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A Sufficient Condition for the Feedback Quasilinearization of Control Mechanical Systems

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Abstract – We derive a sufficient condition for feedback quasilinearizability of control mechanical systems and apply it to show the feedback quasilinearizability of the Acrobot system.

Keywords: Feedback quasilinearizability, Control mechanical system

1. Introduction

The equations of motion of a control mechanical system contain terms quadratically dependent on velocity that are usually called the Coriolis terms. Controller synthesis becomes tractable in the absence of these nonlinear terms [6], so it is useful to find a transformation that eliminates them from the equations of motion.

A control mechanical system is called quasilinearizable if there is a linear transformation of the velocity variables such that the Coriolis terms all vanish after the transformation. There has been active research on quasilinearization [2, 5, 7, 8], but the results were obtained by the zero curvature condition or by some complicated PDE conditions, producing restrictive outcomes. Then, very strong results were finally obtained in [4] where easily verifiable quasilinearizability conditions were derived.

In this paper we consider feedback transformations as well as state transformations, in order to increase possibility of removing the Coriolis terms from the dynamics. A control mechanical system is called feedback quasilinearizable if all Coriolis terms can be eliminated by a linear velocity transformation followed by a feedback transformation. We here obtain a sufficient condition for feedback quasilinearizability and apply it to prove the feedback quasilinearizability of the Acrobot system. We also derive a condition for partial quasilinearizability via a linear velocity transformation in the course of obtaining the result on feedback quasilinearizability.

2. Main Results

2.1 Review of quasilinearization theory

We review the theory of quasilinearization of mechanical systems in [4] from a slightly different viewpoint. We here use a linear bundle map from the tangent bundle TQ of a given configuration space Q to its cotangent bundle T^*Q instead of a linear bundle map from TQ to itself. This different style of presentation, however, does not affect the validity of the results in [4].

Let Q be an n-dimensional manifold and $q=(q^i)$ a local coordinate system on Q; refer to [1], [3] for manifolds theory. Let TQ and T^*Q denote the tangent bundle and the cotangent bundle of Q, respectively. The natural pairing between TQ and T^*Q is denoted by $\langle , \rangle$. The natural local coordinate bases of TQ and T^*Q are used.

$$TQ = span\{\partial_1, \cdots, \partial_n\}$$
$$T^*Q = span\{dq^1, \cdots, dq^n\}.$$

The symbol ∂_i is also used as the operator of partial differentiation with respect to q^i. We use the Einstein summation convention throughout this paper and the following convention for the ranges of various indices:

$$i, j, k, \ell, r, s = 1, \cdots, n;$$
$$a, b, c = 1, \cdots, p.$$

Consider a control mechanical system on the configuration space Q with Lagrangian

$$L(q,\dot{q}) = \frac{1}{2} m_{ij} \dot{q}^i \dot{q}^j - V(q)$$

and p-dimensional control bundle $W \subset T^*Q$, where $m=(m_{ij})$ is the positive definite symmetric mass matrix and $V(q)$ is the potential energy of the system. Since our results are all local, we assume that W is generated by p independent 1-forms as follows:

$$W = span\{W_1, \cdots, W_p\}$$

where each 1-form W_a, $a=1,\cdots,p$, is written in coordinates as

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$$W_a = W_{ia} dq^i .$$

The equations of motion of this control mechanical system are given by

$$\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k + m^{ij}\partial_j V = m^{ij} W_{ja} u^a \tag{1}$$

for $i = 1,\cdots,n,$ where $u = (u^a) \in \Re^p$ is the control vector. Here, m^{ij} denotes the (i,j) entry of the inverse matrix of $m = (m_{ij})$, and Γ^i_{jk} are the Christoffel symbols defined by

$$\Gamma^k_{ij} = \frac{1}{2} m^{k\ell}\left(\frac{\partial m_{\ell j}}{\partial q^i} + \frac{\partial m_{i\ell}}{\partial q^j} - \frac{\partial m_{ij}}{\partial q^\ell}\right).$$

The quadratic terms $\Gamma^i_{jk}\dot{q}^j\dot{q}^k$ in the equations of motion are called Coriolis terms.

