

AUTOMATIC CONTINUITY OF ALMOST MULTIPLICATIVE LINEAR FUNCTIONALS ON FRÉCHET ALGEBRAS

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ABSTRACT. A linear functional T on a Fréchet algebra $(A, (p_n))$ is called *almost multiplicative* with respect to the sequence (p_n) , if there exists $\varepsilon \geq 0$ such that $|Tab - TaTb| \leq \varepsilon p_n(a)p_n(b)$ for all $n \in \mathbb{N}$ and for every $a, b \in A$.

We show that an almost multiplicative linear functional on a Fréchet algebra is either multiplicative or it is continuous, and hence every almost multiplicative linear functional on a functionally continuous Fréchet algebra is continuous.

1. Introduction

Let A be an algebra over the complex field. A subset V of A is *balanced* if $\lambda V \subseteq V$ for all scalars λ such that $|\lambda| \leq 1$. An algebra A is a *Fréchet algebra* if it is a complete metrizable topological linear space and has a neighbourhood basis (V_n) of zero such that V_n is *absolutely convex* (convex and balanced) and V_n is idempotent, i.e., $V_n V_n \subseteq V_n$ for all $n \in \mathbb{N}$. The topology of a Fréchet algebra A can be generated by a sequence (p_n) of separating submultiplicative seminorms, i.e., $p_n(xy) \leq p_n(x)p_n(y)$ for all $n \in \mathbb{N}$ and $x, y \in A$, such that $p_n(x) \leq p_{n+1}(x)$, whenever $n \in \mathbb{N}$ and $x \in A$. If A is unital, then p_n can be chosen such that $p_n(1) = 1$ for all $n \in \mathbb{N}$.

A Fréchet algebra A with the above generating sequence of seminorms (p_n) is denoted by $(A, (p_n))$. A sequence (x_k) in the Fréchet algebra $(A, (p_n))$ converges to $x \in A$ if and only if $p_n(x_k - x) \rightarrow 0$ for each $n \in \mathbb{N}$, as $k \rightarrow \infty$.

Let A be an algebra. An element $x \in A$ is called *quasi-invertible* if there exists $y \in A$ such that

$$x + y - xy = y + x - yx = 0.$$

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The set of all quasi-invertible elements of A is denoted by $q - InvA$. A topological algebra A is called a Q -algebra if $q - InvA$ is open, or equivalently, if $q - InvA$ has an interior point in A [19, Lemma E2].

The (*Jacobson*) radical of A , denoted by $radA$, is the intersection of all maximal left (right) ideals in A . The algebra A is called *semisimple* if $radA = \{0\}$. If A is a commutative Fréchet algebra, then $radA = \bigcap_{\varphi \in M(A)} ker\varphi$, where $M(A)$ is the continuous character space of A , i.e., the space of all continuous non-zero multiplicative linear functionals on A . See, for example, [6, Proposition 8.1.2].

A well-known class of Fréchet algebras is the class of Fréchet Q -algebras. Banach algebras are important examples of Fréchet Q -algebras.

Homomorphisms between different classes of topological algebras, including Fréchet algebras, Q -algebras and Banach algebras, have been widely studied by many authors. For the automatic continuity of homomorphisms between Banach algebras, Fréchet algebras and topological algebras one may refer to the monographs of T. Husain [9], H. G. Dales [3], M. Fragoulopoulou [5], K. Jarosz [10], E. Kaniuth [14], A. Mallios [18], and the interesting article of E. A. Michael [19].

The following theorem is a well-known result, due to Šilov, concerning the automatic continuity of multiplicative linear maps (homomorphisms) between Banach algebras.

Theorem 1.1 ([3, Theorem 2.3.3]). *Let A and B be Banach algebras such that B is commutative and semisimple. Then, every homomorphism $T : A \rightarrow B$ is automatically continuous.*

By the theorem above, every commutative semisimple Banach algebra has a unique complete norm. It was a major open question for many years whether every semisimple Banach algebra has a unique complete norm, even if it is not commutative. This was eventually proved in 1967 by B. E. Johnson, and as a consequence of this result, it was shown that if $T : A \rightarrow B$ is a surjective homomorphism between a Banach algebra A and a semisimple Banach algebra B , then T is automatically continuous.

