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ON k-GRACEFUL LABELING OF SOME GRAPHS†

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Abstract. In this paper, it has been shown that the hairy cycle $C_n \odot rK_1, n \equiv 3(mod4)$, the graph obtained by adding pendant edge to each pendant vertex of hairy cycle $C_n \odot 1K_1, n \equiv 0(mod4)$, double graph of path P_n and double graph of comb $P_n \odot 1K_1$ are k-graceful.

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1. Introduction

The k-graceful labeling is the generalization of graceful labeling that introduced by Slater [14] in 1982 and by Maheo and Thuillier [10] also in 1982. Let $G(V,E)$ be a simple undirected graph with order p and size q, k be an arbitrary natural number, if there exist an injective mapping $f : V(G) \to \{0, 1, ..., q+k-1\}$ that induces bijective mapping $f^* : E(G) \to \{k, k+1, ..., q+k-1\}$ where $f^*(u,v) = |f(u) - f(v)| \ \forall \ (u,v) \in E(G)$ and $u, v \in V(G)$ then f is called k-graceful labeling, while f^* is called an induced edges k-graceful labeling and the graph G is called k-graceful graph. Graphs that are k-graceful for all k are sometimes called arbitrarily graceful.

Maheo and Thuillier [10] have shown that cycle C_n is k-graceful if and only if either $n \equiv 0$ or $1(mod4)$ with k even and $k \leq (n-1)/2$ or $n \equiv 3(mod4)$ with k odd and $k \leq (n^2-1)/2$, while P. Pradhan and et al.[11] have shown that cycle $C_n, n \equiv 0(mod4)$ is k-graceful for all $k \in N$(set of natural numbers). Maheo and Thuillier [10] have also proved that the wheel graph W_{2k+1} is k-graceful and conjecture that W_{2k} is k-graceful when $k \neq 3$ or $k \neq 4$. This conjecture has proved by Liang, Sun and Xu [8]. Liang and Liu [7] have shown that $K_{m,n}$ is k-graceful. Acharya [1] has shown that eulerian graph with q edges is k-graceful

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if either $q \equiv 0$ or $1(mod4)$ with k even or $q \equiv 3(mod4)$ with k odd. Seoud and Elsakhawi [12] have shown that paths and ladders are k-graceful.

Jirimutu [5] has shown that the graph obtained from $K_{1,n}(n \geq 1)$ by attaching $r \geq 2$ edges at each vertex is k-graceful for all $k \geq 2$. After that Jirimutu, Bao and Kong [6] have shown that the graph obtained from $K_{2,n}(n \geq 2)$ and $K_{3,n}(n \geq 3)$ by attaching $r \geq 2$ edges at each vertex is k-graceful for all $k \geq 2$ and Siqinqimuge and Jirimutu [13] have proved that the graph obtained from $K_{4,n}(n \geq 4)$ by attaching $r \geq 2$ edges at each vertex is k-graceful for all $k \geq 2$. Deligen, Zhao and Jirimutu [3] have proved that the graph obtained from $K_{5,n}(n \geq 5)$ by attaching $r \geq 2$ edges at each vertex is k-graceful for all $k \geq 2$. Bu, Zhang and He [2] have shown that an even cycle with a fixed number of pendant edges adjoined to each vertex is k-graceful.

In the following section, it has been shown that the hairy cycle $C_n \odot rK_1, n \equiv 3(mod4)$ is k-graceful and the graph obtained by adding pendant edge to pendant vertex of hairy cycle $C_n \odot 1K_1, n \equiv 0(mod4)$ is also k-graceful.

2. Hairy Cycle

A unicycle graph other than a cycle with the property that the removal of any edge from the cycle reduces G to a caterpillar is called hairy cycle. The corona of cycle C_n and rK_1 i.e. $C_n \odot rK_1$ is the example of hairy cycle.

Theorem 2.1. *The hairy cycle $C_n \odot rK_1, n \equiv 3(mod4)$ is admits k-graceful labeling where $k \leq r$.*

Proof. Let $u_i(i = 1, 2, ..., n)$ be the cycle vertices of hairy cycle $C_n \odot rK_1$ and the vertices of the r-hanged edges connected to each $u_i(i = 1, 2, ..., n)$ are denoted by $u_{it}(t = 1, 2, ..., r)$.