Consider an invertible linear bundle map $A: TQ \to T^*Q$ given by

$$(q,\dot{q}) \mapsto (q, \alpha = A(q)\dot{q}) . \tag{2}$$

In coordinates,

$$\alpha_i = A_{ij}\dot{q}^j ,$$

where $\alpha = \alpha_i dq^i$. Let (B^{ij}) be the inverse matrix of (A_{ij}), i.e., $B^{ik} A_{kj} = \delta^i_j$, where δ^i_j is the Kronecker delta. In (x,α) coordinates on T^*Q, the equations of motion (1) become

$$\dot{q}^i = B^{ij}\alpha_j \tag{3}$$

$$\dot{\alpha}_i = \frac{1}{2}\left(\partial_k A_{ij} + \partial_j A_{ik} - 2A_{i\ell}\Gamma^\ell_{jk}\right) B^{jr} B^{ks}\alpha_r\alpha_s - A_{ij} m^{jk}\partial_k V - A_{ij} m^{jk} W_{ka} u^a , \tag{4}$$

where $i = 1,\cdots,n,$ or in vector form

$$\dot{q} = A^{-1}\alpha$$
$$\dot{\alpha} = f(q,\alpha) - Am^{-1}dV - Am^{-1}Wu$$

where

$$f(q,\alpha) = \left(f_i(q,\alpha)\right) = \left(\frac{1}{2}\left(\partial_k A_{ij} + \partial_j A_{ik} - 2A_{i\ell}\Gamma^\ell_{jk}\right) B^{jr} B^{ks}\alpha_r\alpha_s\right)$$
$$dV = \left(\partial_i V\right),\ W = \left(W_{ia}\right),\ u = \left(u^a\right).$$

Notice that all the Coriolis terms vanish in the $\dot{\alpha}_i$ equations in (4) if and only if

$$\partial_k A_{ij} + \partial_j A_{ik} - 2A_{i\ell}\Gamma^\ell_{jk} = 0 \tag{5}$$

for all $i, j, k,$ in which case the equations of motion become

$$\dot{x}^i = B^{ij}\alpha_j , \tag{6}$$

$$\dot{\alpha}_i = -A_{ij} m^{jk}\partial_k V - A_{ij} m^{jk} W_{ka} u^a , \tag{7}$$

or in vector form

$$\dot{q} = A^{-1}\alpha$$
$$\dot{\alpha} = -Am^{-1}dV - Am^{-1}Wu$$

***Definition 2.1**:* A control mechanical system is said to be quasilinearizable if there is an invertible linear transformation of the form (2) that transforms the equations of motion of the system (1) to the form (6) and (7).

We can regard the configuration space Q of a mechanical system as a Riemannian manifold equipped with the metric $m = (m_{ij})$ that is induced from the kinetic energy of the system. A vector field $X = X^i\partial_i$ on a Riemannian manifold (Q,m) is called a Killing vector field if it satisfies

$$X^k\partial_k m_{ij} + m_{kj}\partial_i X^k + m_{ik}\partial_j X^k = 0 \tag{8}$$

for all $1 \le i \le j \le n.$ Letting $\alpha = mX = m_{jk} X^k dq^j$, we can write (8) as

$$\partial_k\alpha_j + \partial_j\alpha_k - 2\alpha_\ell\Gamma^\ell_{jk} = 0 \tag{9}$$

in terms of the 1-form $\alpha.$ A 1-form that satisfies (9) is called a Killing 1-form. Both (8) and (9) are called the Killing equation. Comparison of (5) and (9) implies that Eq. (5) is the Killing equations in (9) for the 1-form $A_i := A_{ij} dq^j$ for each *i*. Hence, a quasilinearizing transformation consists of *n* pointwise independent Killing 1-forms, where each row of A is a Killing 1-form.