Many authors have investigated automatic continuity of homomorphisms between Banach algebras and Fréchet algebras, and there are many open questions in this area. For example, in 1952, E. A. Michael posed the question as whether each multiplicative linear functional on a (commutative semisimple) Fréchet algebra is automatically continuous [19]. This question, known as the Michael's problem, has been intensively studied, but only partial answers have been obtained so far. For further results on automatic continuity of homomorphisms between certain classes of Fréchet algebras, or partial answers to Michael's problem, one may refer, for example, to [4], [7] and the references therein.

In 1985, K. Jarosz introduced the concept of *almost multiplicative maps* between Banach algebras and investigated the problem of automatic continuity of such maps [10]. For Banach algebras A and B , a linear map $T : A \rightarrow B$ is

called *almost multiplicative* if there exists $\varepsilon \geq 0$ such that

$$\|Tab - TaTb\| \leq \varepsilon \|a\| \|b\|$$

for every $a, b \in A$.

Later, in 1986, B. E. Johnson obtained some results on the continuity of *approximately (almost) multiplicative functionals* [12] and then in 1988, Johnson extended his results to *approximately (almost) multiplicative maps* between Banach algebras [13]. Since then, many authors investigated almost multiplicative and nearly (almost) additive maps (operators) between different classes of Banach algebras. See, for example, [1], [8], [11], [15], [16] and [20]. However, as far as we know, the automatic continuity of such maps (operators) between Fréchet algebras has not been studied yet. We investigate some properties of such maps in Sections two and three.

We first recall some notions in topological algebras.

Definition. A subset V of a complex algebra A is called *m-convex* (multiplicatively convex) if V is convex and idempotent, i.e., $VV \subseteq V$. A topological algebra is locally convex if there is a base of neighborhoods of zero consisting of convex sets.

Since every base of convex neighborhoods of zero consists of absolutely convex sets, we may always assume that a locally convex algebra has a base of neighborhoods of zero consisting of absolutely convex sets.

A topological algebra is a locally multiplicatively convex algebra, or briefly, an *lmc algebra*, if there is a base of neighborhoods of the origin consisting of sets which are absolutely convex and idempotent.

Note that Fréchet algebras are *lmc* algebras. Moreover, an *lmc* algebra A is a *Q*-algebra if and only if the spectral radius r_A is continuous at zero and it is uniformly continuous on A if A is also commutative.

A Fréchet algebra A is a *Q*-algebra if and only if the spectral radius $r_A(x)$ is finite for all $x \in A$. See, for example, [5, Theorem 6.18], [18, III. Proposition 6.2], or [19, Theorem 13.6].

Recall that a (complex) topological algebra A is called *functionally continuous* if every complex homomorphism on A is continuous. Banach algebras and *Q*-algebras are the most well-known classes of functionally continuous algebras. However, there are many functionally continuous topological algebras, which are not *Q*-algebras. The algebra of continuous complex-valued functions on \mathbb{R} , denoted by $C(\mathbb{R})$, with the compact open topology, is an example of a functionally continuous Fréchet algebra [3, Corollary 4.10.14, or page 589], which is not a *Q*-algebra, see, for example, [3, page 187]. If we take the subalgebra $C_c(\mathbb{R})$, consisting of functions f in $C(\mathbb{R})$ with compact support and with the compact open topology, then it is a proper dense ideal in $C(\mathbb{R})$, which is also functionally continuous by [19, Lemma 12.3(b)], but it is not a *Q*-algebra by [5, Examples 6.13].

It is interesting to note that, if we take $C_c(\mathbb{R})$ with the uniform norm topology $\|\cdot\|_\infty$, then it is an incomplete normed algebra, which is a Q -algebra [5, Example 6.8] and hence it is functionally continuous.

2. Automatic continuity of almost multiplicative maps between Fréchet algebras

We first introduce the concept of *almost multiplicative maps (operators)* and *weakly almost multiplicative maps (operators)* between Fréchet algebras.

