+5p^2+12p+18)(p-1))$. Hence $x \mid 20$. Since K is a Frobenius group with kernel H, Lemma 2.1 implies that $p \mid ((p+1)^2((p+1)^2+1)/x - 1)$ and hence $p \mid ((p+1)^2((p+1)^2+1) - x) = (p(p^3+4p^2+7p+6) + (2-x))$. Thus $p \mid (2-x)$ and since $p = (2^{2m+1}-1)$ is a prime, we deduce that $x = 2$ and $|H| = (p+1)^2((p+1)^2+1)/2$. Let L be complement of Frobenius group K. Since H is nilpotent, we deduce that $H_2 \rtimes L$ is Frobenus group. Thus by Lemma 2.1, $|L|$ divides $|H_2| - 1$ and hence $p \mid ((p+1)^2/2 - 1)$. This is a contradiction. □

Lemma 3.3. *$G \cong S$.*

Proof. Since p is an isolated point of $\Gamma(G)$, we deduce that $t(G) > 1$. Thus according to Lemmas 2.3 and 3.2, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is a non-abelian simple group. As 3 divides the order of the finite non-abelian simple groups except $Sz(2^{2n+1})$ and 3 doesn't divide $|K/H|$, we have $K/H \cong Sz(2^{2n+1})$. Hence $|Sz(2^{2n+1})|$

divides $|G|$. Thus $2^{2n+1} \mid 2^{2m+1}$, which yields $2n+1 \leq 2m+1$. On the other hand, since p is an isolated point of $\Gamma(G)$, we deduce that $p \mid |Sz(2^{2n+1})|$. Thus $p \mid (2^{4n+2}+1)$ or $p \mid (2^{2n+1}-1)$. Hence Lemma 2.5 yields $p \mid (2^{2n+1}-1)$. Therefore $2m+1 \leq 2n+1$. So we deduce that $m = n$ and $K/H \cong S$. Now since $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ and $|K/H| = |S| = |G|$, we conclude that $H = 1$ and $G = K \cong S$. $\square$

Acknowledgments The authors would like to express their thanks to the referees for their careful reading, helpful comments and valuable suggestions for the improvement of this paper. Partial support by the Center of Excellence of Algebraic Hyperstructures and its Applications of Tarbiat Modares University (CEAHA) is gratefully acknowledged by the first author (AI).

References

[1] G. Y. Chen, *About Frobenius groups and* 2*-Frobenius groups*, J. Southwest China Normal University **20** (1995), no. 5, 485–487.

[2] G. Frobenius, *Verallgemeinerung des sylowschen satze*, Berliner sitz (1895), 981–993.

[3] G. C. Gerono, *Note sur la reśolution en nombres entiers et positifs de l'équation* $x^m = y^n + 1$ Nouv. Ann. Math **2** (1870), no. 9, 469–471.

[4] D. Gorenstein, *Finite Groups*, Harper and Row, New York, 1980.

[5] S. Guo, S. Liu, and W. Shi, *A new characterization of alternating group* A_{13}, Far East J. Math. Sci. **62** (2012), no. 1, 15–28.

[6] A. R. Khalili Asboei, S. S. Salehi Amiri, A. Iranmanesh, and A. Tehranian, *A characterization of Symmetric group* S_r*, where* r *is prime number*, Ann. Math. Inform. **40** (2012), 13–23.

[7] ______, *A characterization of sporadic simple groups by* NSE *and order*, J. Algebra Appl. **12** (2013), no. 2, 1250158, 3 pp.

[8] A. S. Kondrat'ev, *Prime graph components of finite simple groups*, Math. USSR-Sb. **67** (1990), no. 1, 235–247.

[9] S. Liu and R. Zhang, *A new characterization of* A_{12}, Math. Sci. **6** (2012), no. 1, 1–4.

[10] V. D. Mazurov and E. I. Khukhro, *Unsolved problems in group theory*, The Kourovka Notebook, 16 ed. Inst. Mat. Sibirsk. Otdel. Akad. Novosibirsk, 2006.

[11] C. G. Shao and Q. H. Jiang, *A new characterization of some linear groups by* NSE, J. Algebra Appl. **13** (2014), no.2, 1350094, 9 pages.

[12] C. G. Shao, W. J. Shi, and Q. H. Jiang, *Characterization of simple* K_4*-groups*, Front. Math. China **3** (2008), no. 3, 355–370.

[13] J. S. Williams, *Prime graph components of finite groups*, J. Algebra **69** (1981), no. 2, 487–513.

[14] A. V. Zavarnitsine, *Recognition of the simple groups* $L_3(q)$ *by element orders*, J. Group Theory **7** (2004), no. 1, 81–97.

Ali Iranmanesh
Department of Mathematics
Tarbiat Modares University
Tehran, Iran
E-mail address: iranmana@modares.ac.ir

Hosein Parvizi Mosaed
Department of Mathematics
Science and Research Branch
Islamic Azad University
Tehran, Iran
E-mail address: h.parvizi.mosaed@gmail.com

Abolfazl Tehranian
Department of Mathematics
Science and Research Branch
Islamic Azad University
Tehran 14115-137, Iran
E-mail address: tehranian@srbiau.ac.ir