Let $iso(Q,m)$ denote the set of all Killing vector fields on (Q,m). It is a Lie algebra over $\Re$ under the usual bracket operation on vector fields. Let Δ denote the distribution on Q that is generated by Killing vector fields, i.e.

$$\Delta_q = span\{X(q) \in T_qQ \mid X \in iso(Q,m)\} \tag{10}$$

for each $q \in Q.$ The rank of Δ_q is, by definition, the dimension of Δ_q as a vector subspace of $T_qQ.$ Then the quasilinearizability can be geometrically stated as follows.

***Theorem 2.2** ([4]):* Let q be a point in (Q,m). The quasilinearization of the system (1) is possible around q

if and only if $\Delta_q = T_q Q$, i.e., rank $\Delta_q = \dim Q$.

2.2 Partial quasilinearization and feedback quasilinearization

We now pose the following two main questions for control mechanical systems that are not quasilinearizable:

Q1. (Partial Quasilinearization) How many of the $\dot{\alpha}_i$ equations in (4) can be made free of the Coriolis terms via a transformation of the form (2)?

Q2. (Feedback Quasilinearization) If an affine feedback transformation of the form

$$u = h(q) + \tilde{u} \tag{11}$$

with $h : Q \to \Re^p$ and $u \in \Re^p$, is allowed in addition to the linear transformation of the form (2), when can a given system be transformed to the form (6) and (7), i.e, to the following form

$$\dot{q} = A^{-1}\alpha \tag{12}$$

$$\dot{\alpha} = -Am^{-1}dV - Am^{-1}Wu, \tag{13}$$

which is free of the Coriolis terms?

Definition 2.3: A control mechanical system is called feedback quasilinearizable if its equations of motion can be transformed to the form (12) and (13) via a transformation of the form (2) followed by a feedback transformation of the form (11).

Definition 2.4: A point q in (Q, m) is called regular if the rank of the distribution Δ defined in (10) is constant in a neighborhood of q.

We now provide an answer to the first question we posed in the beginning of this section.

Theorem 2.5 (*Partial Quasilinearization*): Let q_0 be a regular point in (Q, m). Then, at least k $\dot{\alpha}_i$-equations can be made free of the Coriolis terms via an invertible transformation of the form (2) around q_0 if and only if rank $\Delta \geq k$ in a neighborhood of q_0.

Proof: $(\Rightarrow)$ By hypothesis there is a linear transformation $\alpha = A(q)\dot{q}$ such that the first k $\dot{\alpha}_i$-equations can be written as

$$\dot{\alpha}_i = -A_{ij}m^{jk}\partial_k V - A_{ij}m^{jk}W_{ka}u^a$$

for $i = 1, \cdots, k$ in a neighborhood of q_0. In other words Eq. (5) holds for $i = 1, \cdots, k$. Hence, the first k row vectors of A are pointwise independent Killing 1-forms, which implies that rank $\Delta \geq k$ in a neighborhood of q_0.

$(\Leftarrow)$ This direction can be proven similarly.

The above theorem can be also interpreted as follows: k is the maximum number of the $\dot{\alpha}_i$ equations that can be made free of Coriolis terms via a transformation of the form (2) around a regular point q_0 if and only if rank $\Delta = k$ in a neighborhood of q_0.

We now answer the second question posed in the beginning of this section.

Theorem 2.6 (*Feedback Quasilinearization*): A control mechanical system is feedback-quasilinearizable around a regular point q_0 if

$$\Delta_q^0 \subset W_q$$

for each q in a neighborhood of q_0, where Δ^0 is the codistribution on Q that annihilates Δ, i.e., pointwise $\Delta_q^0 = \{\beta \in T_q^*Q \mid \langle \beta_q, X_q \rangle = 0, \forall X_q \in \Delta_q\}$.