Definition. Let $(A, (p_n))$ and $(B, (q_n))$ be Fréchet algebras and $\varepsilon \geq 0$. A linear map $T : (A, (p_n)) \rightarrow (B, (q_n))$ is called multiplicative if $Tab = TaTb$ for every $a, b \in A$, it is called ε -multiplicative with respect to (p_n) and (q_n) , if

$$q_n(Tab - TaTb) \leq \varepsilon p_n(a)p_n(b)$$

for all $n \in \mathbb{N}$ and every $a, b \in A$, and it is called weakly ε -multiplicative with respect to (p_n) and (q_n) , if for every $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ such that

$$q_k(Tab - TaTb) \leq \varepsilon p_{n(k)}(a)p_{n(k)}(b)$$

for every $a, b \in A$. A linear map (operator) $T : A \rightarrow B$ is called almost multiplicative or weakly almost multiplicative if it is ε -multiplicative or weakly ε -multiplicative, respectively, for some $\varepsilon \geq 0$.

It is clear that every ε -multiplicative map is weakly ε -multiplicative, and since (q_n) is a separating sequence of seminorms on B , it is clear that, ε -multiplicative and weakly ε -multiplicative linear maps turn out to be multiplicative, whenever $\varepsilon = 0$. Moreover, any multiplicative map is ε -multiplicative for every $\varepsilon \geq 0$.

Remark 2.1. In the case that $B = \mathbb{C}$, a linear functional T on a Fréchet algebra $(A, (p_n))$ is weakly ε -multiplicative with respect to (p_n) , if there exists $m \in \mathbb{N}$ such that

$$|Tab - TaTb| \leq \varepsilon p_m(a)p_m(b)$$

for every $a, b \in A$. Since the generating sequence (p_n) in the Fréchet algebra $(A, (p_n))$ is an increasing sequence, the inequality

$$|Tab - TaTb| \leq \varepsilon p_n(a)p_n(b),$$

holds for all $n \geq m$.

In this paper we study the automatic continuity of (weakly) almost multiplicative maps (operators) between Fréchet algebras. In particular, we obtain some results on almost multiplicative linear functionals, also called, complex-valued almost multiplicative maps (operators), on Fréchet algebras.

Definition. A Fréchet algebra $(A, (p_n))$ is a uniform Fréchet algebra if

$$p_n(a^2) = (p_n(a))^2$$

for each $n \in \mathbb{N}$ and for all $a \in A$.

Remark 2.2 ([6, p. 73] and [4, p. 8]). Let A and B be Fréchet algebras with generating sequences of seminorms (p_n) and (q_n) , respectively. If $\varphi : A \rightarrow B$ is a linear operator, then φ is continuous if and only if for each $k \in \mathbb{N}$, there exist $n(k) \in \mathbb{N}$ and a constant $c_k > 0$ such that

$$q_k(\varphi(a)) \leq c_k p_{n(k)}(a)$$

for every $a \in A$.

In the case that $(B, (q_n))$ is a commutative uniform Fréchet algebra and $\varphi : A \rightarrow B$ is a continuous algebra homomorphism, we may choose $c_k = 1$ for all k .

The first result is on the composition of two linear operators.

Theorem 2.3. *Let $(A, (p_n))$ and $(B, (q_n))$ be Fréchet algebras and $(D, (r_n))$ be a uniform Fréchet algebra. If $T : A \rightarrow B$ is a weakly almost multiplicative map and $\varphi : B \rightarrow D$ is a continuous homomorphism, then $\varphi \circ T$ is a weakly almost multiplicative map.*

Proof. Let T be a weakly ε -multiplicative map for some $\varepsilon > 0$. Since φ is continuous, by Remark 2.2 it follows that for each $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ such that

$$\begin{aligned} r_k((\varphi \circ T)(xy) - (\varphi \circ T)(x)(\varphi \circ T)(y)) &= r_k(\varphi(Txy - TxTy)) \\ &\leq q_{n(k)}(Txy - TxTy). \end{aligned}$$