Consider the map $f : V(C_n \odot rK_1) \to \{0, 1, ..., n(r+1)+k-1\}$ defined as follows:

$$f(u_i) = \begin{cases} \dfrac{(i-1)(r+1)}{2}, & i \text{ is odd} \\ n(r+1) + k - i - \dfrac{i(r-1)}{2}, & i \text{ is even and } i \leq \dfrac{n+1}{2} \\ n(r+1) + k - i - \dfrac{i(r-1)}{2} - 1, & i \text{ is even and } i > \dfrac{n+1}{2} \end{cases}$$

and

$$f(u_{it}) = \begin{cases} i - 1 + \dfrac{(i-2)(r-1)}{2} + (t-1), & i \text{ is even and } 1 \leq t \leq r \\ n(r+1) + k - i - \dfrac{(i-1)(r-1)}{2} - (t-1), & i \text{ is odd, } i \leq \dfrac{n+1}{2} \text{ and } 1 \leq t \leq r \\ n(r+1) + k - i - \dfrac{(i-1)(r-1)}{2} - (t-1), & i \text{ is odd, } i = \dfrac{n+3}{2} \text{ and } 1 \leq t < k \\ n(r+1) + k - i - \dfrac{(i-1)(r-1)}{2} - t, & i \text{ is odd, } i = \dfrac{n+3}{2} \text{ and } k \leq t \leq r \\ n(r+1) + k - i - \dfrac{(i-1)(r-1)}{2} - t, & i \text{ is odd, } i > \dfrac{n+3}{2} \text{ and } 1 \leq t \leq r \end{cases}$$

It is easy to check that f is injective mapping from $V(C_n \odot rK_1)$ to $\{0, 1, ..., n(r+1)+k-1\}$. Now we prove that the induced mapping $f^* : E(C_n \odot rK_1) \to \{k, k+1, ..., n(r+1)+k-1\}$ where $f^*(u,v) = |f(u) - f(v)|$ is a bijective mapping for all edges $(u,v) \in E(C_n \odot rK_1)$. Let

$$\begin{aligned} A_i &= \{|f(u_i) - f(u_{it})| : t = 1, 2, ..., r\}, \quad i = 1, 2, ..., n \\ B_i &= \{|f(u_{i+1}) - f(u_i)| : i = 1, 2, ..., n-1\}, \\ B_n &= \{|f(u_n) - f(u_1)|\} \end{aligned}$$

The edge label induced by f^* is as follows.