Proof: Let k be the constant rank of Δ around q_0. Then there exist k Killing vector fields $X_1, \cdots, X_k$ that span Δ pointwise around q_0. Choose $(n-k)$ more vector fields $X_{k+1}, \cdots, X_n$ such that the set of vector fields $\{X_1, \cdots, X_n\}$ span TQ around q_0. One can find $(n-k)$ 1-forms $\beta_{k+1}, \cdots, \beta_n$ in Δ^0 around q_0 such that

$$\langle \beta_i, X_j \rangle = \begin{cases} 0, & if \quad 1 \leq j \leq k \\ \delta_{ij}, & if \quad k+1 \leq j \leq n \end{cases}$$

for $k+1 \leq i \leq n$. Since $\Delta^0 \subset W$ by hypothesis, there exist vectors $u_{k+1}, \cdots, u_n$ in $\Re^p$ such that

$$\beta_i = Wu_i$$

for $k+1 \leq i \leq n$, where β_i and u_i are assumed to be in column vector form. Let

$$A = \begin{bmatrix} mX_1 & \cdots & mX_n \end{bmatrix}^T = \begin{bmatrix} X_1^T m \\ \vdots \\ X_n^T m \end{bmatrix}$$

the first k rows of which are Killing 1-forms since $X_1, \cdots, X_k$ are Killing vector fields. Let

$$\alpha = A\dot{q}$$

or in coordinates $\alpha_i = A_{ij}\dot{q}^j$. Change coordinates from $\dot{q}$ to α to transform (1) to (3) and (4), where the first k $\dot{\alpha}_i$-equations in (4) become free of the Coriolis terms. Apply the following control $u \in \Re^p$

$$u = \begin{bmatrix} u_{k+1} & \cdots & u_n \end{bmatrix} \begin{bmatrix} f_{k+1} \\ \vdots \\ f_n \end{bmatrix} + \tilde{u},$$

where

$$f_i = \frac{1}{2}\left(\partial_k A_{ij} + \partial_j A_{ik} - 2A_{i\ell}\Gamma^{\ell}_{jk}\right)B^{jr}B^{ks}\alpha_r\alpha_s$$

for $k+1 \le i \le n$. It is then easy to see that the system (3) and (4) is transformed via this feedback control to the system (12) and (13). Therefore, the system is feedback quasilinearizable around q_0.

3. Example

Consider the Acrobot system in Fig. 1, where there is an actuation u on the outer joint. Let M_1 and M_2 be the masses of the bobs and ℓ_1 and ℓ_2 the lengths of the massless rods. The gravitational acceleration is denoted by g. Let θ_1 denote the angle of the first rod measured counter-clockwise from the upward vertical, and θ_2 the angle measured counterclockwise from the ray containing the first rod to the second rod.

The Lagrangian of the system is given by

$$L = \frac{1}{2}m_{11}\dot{\theta}_1^2 + m_{12}\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}m_{22}\dot{\theta}_2^2 \\ -(M_1+M_2)g\ell_1\cos\theta_1 - M_2 g\ell_2\cos(\theta_1+\theta_2)$$

where

$$m_{11} = M_1\ell_1^2 + M_2(\ell_1^2 + \ell_2^2 + 2\ell_1\ell_2\cos\theta_2), \\ m_{12} = M_2(\ell_2^2 + \ell_1\ell_2\cos\theta_2), \\ m_{22} = M_2\ell_2^2.$$

The scalar curvature R_S of the metric $m = (m_{ij})$ is computed as

$$R_S = \frac{2m_1\cos\theta_2}{\ell_1\ell_2(M_1+M_2-M_2\cos^2\theta_2)^2},$$

which is not constant. Hence, the system is not quasilinearizable by Theorem III.1 in [4].

Let us now investigate feedback quasilinearizability of this system. The Acrobat has only one Killing vector field up to a scalar factor and it is given by $X = \partial_1$, which can be easily obtained using software Maple. Hence,

$$\Delta = span\{\partial_1\}, \Delta^0 = span\{d\theta_2\}.$$

The control bundle of the Acrobot is given by

$$W = span\{d\theta_2\}.$$

Since $\Delta^0 \subset W$, the Acrobot is feedback quasilinearizable by Theorem 2.6.

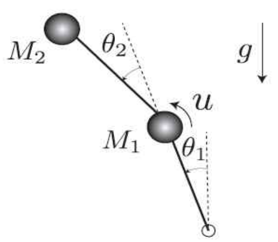

Fig. 1. The Acrobat system

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