Since T is weakly ε -multiplicative, there exists $m(k) \in \mathbb{N}$ such that

$$q_{n(k)}(Txy - TxTy) \leq \varepsilon p_{m(k)}(x)p_{m(k)}(y).$$

Therefore,

$$r_k((\varphi \circ T)(xy) - (\varphi \circ T)(x)(\varphi \circ T)(y)) \leq \varepsilon p_{m(k)}(x)p_{m(k)}(y),$$

and consequently, $\varphi \circ T$ is a weakly ε -multiplicative map. □

Corollary 2.4. *Let $(A, (p_n))$ and $(B, (q_n))$ be Fréchet algebras such that B is commutative. If $T : A \rightarrow B$ is weakly almost multiplicative and $\varphi \in M(B)$, then $\varphi \circ T$ is a weakly almost multiplicative map and, in fact, it is an almost multiplicative linear functional by Remark 2.1.*

We recall that if $(A, (p_n))$ is a Fréchet algebra and $T : (A, (p_n)) \rightarrow \mathbb{C}$ is a weakly ε -multiplicative linear functional, then there exists $m \in \mathbb{N}$ such that

$$(1) \quad |Tab - TaTb| \leq \varepsilon p_m(a)p_m(b)$$

for every $a, b \in A$. In the sequel we use this fixed m for every weakly ε -multiplicative linear functional.

Lemma 2.5. *Let $(A, (p_n))$ be a Fréchet algebra and $T : (A, (p_n)) \rightarrow \mathbb{C}$ be a weakly ε -multiplicative linear functional. Then, for every $x, y, z \in A$ we have*

$$(2) \quad |Tz| \cdot |Txy - TxTy| \leq \varepsilon(2p_m(x) + |Tx|)p_m(y)p_m(z),$$

where m is a natural number satisfying (1).

Proof. Using (1), for every $x, y, z \in A$ we have

$$\begin{aligned}
 |Tz| \cdot |Txy - TxTy| &= |TxyTz - TxTyTz| \\
 &\leq |TxyTz - Txyz| + |Txyz - TxTyTz| \\
 &\quad + |TxTyTz - TxTyTz| \\
 &\leq \varepsilon p_m(xy)p_m(z) + \varepsilon p_m(x)p_m(yz) \\
 &\quad + |Tx|\varepsilon p_m(y)p_m(z) \\
 &\leq \varepsilon p_m(x)p_m(y)p_m(z) + \varepsilon p_m(x)p_m(y)p_m(z) \\
 &\quad + |Tx|\varepsilon p_m(y)p_m(z) \\
 &\leq \varepsilon(2p_m(x) + |Tx|)p_m(y)p_m(z). \quad \square
 \end{aligned}$$

Lemma 2.6. *Let $(A, (p_n))$ be a Fréchet algebra and $T : (A, (p_n)) \rightarrow \mathbb{C}$ be a weakly ε -multiplicative linear functional. Then, at least one of the following holds:*

- (i) T is multiplicative,
- (ii) for every $a \in A$, if $p_m(a) = 0$, then $Ta = 0$.

Proof. Let T be weakly ε -multiplicative for some $\varepsilon \geq 0$ and let (ii) do not hold. Then there exists some $a \in A$ such that $p_m(a) = 0$ and $Ta \neq 0$. If we take $z = a$ in (2), we have $Txy = TxTy$ for all $x, y \in A$ and hence T is multiplicative. \square

By applying the above two lemmas we obtain the following interesting result, which is related to the Michael's problem somehow.

Theorem 2.7. *Let $(A, (p_n))$ be a Fréchet algebra and $T : (A, (p_n)) \rightarrow \mathbb{C}$ be a weakly almost multiplicative linear functional. Then, at least one of the following holds:*

- (i) T is multiplicative.
- (ii) T is continuous.