$$\begin{aligned}
A_1 &= \{|f(u_1) - f(u_{1t})| : t = 1, 2, ..., r\} \\
&= \{n(r+1)+k-1, n(r+1)+k-2, ..., n(r+1)k - r\} \\
B_1 &= \{|f(u_2) - f(u_1)|\} = \{n(r+1)+k-(r+1)\} \\
A_2 &= \{|f(u_2) - f(u_{2t})| : t = 1, 2, ..., r\}, \\
&= \{n(r+1)+k-(r+2), n(r+1)+k-(r+3), ..., n(r+1)k-(2r+1)\} \\
B_2 &= \{|f(u_3) - f(u_2)|\} = \{n(r+1)+k-(2r+2)\} \\
A_3 &= \{|f(u_3) - f(u_{3t})| : t = 1, 2, ..., r\}, \\
&= \{n(r+1)+k-(2r+3), n(r+1)+k-(2r+4), ..., n(r+1)k-(3r+2)\} \\
B_3 &= \{|f(u_4) - f(u_3)|\} = \{n(r+1)+k-(3r+3)\} \\
A_4 &= \{|f(u_4) - f(u_{4t})| : t = 1, 2, ..., r\}, \\
&= \{n(r+1)+k-(3r+4), n(r+1)+k-(3r+5), ..., n(r+1)k-(4r+3)\} \\
B_4 &= \{|f(u_5) - f(u_4)|\} = \{n(r+1)+k-(4r+4)\} \\
A_{\frac{n+1}{2}} &= \{|f(u_{\frac{n+1}{2}}) - f(u_{\frac{n+1}{2}t})| : t = 1, 2, ..., r\}, \\
&= \{n(r+1)+k-(\frac{n-1}{2}r + \frac{n+1}{2}), n(r+1)+k-(\frac{n-1}{2}r + \frac{n+1}{2}+1), ..., \\
&\qquad n(r+1)k - (\frac{n+1}{2}r + \frac{n+1}{2} - 1)\} \\
B_{\frac{n+1}{2}} &= \{|f(u_{\frac{n+1}{2}+1}) - f(u_{\frac{n+1}{2}})|\} = \{n(r+1)+k-(\frac{n+1}{2}r + \frac{n+1}{2})\} \\
A_{\frac{n+1}{2}+1} &= \{|f(u_{\frac{n+1}{2}+1}) - f(u_{(\frac{n+1}{2}+1)t}| : t = 1, 2, ..., r\}, \\
&= \{n(r+1)+k-(\frac{n+1}{2}r + \frac{n+1}{2}+1), n(r+1)+k-(\frac{n+1}{2}r + \frac{n+1}{2}+2), ..., \\
&\quad n(r+1)+k-(\frac{n+1}{2}r + \frac{n+1}{2}+k-1), n(r+1)+k-(\frac{n+1}{2}r + \frac{n+1}{2}+k+1), \\
&\qquad ..., n(r+1)+k-((\frac{n+1}{2}+1)r + \frac{n+1}{2}+1)\} \\
B_{\frac{n+1}{2}+1} &= \{|f(u_{\frac{n+1}{2}+2}) - f(u_{\frac{n+1}{2}+1})|\} = \{n(r+1)+k-(\frac{n+1}{2}r + \frac{n+1}{2}+2)\} \\
A_n &= \{|f(u_n) - f(u_{nt})| : t = 1, 2, ..., r\}, \\
&= \{n(r+1)+k-((n-1)r+n+1), n(r+1)+k-((n-1)r+n+2), ..., n(r+1)k \\
&\qquad -(nr+n)\} \\
&= \{k+r-1, k+r-2, ..., k\} \\
B_n &= \{|f(u_n) - f(u_1)|\} = \{\frac{(n-1)(r+1)}{2}\} = \{n(r+1)+k-(\frac{n+1}{2}r + \frac{n+1}{2}+k)\}
\end{aligned}$$

We tide up the elements of each set and have a union

$$\left(\bigcup_{i=1}^{n} A_i\right) \bigcup \left(\bigcup_{i=1}^{n} B_i\right) = A_1 \bigcup A_2 \bigcup ... \bigcup A_n \bigcup B_1 \bigcup B_2 \bigcup ... \bigcup B_n$$
$$= \{k, k+1, ..., n(r+1)+k-1\}.$$

So the induced mapping f^* is a bijective mapping from $V(C_n \odot rK_1)$ onto $\{k, k+1, ..., n(r+1)+k-1\}$. Thus, the hairy cycle $C_n \odot rK_1, n \equiv 3(mod4)$ is admits k-graceful labeling. For example, 3-graceful labeling of hairy cycle $C_7 \odot 4K_1$, has shown in Fig. 1. □

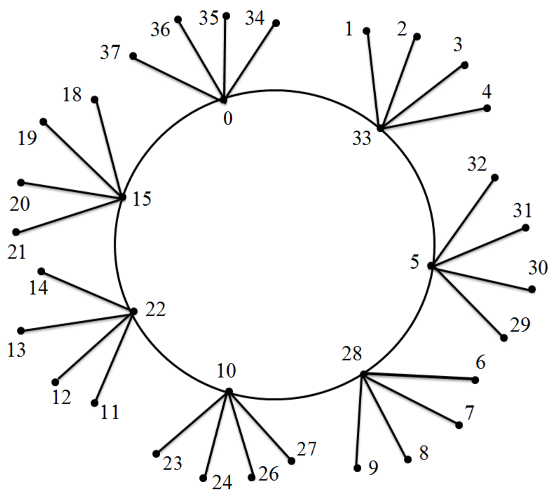

FIGURE 1. 3-graceful labeling of hairy cycle $C_7 \odot 4K_1$

Theorem 2.2. *The graph obtained by adding pendant edge to each pendant vertex of hairy cycle $C_n \odot 1K_1, n \equiv 0(mod4)$ admits k-graceful labeling.*