Proof. Let T be weakly ε -multiplicative for some $\varepsilon > 0$. Then, by (1), we have

$$|Tab - TaTb| \leq \varepsilon p_m(a)p_m(b)$$

for every $a, b \in A$. Set $c = \frac{1+\sqrt{1+4\varepsilon}}{2}$. If

$$(3) \quad |Ta| \leq cp_m(a)$$

for all $a \in A$, then T is continuous. If (3) does not hold for some $a_0 \in A$, we have

$$(4) \quad |Ta_0| > cp_m(a_0).$$

Therefore, $Ta_0 \neq 0$. If $p_m(a_0) = 0$, then by Lemma 2.6, T is multiplicative. Now, let $p_m(a_0) \neq 0$ in (4). By considering $\frac{a_0}{p_m(a_0)}$ instead of a_0 in (4), we may assume that $p_m(a_0) = 1$ and $|Ta_0| > c$. Therefore, we can write $|Ta_0| = c + r$ for some $r > 0$. Since $c > 1$ and $c^2 - c = \varepsilon$, by a method similar to [2,

Theorem 1], and using the fact that T is weakly ε -multiplicative, one can show by induction that

$$(5) \quad |Ta_0^{2^n}| > c + (n + 1)r$$

for all $n \in \mathbb{N}$. To prove that T is multiplicative, let $x, y \in A$ and note that, $Ta_0^{2^n} \neq 0$ for each $n \in \mathbb{N}$, by (5). By taking $z = a_0^{2^n}$ in (2), it follows from (5) that

$$\begin{aligned} |Txy - TxTy| &\leq \frac{\varepsilon(2p_m(x) + |Tx|)p_m(y)p_m(a_0^{2^n})}{|Ta_0^{2^n}|} \\ &\leq \frac{\varepsilon(2p_m(x) + |Tx|)p_m(y)p_m^{2^n}(a_0)}{|Ta_0^{2^n}|} \\ &= \frac{\varepsilon(2p_m(x) + |Tx|)p_m(y)}{|Ta_0^{2^n}|} \\ (6) \quad &\leq \frac{\varepsilon(2p_m(x) + |Tx|)p_m(y)}{c + (n + 1)r}. \end{aligned}$$

Hence, by letting $n \rightarrow \infty$ in (6), we obtain $Txy = TxTy$, which completes the proof. \square

Corollary 2.8. *Let $(A, (p_n))$ be a functionally continuous Fréchet algebra. Then, every (weakly) almost multiplicative linear functional is automatically continuous.*

We now extend the above corollary and obtain the following interesting result.

Theorem 2.9. *Let $(A, (p_n))$ and $(B, (q_n))$ be Fréchet algebras such that A is functionally continuous and B is semisimple and commutative. Then, every weakly almost multiplicative map $T : A \rightarrow B$ is automatically continuous.*

Proof. Let T be weakly almost multiplicative and $\varphi \in M(B)$. Then, $\varphi \circ T : A \rightarrow \mathbb{C}$ is weakly almost multiplicative by Corollary 2.4 and hence it is continuous by Corollary 2.8.

Now, suppose that $a_n \rightarrow 0$ in A and $Ta_n \rightarrow b$ in B . Then, by the continuity of $\varphi \circ T$, we have $(\varphi \circ T)(a_n) \rightarrow 0$. On the other hand, by the continuity of $\varphi : B \rightarrow \mathbb{C}$, $(\varphi \circ T)(a_n) = \varphi(Ta_n) \rightarrow \varphi(b)$. Consequently, $\varphi(b) = 0$ and since $\varphi \in M(B)$ was arbitrary, we get

$$b \in \bigcap_{\varphi \in M(B)} \ker \varphi = \text{rad}B = \{0\}.$$

Therefore, by the Closed Graph Theorem, T is continuous. \square

Remark 2.10. Note that the topology of A , generated by the sequence (p_n) , coincides with the topology of A , generated by the sequence $(p_n)_{n=m}^\infty$. Hence, in the definition of weakly ε -multiplicative, we may assume that the inequality

$$|Tab - TaTb| \leq \varepsilon p_n(a)p_n(b),$$

holds for all $n \in \mathbb{N}$. Therefore, the notion of weakly ε -multiplicative linear functionals is, in fact, the same as ε -multiplicative linear functionals on A .

Finally, we present an example of an almost multiplicative linear functional on a Fréchet algebra, which is not multiplicative and hence it is continuous by Theorem 2.7.