Proof. The order and size of the graph G obtained by adding pendant edge to each pendant vertex of hairy cycle $C_n \odot 1K_1, n \equiv 0(mod4)$ are respectively $3n$ and $3n$. Let $u_1, u_2, ..., u_n$ be the cycle vertices of $C_n \odot 1K_1, v_1, v_2, ..., v_n$ be the vertices adjacent to $u_1, u_2, ..., u_n$ and $w_1, w_2, ..., w_n$ be the vertices adjacent to $1K_1, v_1, v_2, ..., v_n$ respectively. Obviously

$$d(u_i) = 3, i = 1, 2, ..., n$$
$$d(v_i) = 2, i = 1, 2, ..., n$$
$$d(w_i) = 1, i = 1, 2, ..., n$$

Consider a labeling map $f : V(G) \to \{0, 1, ..., 3n + k - 1\}$ defined as follows:

$$f(u_i) = \begin{cases} \dfrac{3(i-1)}{2}, & i \text{ is odd} \\ 3n + k - \dfrac{3i}{2}, & i \text{ is even and } i \leq \dfrac{n}{2} \\ 3n + k - 1 - \dfrac{3i}{2}, & i \text{ is even and } i > \dfrac{n}{2}. \end{cases}$$

$$f(v_i) = \begin{cases} f(u_{i+1}) + 2, & i \text{ is odd} \\ f(u_{i-1}) + 2, & i \text{ is even} \end{cases}$$

$$f(w_i) = f(u_i) + 1$$

It is clear that f is injective and the induced labeling map $f^* : E(G) \to \{k, k+1, ..., 3n + k - 1\}$ defined as $f^*(u, v) = |f(u) - f(v)|\ \forall\ (u, v) \in E(G)$ and $u, v \in V(G)$, where u and v are adjacent vertices of G, is bijective. Thus f is k-graceful labeling of the graph G. For example, the graph obtained by adding pendant edge to each pendant vertex of $C_{16} \odot 1K_1$ and its 3-graceful labeling are shown in Fig. 2 and Fig. 3 respectively. □

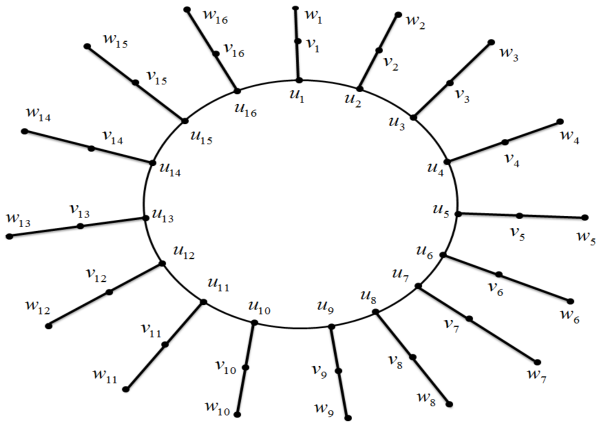

FIGURE 2

3. Double graph:

Let G' be a copy of simple graph G, let u_i be the vertices of G and v_i be the vertices of G' correspond with u_i. A new graph denoted by $D(G)$ is called the double graph of G[9] if
$V(D(G)) = V(G) \bigcup V(G')$ and
$E(D(G)) = E(G) \bigcup E(G') \bigcup \{u_i v_j : u_i \in V(G), v_j \in V(G') \; and \; u_i u_j \in E(G)\}$

Theorem 3.1. *Double graph of path $P_n (n > 1)$ is k-graceful.*

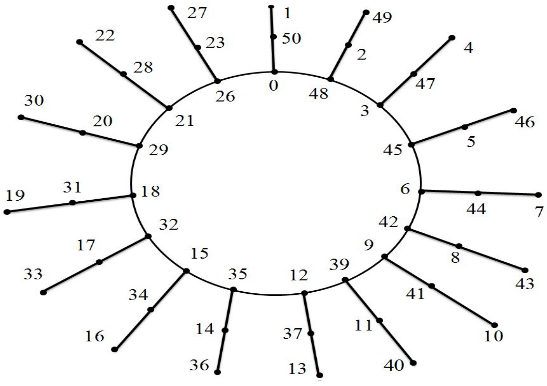

FIGURE 3

Proof. Let P_n be a path with n vertices $u_1, u_2, ..., u_n$ and v_i be the copy of u_i, then the path $P'_n = v_1, v_2, ..., v_n$ be copy of P_n. Double graph of path P_n denoted by $D(P_n)$ have order and size $2n$ and $4(n-1)$ respectively. In the following Fig. 4, Fig. 5 and Fig. 6, we have shown path P_9, P'_9 and double graph $D(P_9)$ respectively.

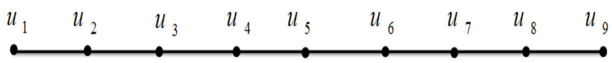

FIGURE 4. Path P_9

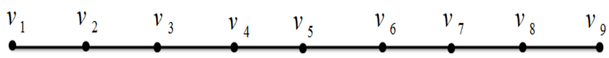

FIGURE 5. Path P'_9

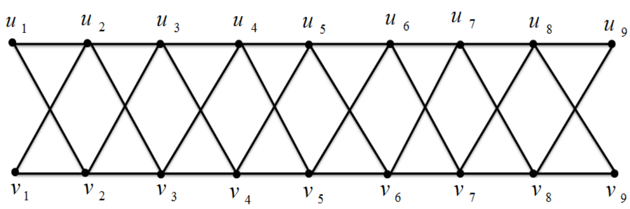

FIGURE 6. Double graph $D(P_9)$

Consider the mapping $f : V(D(P_n)) \rightarrow \{0, 1, ..., 4(n-1) + k - 1\}$ defined as follows:

$$f(u_i)=\begin{cases}\dfrac{(i-1)}{2}, & i \text{ is odd}\\ 4(n-1)+k-\dfrac{i}{2}, & i \text{ is even}\end{cases}$$

$$f(v_i)=\begin{cases}n-1+\dfrac{(i-1)}{2}, & i \text{ is odd}\\ 2(n-1)+k-\dfrac{i}{2}, & i \text{ is even}\end{cases}$$

It is clear that f is injective and the induced labeling map $f^* : E(D(P_n)) \to \{k, k+1, ..., 4(n-1)+k-1\}$ defined as $f^*(u,v) = |f(u) - f(v)|\ \forall\ (u,v) \in E(D(P_n))$ and $u, v \in V(D(P_n))$, where u and v are adjacent vertices of $D(P_n)$, is bijective. Thus f is k-graceful labeling of the double graph $D(P_n)$. Hence the double graph $D(P_n)$ is k-graceful. In the following Fig. 7, we have shown the 3-graceful labeling of the double graph $D(P_9)$.

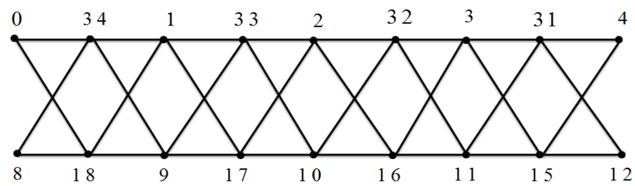

FIGURE 7. 3-graceful labeling of the double graph $D(P_9)$

□

Theorem 3.2. *Double graph of comb graph $P_n \odot 1K_1 (n > 1)$ is k-graceful.*

Proof. Let $\{v_1, v_2, ..., v_n\}$ be the set of path vertices and $\{u_1, u_2, ..., u_n\}$ be the set of pendant vertices of comb graph $P_n \odot 1K_1$ such that v_i is adjacent to $u_i, i = 1, 2, ..., n$. Similarly, let $\{v'_1, v'_2, ..., v'_n\}$ be the set of path vertices and $\{u'_1, u'_2, ..., u'_n\}$ be the set of pendant vertices of comb graph $(P_n \odot 1K_1)'$ such that v'_i is adjacent to $u'_i, i = 1, 2, ..., n$. Double graph of comb $P_n \odot 1K_1$ denoted by $D(P_n \odot 1K_1)$ have order and size $4n$ and $4(2n-1)$ respectively. In the following Fig. 8, and Fig. 9, we have shown comb graph $P_7 \odot 1K_1$ and double graph $D(P_7 \odot 1K_1)$ respectively.