Example 2.11. We consider the Fréchet algebra $(C(\mathbb{R}), (p_n))$, with the sequence of seminorms $p_n(f) = \sup\{|f(x)| : x \in [-n, n]\}$, which is not a Q -algebra but it is functionally continuous. Take $r = \frac{1+\sqrt{1+4\varepsilon}}{2}$ for $\varepsilon > 0$. For a fixed $x_0 \in \mathbb{R}$ we define a map $T : C(\mathbb{R}) \rightarrow \mathbb{C}$ by $T(f) = rf(x_0)$, which is obviously a continuous linear functional. Since $T(fg) = rf(x_0)g(x_0)$ and $T(f)T(g) = r^2f(x_0)g(x_0)$, it follows that T is not multiplicative. However, there exists $m \in \mathbb{N}$ such that $x_0 \in [-m, m]$ and hence $x_0 \in [-n, n]$ for all $n \geq m$. Therefore,

$$\begin{aligned} |T(fg) - T(f)T(g)| &= |(r - r^2)f(x_0)g(x_0)| \\ &= (r^2 - r)|f(x_0)g(x_0)| \\ &\leq \varepsilon \|f\|_{[-m, m]} \|g\|_{[-m, m]} \\ (7) \qquad \qquad \qquad &= \varepsilon p_m(f)p_m(g) \leq \varepsilon p_n(f)p_n(g). \end{aligned}$$

It follows from (7) that T is weakly ε -multiplicative and hence, it is almost multiplicative by Remark 2.10.

References

- [1] E. Ansari-Piri and N. Eghbali, *The nearly additive maps*, Bull. Korean Math. Soc. **46** (2009), no. 2, 199–207.
- [2] J. Baker, *The stability of the cosine equation*, Proc. Amer. Math. Soc. **80** (1980), no. 3, 411–416.
- [3] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Mathematical Society, Monograph **24**, Clarendon Press, Oxford, 2000.
- [4] M. Fragoulpoulou, *Uniqueness of topology for semisimple LFQ-algebras*, Proc. Amer. Math. Soc. **117** (1993), no. 4, 963–969.
- [5] ———, *Topological Algebras with Involution*, Elsevier, 2005.
- [6] H. Goldmann, *Uniform Fréchet Algebras*, North Holland, The Netherlands, 1990.
- [7] T. G. Honary, *Automatic continuity of homomorphisms between Banach algebras and Fréchet algebras*, Bull. Iranian Math. Soc. **32** (2006), no. 2, 1–11.
- [8] R. A. J. Howey, *Approximately multiplicative functionals on algebras of smooth functions*, J. London Math. Soc. (2) **68** (2003), no. 3, 739–752.
- [9] T. Husain, *Multiplicative Functionals on Topological Algebras*, Pitman Books Limited, Boston, London, Melbourne, 1983.
- [10] K. Jarosz, *Perturbations of Banach Algebras*, Lecture Notes in Mathematics, Vol. **1120**, Springer-Verlag, Berlin, 1985.
- [11] ———, *Almost multiplicative functionals*, Studia Math. **124** (1997), no. 1, 37–58.
- [12] B. E. Johnson, *Approximately multiplicative functionals*, J. London Math. Soc. **34** (1986), no. 2, 489–510.
- [13] ———, *Approximately multiplicative maps between Banach algebras*, J. London Math. Soc. **37** (1988), no. 2, 294–316.
- [14] E. Kaniuth, *A Course in Commutative Banach Algebras*, Graduate Texts in Mathematics **246**, Springer, 2000.

- [15] G. Krishna Kumar and S. H. Kulkarni, *Linear maps preserving pseudospectrum and condition spectrum*, Banach J. Math. Anal. **6** (2012), no. 1, 45–60.
- [16] S. H. Kulkarni and D. Sukumar, *Almost multiplicative functions on commutative Banach algebras*, Studia Math. **197** (2010), no. 1, 93–99.
- [17] H. Lin, *Almost multiplicative morphisms and some applications*, J. Operator Theory **37** (1997), no. 1, 121–154.
- [18] A. Mallios, *Topological Algebras*, North Holland, 1986.
- [19] E. A. Michael, *Locally multiplicatively convex topological algebras*, Mem. Amer. Math. Soc. **1952** (1952), no. 11, 1–82.
- [20] P. Šemrl, *Almost multiplicative functions and almost linear multiplicative functionals*, Aequationes Math. **63** (2002), no. 1-2, 180–192.

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