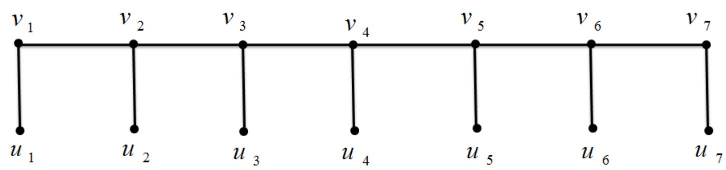

FIGURE 8. Comb graph $P_7 \odot 1K_1$

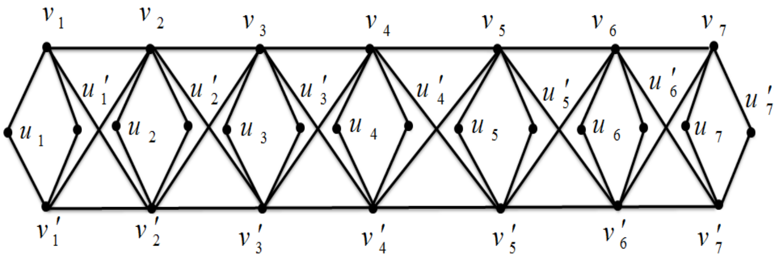

FIGURE 9. Double graph $D(P_7 \odot 1K_1)$

Consider the mapping $f : V(D(P_n)) \to \{0, 1, ..., 4(2n-1)+k-1\}$ defined as follows:

$$f(v_i) = \begin{cases} i-1, & i \text{ is odd} \\ 4(2n-1)+k-i, & i \text{ is even} \end{cases}$$

$$f(u_i) = \begin{cases} i-1, & i \text{ is even} \\ 4(2n-1)+k-i, & i \text{ is odd} \end{cases}$$

$$f(v_i') = \begin{cases} 2(n-1)+i, & i \text{ is odd} \\ 2(2n-1)+k-i, & i \text{ is even} \end{cases}$$

$$f(u_i') = \begin{cases} 2(n-1)+i, & i \text{ is even} \\ 2(2n-1)+k-i, & i \text{ is odd} \end{cases}$$

It is clear that f is injective and the induced labeling map $f^* : E(D(P_n \odot 1K_1)) \to \{k, k+1, ..., 4(2n-1)+k-1\}$ defined as $f^*(u,v) = |f(u)-f(v)| \; \forall \; (u,v) \in E(D(P_n \odot 1K_1))$ and $u, v \in V(D(P_n \odot 1K_1))$, where u and v are adjacent vertices of $D(P_n \odot 1K_1)$, is bijective. Thus f is k-graceful labeling of the double graph $D(P_n \odot 1K_1)$. Hence the double graph $D(P_n \odot 1K_1)$ is k-graceful. In the following Fig. 10, we have shown the 4-graceful labeling of the double graph $D(P_7 \odot 1K_1)$.

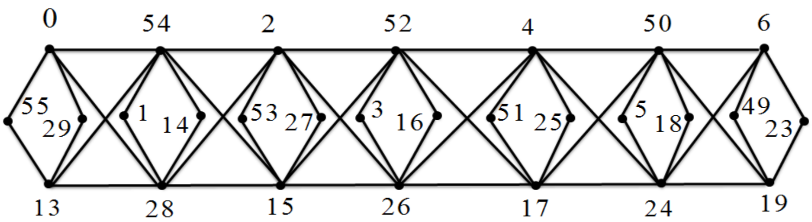

FIGURE 10. 4-graceful labeling of the double graph $D(P_7 \odot 1K_1)$

□

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